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Hybrid Taylor-Lucas Collocation Method for Numerical Solution of High-Order Pantograph Type Delay Differential Equations with Variables Delays

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Abstract: In this study we consider a higher-order linear nonhomogenous pantograph type delay differential equation with variable coefficients and variables delays, and propose a new collocation method based on hybrid Taylor and Lucas polynomials. The presented method transforms the delay differential equation with the initial and boundary conditions to a system of linear algebraic equations with the unknown Lucas coefficients; by finding Lucas coefficients easily, Lucas polynomial solutions are obtained. Also an error estimation technique based on residual function is developed for our method and applied to exiting problems.

Keywords: Taylor and Lucas polynomials; delay differential equation; variable delays; collocation method; residual error estimation

1 Introduction

Delay differential equations are frequently used to model a wide class of problems in many scientific fields such as engineering, chemical reactions, mathematical physics, biology, ecology and economics. Most of these equations have no analytic solution and numerical methods may be required to obtain approximate solutions. So, methods of solution for delay differential equations have attracted the attention of many researchers. In recent years, specially, there has been a great deal of work on one and higher-order nonhomogenous pantograph-type delay differential equations with variable coefficients and variable delays [1,2,3,4,5,6,7,8]

$$x^{(m)}(t)=\sum_{i=0}^{m-1}\sum_{j=1}^{j}P_{ij}(t)x^{(i)}(t-\tau_j(t))+f(t) \qquad (1)$$

where $P_{ij}(t)$, $f(t)$ and usually also $\tau_j(t)\geq 0$ are assumed continuous on $0\leq a\leq t\leq b$. These type equations, generally, have been used to describe fluid and elastic mechanical phenomena, and dynamical systems. We refer the reader to [2,3] and the references cited therein.

In addition, time delays occur naturally, in differential equations arising in many fields of applied mathematic. (see [5] and references therein). In population Dynamics, they may model the gestation or maturation time of a species, or the time taken for food resources to regenerate. In signal processing, delay of microseconds between the out put and reception of a signal may be important, and the time taken to receive and respond to feedback is significant in many industrial processes. On the other hand, differential equations with variable delays (1) have been intensively studied over the past 20 years. The foundations of such equations (in the case of first order) were developed in literature: oscillation properties [4], periodic solutions [3], global attractivity [5,6], the existence of positive solutions [7], asymptotically stability [8], asymptotic behaviour of solutions [2], stability criteria [4].

In this study, by means of matrix method based on collocation points which have been used by Sezer and coworkers [9,10,11,12,13,14,15,16,17,18,19,20,21,22, 23,24,25,26] and Koç et al. [27] for differential-difference and delay differential equations with constant delay, we develop a new method, called Taylor-Lucas collocation, to find the approximate solution of the differential equation with variable delay (1) under

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the initial and boundary conditions defined as

$$\sum_{i=0}^{m-1}\left(a_{r\,i}x^{(i)}(a)+b_{r\,i}x^{(i)}(b)\right)=\lambda_r,\ r=0,1,...,m-1 \quad (2)$$

where $a_{r\,i}$, $b_{r\,i}$ and λ_r are appropriate constants. Our aim is to find an approximate solution expressed in the truncated series form

$$x(t)\cong x_N(t)=\sum_{n=0}^{N}a_n\,L_n(t),\ 0\le a\le t\le b<\infty \quad (3)$$

where a_n, $n=0,1,...,N$ are unknown coefficients to be determined and $L_n(t)$, $n=0,...,N$, $N\ge m$ are the Lucas polynomials, originally studied in 1970 by Bichnell, which are defined recursively as follows [28,29,30]

$L_{n+1}(t)=t\,L_n(t)+L_{n-1}(t)$, $\quad n\ge 1$ with $L_0(t)=2$ and $L_1(t)=t$.

