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New Sets of Euler-Type Polynomials

Gabriella Bretti[1], *Pierpaolo Natalini*[2,*] *and Paolo Emilio Ricci*[3]

[1] Istituto per le Applicazioni del Calcolo (IAC) "M. Picone" Consiglio Nazionale delle Ricerche (CNR - National Research Council) Via dei Taurini, Roma, Italia
[2] Dipartimento di Matematica e Fisica-Università degli Studi Roma Tre Largo San Leonardo Murialdo, Roma, Italia
[3] International Telematic University UniNettuno, Corso Vittorio Emanuele II, 39, 00186 - Roma, Italia

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Abstract: In recent papers, new sets of Sheffer and Brenke polynomials based on higher order Bell numbers have been studied, and several integer sequences related to them have been introduced. In the article other types of Sheffer polynomials are considered, by introducing two sets of Euler-type polynomials.

Keywords: Sheffer polynomials, combinatorial analysis, Bernoulli numbers, Euler-type polynomials

1 Introduction

In recent articles [5,12], the present authors have studied new sets of Sheffer [14] and Brenke [4] polynomials related to higher order Bell numbers. Furthermore, several integer sequences associated with the considered polynomials sequences, both of exponential and logarithmic type, have been highlighted.
It is worth to note that exponential and logarithmic polynomials have been recently studied in the multivariate case [9,10,11].
In this article we show new sets of Sheffer polynomials, by introducing two families of Euler-type polynomials.

2 Sheffer polynomials

The Sheffer polynomials $\{s_n(x)\}$ are introduced [14] by means of the exponential generating function [15] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\,, \tag{1}$$

where

$$\begin{aligned} A(t) &= \textstyle\sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \qquad (a_0 \neq 0)\,, \\ H(t) &= \textstyle\sum_{n=0}^{\infty} h_n \frac{t^n}{n!}\,, \qquad (h_0 = 0)\,. \end{aligned} \tag{2}$$

According to a different characterization (see [13, p. 18]), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \textstyle\sum_{n=0}^{\infty} g_n \frac{t^n}{n!}\,, \qquad (g_0 \neq 0)\,, \\ f(t) &= \textstyle\sum_{n=0}^{\infty} f_n \frac{t^n}{n!}\,, \qquad (f_0 = 0,\ f_1 \neq 0)\,. \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e. such that $f\left(f^{-1}(t)\right) = f^{-1}(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]}\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x)\frac{t^n}{n!}\,, \tag{4}$$

so that

$$A(t) = \frac{1}{g[f^{-1}(t)]}\,, \qquad H(t) = f^{-1}(t)\,. \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x)\frac{t^n}{n!}\,. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [2,3].

3 Euler-type polynomials

Here we introduce a Sheffer polynomial set connected with the classical Euler polynomials.

* Corresponding author e-mail: natalini@mat.uniroma3.it

Assuming:

$$A(t)=\frac{1}{\cosh t}, \qquad H(t)=\sinh t\,, \tag{1}$$

we consider the Euler-type polynomials $\tilde{E}_n(x)$, defined by the generating function

$$G(t,x)=\frac{1}{\cosh t}\exp\left[x\sinh t\right]=\sum_{k=0}^{\infty}\tilde{E}_k(x)\frac{t^k}{k!}\,. \tag{2}$$

Note that the Euler numbers are recovered, since we have:

$$G(t,0)=\frac{2}{e^t+e^{-t}}=\sum_{k=0}^{\infty}\tilde{E}_k(0)\frac{t^k}{k!}\,, \tag{3}$$

so that $\tilde{E}_n(0)=E_n$.
In what follows, we use the expansions

$$\sinh t=\sum_{k=0}^{\infty}\frac{t^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k+1}}{2}\right)\frac{t^k}{k!}\,, \tag{4}$$

$$\cosh t=\sum_{k=0}^{\infty}\frac{t^{2k}}{(2k)!}=\sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k}}{2}\right)\frac{t^k}{k!}\,. \tag{5}$$

Theorem 1.- *For any* $k\geq 0$, *the polynomials* $\tilde{E}_k(x)$ *satisfy the differential identity:*

$$\tilde{E}'_k(x)=\sum_{h=0}^{k}\binom{k}{h}\left(\frac{1+(-1)^{k-h+1}}{2}\right)\tilde{E}_h(x)\,. \tag{6}$$

Proof. - We write equation (2) in the form

$$\exp\left[x\sinh t\right]=\cosh t\sum_{k=0}^{\infty}\tilde{E}_k(x)\frac{t^k}{k!}\,,$$

by differentiating with respect to x, we find

$$\sinh t\exp\left[x\sinh t\right]=\cosh t\sum_{k=1}^{\infty}\tilde{E}'_k(x)\frac{t^k}{k!}\,,$$

i.e.,

$$\sinh t\;G(t,x)=\sinh t\sum_{k=0}^{\infty}\tilde{E}_k(x)\frac{t^k}{k!}=\sum_{k=1}^{\infty}\tilde{E}'_k(x)\frac{t^k}{k!}\,.$$

Therefore, by using equation (4), we have

$$\sinh t\;G(t,x)=\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}\left(\frac{1+(-1)^{k-h+1}}{2}\right)\tilde{E}_h(x)\frac{t^k}{k!}=\sum_{k=1}^{\infty}\tilde{E}'_k(x)\frac{t^k}{k!}\,. \tag{7}$$

and putting $\tilde{E}'_0(x)=0$, equation (6) follows.