Their explicit form for $n\ge 1$ is

$$L_n(t)=\sum_{k=0}^{[\frac{n}{2}]}\frac{n}{n-k}\binom{n-k}{k}t^{n-2k} \quad (4)$$

where $[t]$ is the largest integer smaller than or equal to t and $\binom{n}{m}$ is the binomial coefficient; on the other hand, in order to find solutions of Eq.(1) under the conditions (2), we can use the collocation points defined by

$$t_s=a+\frac{b-a}{N}s,\ s=0,1,...,N \quad (5)$$

The paper is organized as follows. In Section 2, we derive the matrix forms of each term fin the equation (1) along with the can condition (2) and construct the fundamental matrix equation. Using these relations and the collocation points (4), we construct the Taylor-Lucas method for solution in Section 3. We develop the residual error analysis for our method in Section 4. To support our finding, we present the results of numerical experiments using Maple 12 in Section 5. We end with a brief conclusion in Section 6.

2 Fundamental Matrix Operations and Method

Firstly, we can convert the desired approximate solution (3) to the matrix form

$$x(t)\cong x_N(t)=\mathbf{L}(t)\,\mathbf{A} \quad (6)$$

where

$$\mathbf{L}(t)=[L_0(t)\quad L_1(t)\quad \ldots\quad L_N(t)]$$

$$\mathbf{A}=[a_0\quad a_1\quad \ldots\quad a_N]^T$$

Now we clearly write the matrix form $L(t)$, by using the Lucas polynomials $L_n(x)$ given by (4), as

$$\mathbf{L}(t)=\mathbf{T}(t)\mathbf{M}^T \quad (7)$$

where

$$\mathbf{T}(t)=\left[1\quad t\quad t^2\ldots t^N\right]$$

and if N is odd

$$\mathbf{M}=\begin{bmatrix} 2 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0\\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & \cdots & 0\\ 0 & \frac{3}{2}\binom{2}{1} & o & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{n-1}{(n-1)/2}\binom{(n-1)/2}{(n-1)/2} & 0 & \frac{n-1}{(n+1)/2}\binom{(n+1)/2}{(n-3)/2} & \cdots & 0\\ 0 & \frac{n}{(n+1)/2}\binom{(n+1)/2}{(n-1)/2} & 0 & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}$$

İf N is even

$$\mathbf{M}=\begin{bmatrix} 2 & 0 & 0 & \cdots & 0\\ 0 & \frac{1}{1}\binom{1}{0} & 0 & \cdots & 0\\ \frac{2}{1}\binom{1}{1} & 0 & \frac{2}{2}\binom{2}{0} & \cdots & 0\\ 0 & \frac{3}{2}\binom{2}{1} & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & \frac{n-1}{n/2}\binom{n/2}{(n-2)/2} & 0 & \cdots & 0\\ \frac{n}{n/2}\binom{n/2}{n/2} & 0 & \frac{n}{(n+2)/2}\binom{(n+2)/2}{(n-2)/2} & \cdots & \frac{n}{n}\binom{n}{0} \end{bmatrix}$$

By using the relations (6) and (7), we obtain the matrix form

$$x_N(t)=\mathbf{T}(t)\,\mathbf{M}^T\mathbf{A} \quad (8)$$

Also, it is clearly seen that the relation between the matrix $\mathrm{T}(t)$ and its kth-order derivative $\mathrm{T}^{(k)}(t)$ is

$$\mathbf{T}^{(k)}(t)=\mathbf{T}(t)\mathbf{B}^k \quad (9)$$

where

$$\mathbf{B}=\begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & N\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By using the matrices (8) and (9), we have the matrix relation

$$x_N^{(k)}(t)=\mathbf{T}(t)\,\mathbf{B}^k\mathbf{M}^T\mathbf{A},\qquad k=0,1,2,... \quad (10)$$

By putting $t\to t-\tau_j(t)$ in the relation (10), we gain the recurrence relation

$$x_N^{(k)}(t-\tau_j(t))=\mathbf{T}(t)\mathbf{S}^T\left(\tau_j(t)\right)\mathbf{B}^k\mathbf{M}^T\mathbf{A} \quad (11)$$

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$$\mathbf{S}\left(\tau_j(t)\right)=\begin{bmatrix} \binom{0}{0}\left(-\tau_j(t)\right)^0 & 0 & 0 & \cdots & 0 \\ \binom{1}{0}\left(-\tau_j(t)\right)^1 & \binom{1}{1}\left(-\tau_j(t)\right)^0 & 0 & \cdots & 0 \\ \binom{2}{0}\left(-\tau_j(t)\right)^2 & \binom{2}{1}\left(-\tau_j(t)\right)^1 & \binom{2}{1}\left(-\tau_j(t)\right)^0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & N \\ \binom{N}{0}\left(-\tau_j(t)\right)^N & \binom{N}{1}\left(-\tau_j(t)\right)^{N-1} & \binom{N}{2}\left(-\tau_j(t)\right)^{N-2} & \cdots & \binom{0}{0}\left(-\tau_j(t)\right)^0 \end{bmatrix}$$

Note that the matrix $\mathbf{T}(t-\tau_j(t))$ can be written as

$$\mathbf{T}(t-\tau_j(t))=\mathbf{T}(t)\mathbf{S}^T(\tau_j(t)).$$

By substituting the relations (10) and (11) into Eq.(1) we obtain the matrix equation

$$\mathbf{T}(t)\,\mathbf{B}^m\mathbf{M}^T\mathbf{A}=\sum_{i=0}^{m-1}\sum_{j=0}^{J}\mathbf{P}_{ij}(t)\mathbf{T}(t)\mathbf{S}^T(\tau_j(t))\,\mathbf{B}^i\mathbf{M}^T\mathbf{A}+f(t)$$

and then, by placing the collocation points (5), the system of the matrix equations

$$\left\{\mathbf{T}(t_s)\,\mathbf{B}^m-\sum_{i=0}^{m-1}\sum_{j=0}^{J}\mathbf{P}_{ij}(t)\mathbf{T}(t_s)\mathbf{S}^T(\tau_j(t_s))\,\mathbf{B}^i\right\}\mathbf{M}^T\mathbf{A}-f(t_s),\ s=0,1,2,...,N.$$

The compact form of this system can be written as

$$\left(\mathbf{T}\,\mathbf{B}^m-\sum_{i=0}^{m-1}\sum_{j=1}^{J}\mathbf{P}_{ij}\mathbf{T}\,\mathbf{S}_j\,\mathbf{B}^i\right)\mathbf{M}^T\mathbf{A}=\mathbf{F} \qquad (12)$$

where

$$\mathbf{T}=\begin{bmatrix} T(t_0)\\ T(t_1)\\ \vdots \\ T(t_N)\end{bmatrix}=\begin{bmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^N \\ 1 & t_1 & t_1^2 & \cdots & t_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_N & t_N^2 & \cdots & t_N^N\end{bmatrix},\mathbf{P}_{ij}=\begin{bmatrix} P_{ij}(t_0) & 0 & \cdots & 0 \\ 0 & P_{ij}(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & P_{ij}(t_N)\end{bmatrix},$$

$$\overline{\mathbf{T}}=\begin{bmatrix} T(t_0) & 0 & \cdots & 0 \\ 0 & T(t_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T(t_N)\end{bmatrix},\mathbf{F}=\begin{bmatrix} f(t_0)\\ f(t_1)\\ \vdots \\ f(t_N)\end{bmatrix},\mathbf{S}_j=\begin{bmatrix} S^T(\tau_j(t_0))\\ S^T(\tau_j(t_1))\\ \vdots \\ S^T(\tau_j(t_N))\end{bmatrix}.$$

In Eq. (12), the full dimensions of the matrices $\mathbf{T}$, $\mathbf{B}$, $\mathbf{P}_{ij}$, $\overline{\mathbf{T}}$, $\mathbf{S}_j$, $\mathbf{M}$, $\mathbf{A}$ and $\mathbf{F}$, respectively, are $(N+1)\times(N+1)$, $(N+1)\times(N+1)$, $(N+1)\times(N+1)$, $(N+1)\times(N+1)^2$, $(N+1)^2\times(N+1)$, $(N+1)\times(N+1)$, $(N+1)\times 1$ and $(N+1)\times 1$

The fundamental matrix equation (12) can be expressed in the form

$$\mathbf{W}\mathbf{A}=\mathbf{F} \quad \text{or} \quad [\mathbf{W};\mathbf{F}] \qquad (13)$$

where

$$\mathbf{W}=\left(\mathbf{T}\,\mathbf{B}^m-\sum_{i=0}^{m-1}\sum_{j=1}^{J}\mathbf{P}_{ij}\overline{\mathbf{T}}\mathbf{S}_j\mathbf{B}^i\right)\mathbf{M}^T=\left[w_{pq}\right],\ p,q=0,1,...,N$$