Theorem 2.- *For any* $k\geq 0$, *the polynomials* $\tilde{E}_k(x)$ *satisfy the following recurrence relation:*

$$\tilde{E}_{k+1}(x)=\sum_{h=0}^{k}\binom{k}{h}\left[\left(\frac{1+(-1)^{k-h}}{2}\right)x-\tau_{k-h}\right]\tilde{E}_h(x)\,, \tag{8}$$

where

$$\tau_k=\left(\frac{1+(-1)^{k+1}}{2}\right)\frac{2^{k+1}\left(2^{k+1}-1\right)B_{k+1}}{k+1}$$

are the coefficients of the McLaurin expansion of the function $\tanh t$, *and* B_k *are the Bernoulli numbers.*

Proof. - Differentiating $G(t,x)$ with respect to t, we have

$$\frac{\partial G(t,x)}{\partial t}=-\frac{\sinh t}{\cosh t}\,G(t,x)+x\cosh t\;G(t,x)=\sum_{n=1}^{\infty}\tilde{E}_n(x)\frac{t^{n-1}}{(n-1)!}\,. \tag{9}$$

that is,

$$x\cosh t\;G(t,x)-\tanh t\;G(t,x)=\sum_{n=0}^{\infty}\tilde{E}_{n+1}(x)\frac{t^n}{n!}\,.$$

Putting

$$\tanh t=\sum_{k=1}^{\infty}\frac{2^{2k}\left(2^{2k}-1\right)B_{2k}}{(2k)!}\,t^{2k-1}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k+1}}{2}\right)\frac{2^{k+1}\left(2^{k+1}-1\right)B_{k+1}}{k+1}\frac{t^k}{k!}=\sum_{k=0}^{\infty}\tau_k\frac{t^k}{k!}\,,$$

we find

$$x\sum_{k=0}^{\infty}\left(\frac{1+(-1)^{k}}{2}\right)\frac{t^k}{k!}\sum_{k=0}^{\infty}\tilde{E}_k(x)\frac{t^k}{k!}-\sum_{k=0}^{\infty}\tau_k\frac{t^k}{k!}\sum_{k=0}^{\infty}\tilde{E}_k(x)=\sum_{n=0}^{\infty}\tilde{E}_{k+1}(x)\frac{t^k}{k!}\,,$$

and therefore

$$\sum_{k=0}^{\infty}\tilde{E}_{k+1}(x)\frac{t^k}{k!}=\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}\left(\frac{1+(-1)^{k-h}}{2}\right)x\tilde{E}_h(x)\frac{t^k}{k!}-\sum_{k=0}^{\infty}\sum_{h=0}^{k}\binom{k}{h}\tau_{k-h}\tilde{E}_h(x)\frac{t^k}{k!}\,,$$

so that the recurrence relation (8) follows.

We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that

$$\hat{P}\left(p_n(x)\right)=np_{n-1}(x)$$
$$\hat{M}\left(p_n(x)\right)=p_{n+1}(x)\,, \qquad (n=1,2,\dots). \tag{10}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.
This definition traces back to a paper by J.F. Steffensen [16], recently improved by G. Dattoli [6] and widely used in several applications (see e.g. [7,8] and the references therein).
Y. Ben Cheikh [1] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, in the same article (Corollary 3.2), the following result is proved

Theorem 3.- *Let* $(p_n(x))$ *denote a Boas-Buck polynomial set, i.e. a set defined by the generating function*

$$A(t)\psi(xH(t))=\sum_{n=0}^{\infty}p_n(x)\frac{t^n}{n!}\,, \tag{11}$$

where

$$A(t)=\sum_{n=0}^{\infty}\tilde{a}_n t^n\,, \qquad (\tilde{a}_0\neq 0)\,,$$

$$\psi(t)=\sum_{n=0}^{\infty}\tilde{\gamma}_n t^n\,, \qquad (\tilde{\gamma}_n\neq 0 \quad \forall n)\,, \tag{12}$$

with $\psi(t)$ *not a polynomial, and lastly*

$$H(t)=\sum_{n=0}^{\infty}\tilde{h}_n t^{n+1}\,, \qquad (\tilde{h}_0\neq 0)\,. \tag{13}$$

Let $\sigma \in \Lambda^{(-)}$ the lowering operator defined by

$$\sigma(1) = 0, \qquad \sigma(x^n) = \frac{\tilde{\gamma}_{n-1}}{\tilde{\gamma}_n}\, x^{n-1}, \qquad (n = 1, 2, \ldots). \quad (14)$$

Put

$$\sigma^{-1}(x^n) = \frac{\tilde{\gamma}_{n+1}}{\tilde{\gamma}_n}\, x^{n+1} \quad (n = 0, 1, 2, \ldots). \quad (15)$$