By using the relation (10), we obtain the corresponding matrix forms for the conditions (2) as

$$\mathbf{U}_r\mathbf{A}=[\lambda_r] \quad \text{or} \quad [\mathbf{U}_r;\lambda_r] \qquad (14)$$

such that

$$\mathbf{U}_r=\sum_{i=0}^{m-1}\left(a_{ri}\mathbf{T}(a)+b_{ri}\mathbf{T}(b)\right)\mathbf{B}^i\mathbf{M}^T=\left[\,u_{r_0}\ u_{r_1}\ u_{r_2}\ \cdots\ u_{r_N}\,\right],r=0,1,...,m-1$$

Consequently, in order to obtain the solution of Eq. (1) under the conditions (2), we replace the m row matrices (14) by the last rows of the augmented matrix (13). Then we obtain the row augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{F}} \quad \text{or} \quad \left[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}\right] \qquad (15)$$

If $rank\,\widetilde{\mathbf{W}}=rank\left[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}\right]=N+1$, then we can write $\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{F}}$. Thus the matrix $\mathbf{A}$ (thereby the Lucas coefficients $a_0,a_1,...,a_N$) is uniquely determined. Also the Eq. (1) has a unique solution under the conditions (2).

3 Error Analysis Technique Based on Residual Function: Accuracy of Solutions

We can easily check the accuracy of the obtained solutions as follows. Since the truncated Lucas series in (3) is an approximate solution of Eq. (1), when the function $x_N(t)$ and its derivatives are substituted in Eq. (1), the resulting equation must be approximately satisfied; that is, for $t=t_i\in[0,b]$, $i=0,1,...,N$

$$R_N(t_i)=x^{(m)}(t)-\sum_{i=0}^{m-1}\sum_{j=1}^{J}\mathrm{P}_{ij}(t)x^{(i)}(t-\tau_j(t))-f(t)\cong 0$$

or

$$R_N(t_i)\leq 10^{-k_i},\ (k_i \text{ is any positive integer}).$$

If $\max 10^{-k_i}\leq 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $R_N(t_i)$ at each of the points becomes smaller than the prescribed 10^{-k}.

On the other hand, by means of the residual function defined by $R_N(t)$ and the mean value of the function $|R_N(t)|$ on the interval $[0,b]$, the accuracy of the solution can be controlled and the error can be estimated [21,22, 31,32,33]. If $R_N(t)\to 0$ when N is sufficiently large enough, then the error decreases. Also, by using the Mean-Value Theorem, we can estimate the upper bound of the mean error $\overline{R}$ as follows:

$$\left|\int_0^b R_N(t)\,dt\right|\leq\int_0^b|R_N(t)|\,dt$$

and

$$\int_0^b R_N(t)\,dt=bR_N(c)\Rightarrow\left|\int_0^b R_N(t)\,dt\right|=b|R_N(c)|\Rightarrow b|R_N(c)|\leq\int_0^b|R_N(t)|\,dt$$

$$\Rightarrow|R_N(c)|\leq\frac{\int_0^b R_N(t)\,dt}{b}=\overline{R_N}\ (0\leq c\leq b)$$

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4 Illustrative Examples

In this section, the efficiency of the method is demonstrated with the numerical results of some examples. In tables and figures, we give the values of the exact solution $x(t)$, the Lucas polynomial solution at the selected points of the given interval, the absolute error function $|x(t)-x_N(t)|$ and the estimated upper bound of mean error $\overline{R_N}$ at the selected points of the given interval. All examples have been solved by a computer code written in Maple.

Example 1. Let us first consider the first-order pantograph type delay differential equations with e^t variable delays

$$x'(t) = -2x(t) + 2x(t-e^t) + 2e^t + 1,\ 0 \le t \le 2$$

with the initial condition $x(0) = 0$.
We approximate the solution $x(t)$ by the polynomial

$$x(t) \cong x_N(t) = \sum_{n=0}^{2} a_n\, L_n(t),\ 0 \le t \le 2$$

where $f(t) = 2e^t + 1$, $\begin{cases} P_{01}(t) = -2, & \tau_1(t) = 0 \\ P_{02}(t) = 2, & \tau_2(t) = e^t \end{cases}$
and the collocation points (5) for $a = 0, b = 2$ and $N = 2$ are computed as $\{x_0 = 0,\ x_1 = 1,\ x_2 = 2\}$.
Following the procedure in Section 2, the fundamental matrix equation of the given equation becomes

$$\left(\mathbf{T}\,\mathbf{B} - \sum_{j=1}^{2} \mathbf{P}_{0j}\,\overline{\mathbf{T}}\,\mathbf{S}_j\,\mathbf{B}^0\right)\mathbf{M}^T\mathbf{A} = \mathbf{F},\ (\mathbf{B}^0 : unitmatrix)$$

where

$$\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}, \mathrm{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{P}_{01} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \mathbf{P}_{02} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\overline{\mathbf{T}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \end{bmatrix}, \mathbf{S}(\tau_1(t)) = \begin{pmatrix} 1 & 0 & 0 \\ -\tau_1(t) & 1 & 0 \\ \tau_1^2(t) & -2\tau_1(t) & 1 \end{pmatrix}$$

$$\mathbf{S}_1 = \begin{bmatrix} \mathrm{S}^T(\tau_1(0)) \\ \mathrm{S}^T(\tau_1(1)) \\ \mathrm{S}^T(\tau_1(2)) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{S}_2 = \begin{bmatrix} \mathrm{S}^T(\tau_2(0)) \\ \mathrm{S}^T(\tau_2(1)) \\ \mathrm{S}^T(\tau_2(2)) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & -e & e^2 \\ 0 & 1 & -2e \\ 0 & 0 & 1 \\ 0 & -e^2 & e^4 \\ 0 & 1 & -2e^2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 3 \\ 2e+1 \\ 2e^2+1 \end{bmatrix}$$

The augmented matrix for this fundamental matrix equation is

$$\mathbf{W} = \begin{bmatrix} 0 & 3 & 0 & ; & 3 \\ 0 & 2e+1 & 2-2e^2+4e & ; & 2e+1 \\ 2 & 0 & 2 & ; & 0 \end{bmatrix}$$

Solving this system, $\mathbf{A}$ is obtained as $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$
From Eq. (8), $x(t)$ is obtained as

$$x(t) = \mathbf{T}(t)\mathbf{M}^T\mathbf{A} = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Thus, the solution of the problem becomes

$$x(t) = t$$

which is the exact solution. Hence, it is seen that the present method is accurate, efficient, and applicable.

Example 2. Lastly, let us consider the first-order pantograph type delay differential equations with $\ln(t+1)$ variable delays

$$\begin{cases} x'(t) + tx(t-\ln(t+1)) + x(t) = \left(t^2+1\right)e^{-t} \\ x(0) = 1, \quad 0 \le t \le 2 \end{cases},$$

The exact solution of the problem is $x(t) = e^{-t}$.

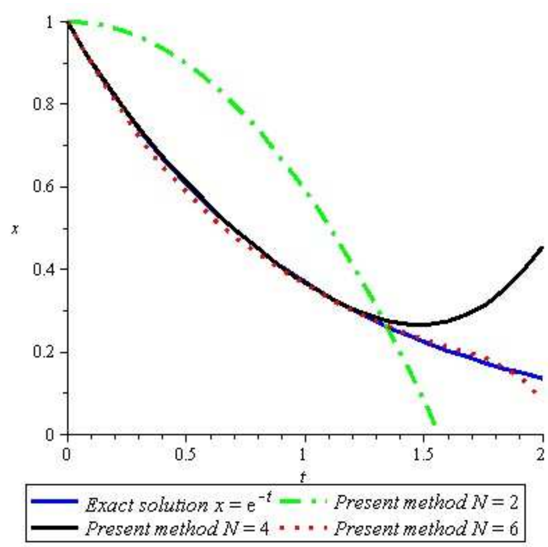

Fig. 1: Numerical and Exact solutions of Example 2 for $N = 2, 4, 6$.

Following the procedure in Section 2, the polynomial solutions are obtained, for $N = 2, 4, 6$

$$x_2(t) = 0.999999998 - 0.4085896211t,$$

$x_4(t) = 1.000000001 - 0.9999999992t + 0.5611021418t^2 - 0.2927828997t^3 + 0.09704064924t^4$,
$x_6(t) = 1.000000067 - 1.000000072t - 0.010170322t^2 + 1.321585041t^3 - 1.612385853t^4 + 0.8143793089t^5 - 0.1517978235t^6$.

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Table 1: Numerical results of the exact and the approximate solution for $N = 6$ and the Absolute errors of Example 3.

t	Exact Solution $x(t) = t^2$	The solution for $N = 6$	Absolute errors for $N = 6$
0	0	-1.683e-10	1.683e-10
0.2	0.04	0.03999999763	2.371e-09
0.4	0.16	0.15999999763	4.870e-09
0.6	0.36	0.3599999921	8.017e-09
0.8	0.64	0.6399999877	1.229e-08
1	1	0.9999999817	1.830e-08
1.2	1.44	1.439999973	2.692e-08
1.4	1.96	1.9599999961	3.932e-08
1.6	2.56	2.559999943	5.721e-08
1.8	3.24	3.239999918	8.301e-08
2	4	3.999999879	1.201e-07

Table 2: Numerical results of the exact and the approximate solutions for $N = 4, 6, 10$ of Example 4.

t	Exact Solution	Present method for N=4	Present method for N=6	Present method for N=10
0	1	1	1	1
0.1	1.004837418	1.004837265	1.004837078	1.004837399
0.2	1.018730753	1.018729586	1.018726864	1.018730968
0.3	1.040818221	1.040818528	1.04086171	1.040822594
0.4	1.070320046	1.070340038	1.070305485	1.070346195
0.5	1.106530660	1.106624444	1.106551072	1.106628727
0.6	1.148811636	1.149096445	1.148967088	1.149093170
0.7	1.196585304	1.197275127	1.197077688	1.197262516
0.8	1.249328964	1.250773951	1.250509141	1.250763774
0.9	1.306569660	1.309300755	1.30899128	1.309331161
1	1.367879441	1.372657757	1.372362866	1.372807874

From Figure 1, it is obvious that the results get better as N increases.

Example 3. Let us now consider the second-order pantograph type delay differential equations with $\sin t$ variable delays

$$\begin{cases} x''(t) = x'(t) + 2tx(t) - x'(t - \sin t) + 2 - 2.t^3 - 2\sin t \\ x(0) = 0, \quad x'(0) = 0, \quad 0 \le t \le 2 \end{cases},$$

where

$$\begin{cases} P_{01}(t) = 2t, \quad \tau_1(t) = 0 \\ P_{02}(t) = 0, \quad \tau_2(t) = 0 \\ P_{11}(t) = 1, \quad \tau_1(t) = 0 \\ P_{12}(t) = -1, \quad \tau_2(t) = \sin t \\ f(t) = 2 - 2t^3 - 2\sin t \end{cases}$$

The exact solution of the problem is $x(t) = t^2$. Following the procedure in Section 2, the fundamental matrix equation becomes

$$\left(\mathbf{T}\,\mathbf{B}^2 - \mathbf{P}_{01}\,\overline{\mathbf{T}}\,\mathbf{S}_1 - \mathbf{P}_{11}\overline{\mathbf{T}}\,\mathbf{S}_1\mathbf{B} - \mathbf{P}_{02}\overline{\mathbf{T}}\,\mathbf{S}_2 - \mathbf{P}_{12}\overline{\mathbf{T}}\,\mathbf{S}_2\mathbf{B}\right)\mathbf{M}^T\mathbf{A} = \mathbf{F}.$$

The polynomial solution of the problem is obtained as

$$x_6(t) = -1.68327544\times10^{-10} - 1.070485731\times10^{-8}t + 0.9999999992\,t^2 \\ -2.84759534\times10^{-9}t^3 - 5.000094533\times10^{-9}t^4 + 2.199145176\times10^{-9}t^5 \\ -9.831861521\times10^{-10}t^6$$

Also, by using the Mean–Value Theorem, the upper bound of the mean error $\overline{R_6}$ is obtained as

$$\overline{R_6} = \frac{\int_0^2 |R_6(t)|}{2} = \frac{7\times10^{-8}}{2} = 3.5\times10^{-8}$$

The approximate solutions obtained by using the collocation points $t_s = a + \frac{b-a}{N}s$, $s = 0, 1, ..., N$, in $[0, 2]$ for $N = 6$ are compared with exact solution in Figure 2. Absolute errors in $[0, 2]$ for $N = 6$ are given in Table 1.

Example 4. Lastly, let us consider the third-order pantograph type delay differential equations with t^2 variable delays

$$\begin{cases} x'''(t) - x''(t - t^2) + x(t) = t - e^{t^2 - t} \\ x(0) = 1, \quad x'(0) = 0, \quad x''(0) = 1, \quad 0 \le t \le 2 \end{cases},$$

The exact solution of the problem is $x(t) = t + e^{-t}$. Following the procedure in Section 2, the polynomial solution is obtained as for $N = 4, 6, 10$

$$x_4(t) = 1 - 1\times10^{-10}t + 0.5000000001t^2 - 0.1666666667t^3 + 0.03932442423t^4$$

$$x_6(t) = 1 - 1.17\times10^{-10}t + 0.4999999999t^2 - 0.1666666667t^3 + 0.03727144272t^4 \\ +0.001758089570t^5 + 3.749993833\times10^{-12}t^6$$

$$x_{10}(t) = 1 - 3.69\times10^{-11}t + 0.5t^2 - 0.167t^3 + 0.412t^4 - 0.007t^5 \\ +0.006t^6 - 0.001t^7 - 0.0008t^8 + 0.0003t^9 \\ -0.00001t^{10}.$$

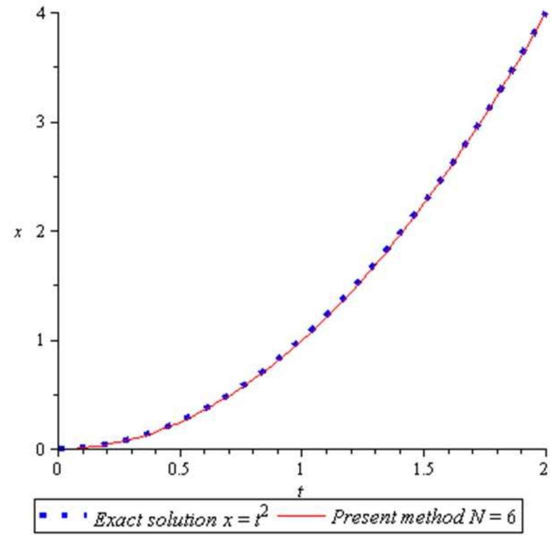

Fig. 2: Numerical and Exact solution of Example 3 for $N = 6$.

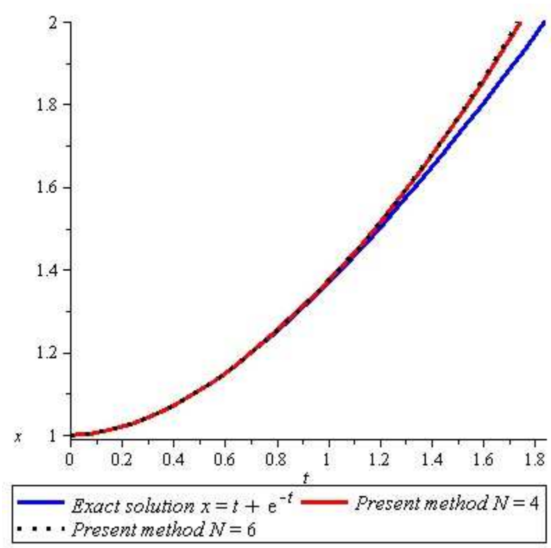

Fig. 3: Numerical and Exact solution of Example 4 for $N = 4, 6$.

Figure 3 and Table 2 show the convergence of the Lucas polynomial solution to the exact solution. It is observed that there is a very good agreement even for a low truncation limit such as $N = 4$ and $N = 6$ on the other hand, taking results in a very accurate solution in the interval $[0, 2]$.

5 Conclusions

The proposed practical matrix method is used to solve high-order pantograph type delay differential equations with variables delays. Comparison of the results obtained by present method with those obtained by exact solutions reveals that the present method is very effective and convenient. The numerical results show that the accuracy improves with increasing N. Also the proposed technique can be used to test reliability of the solutions of the other problems. Hybrid Taylor-Lucas collocation method provides two main advances: it is very simple to construct the main matrix equations and it is very easy for computer programming.

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