Denoting, as before, by $f(t)$ the compositional inverse of $H(t)$, the Boas-Buck polynomial set $\{p_n(x)\}$ is quasi-monomial under the action of the operators

$$\hat{P} = f(\sigma), \qquad \hat{M} = \frac{A'[f(\sigma)]}{A[f(\sigma)]} + xD_x H'[f(\sigma)]\sigma^{-1}, \quad (16)$$

where prime *denotes the ordinary derivatives with respect to t.*

Note that in our case we are dealing with a Sheffer polynomial set, so that since we have $\psi(t) = e^t$, the operator σ defined by equation (14) simply reduces to the derivative operator D_x. Furthermore, we have:

$$A(t) = \frac{1}{\cosh t}, \qquad \frac{A'(t)}{A(t)} = -\tanh t,$$

$$H(t) = \sinh t = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}, \qquad \left(\tilde{h}_k = \left(\frac{1+(-1)^{k+1}}{2}\right) \frac{1}{(2k+1)!}\right),$$

$$H'(t) = \cosh t, \qquad f(t) = H^{-1}(t) = \log(t + \sqrt{t^2+1}),$$

so that we have the theorem

Theorem 4.- *The Euler-type polynomial set $\{\tilde{E}_n(x)\}$ is quasi-monomial under the action of the operators*

$$\hat{P} = \log(D_x + \sqrt{D_x^2 + 1})$$
$$\hat{M} = -\tanh(\operatorname{settsinh} D_x) + x \operatorname{settsinh} D_x, \quad (17)$$

(by $\operatorname{settsinh} t = \log(t + \sqrt{t^2+1})$ we denote the inverse of the function $\sinh t$),
i.e.

$$\hat{P} = \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{4^k (k!)^2 (2k+1)} D_x^{2k+1},$$

$$\hat{M} = -\frac{D_x}{\sqrt{1+D_x^2}} + x\sqrt{1+D_x^2} = (xD_x^2 - D_x + x)(1+D_x^2)^{-1/2}, \quad (18)$$

$$\hat{M} = (xD_x^2 - D_x + x) \sum_{k=0}^{\infty} \binom{-1/2}{k} D_x^{2k}.$$

There is no problem about the convergence of the above series, since they reduce to finite sums when applied to polynomials.

According to the results of monomiality principle [6], the quasi-monomial polynomials $\{p_n(x)\}$ satisfy the differential equation

$$\hat{M}\hat{P}\, p_n(x) = n\, p_n(x). \quad (19)$$

In the present case, we have

Theorem 5.- *The Euler-type polynomials $\{\tilde{E}_n(x)\}$ satisfy the differential equation*

$$\left\{ \left[(xD_x^2 - D_x + x) \sum_{k=0}^{\infty} \binom{-1/2}{k} D_x^{2k} \right] \sum_{k=0}^{\infty} (-1)^k \frac{(2k)!}{4^k (k!)^2 (2k+1)} D_x^{2k+1} \right\} \tilde{E}_n(x) = n \tilde{E}_n(x), \quad (20)$$

i.e.

$$(xD_x^2 - D_x + x) \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \sum_{h=0}^{k} (-1)^h \binom{-1/2}{k-h} \frac{(2h)!}{4^h (h!)^2 (2h+1)} D_x^{2k+1} \tilde{E}_n(x) = n \tilde{E}_n(x). \quad (21)$$

Note that, for any fixed n, the Cauchy product of series expansions in equation (20) reduces to a finite sum, with upper limit $\left[\frac{n-1}{2}\right]$, when it is applied to a polynomial of degree n, because the successive addends vanish.

*Remark.*The first few Euler-type polynomials are as follows:

$\tilde{E}_0(x) = 1$
$\tilde{E}_1(x) = x$
$\tilde{E}_2(x) = x^2 - 1$
$\tilde{E}_3(x) = x^3 - 2x$
$\tilde{E}_4(x) = x^4 - 2x^2 + 5$
$\tilde{E}_5(x) = x^5 + 16x$
$\tilde{E}_6(x) = x^6 + 5x^4 + 31x^2 - 61$
$\tilde{E}_7(x) = x^7 + 14x^5 + 56x^3 - 272x$
$\tilde{E}_8(x) = x^8 + 28x^6 + 126x^4 - 692x^2 + 1385$
$\tilde{E}_9(x) = x^9 + 48x^7 + 336x^5 - 1280x^3 + 7936x$
$\tilde{E}_{10}(x) = x^{10} + 75x^8 + 882x^6 - 1490x^4 + 25261x^2 - 50521$

4 Circular Euler-type polynomials

Assuming:

$$A(t) = \frac{1}{\cos t}, \qquad H(t) = \sin t, \quad (1)$$

we consider the circular Euler-type polynomials $\tilde{S}_n(x)$, defined by the generating function

$$G(t,x) = \frac{1}{\cos t} \exp[x \sin t] = \sum_{k=0}^{\infty} \tilde{S}_k(x) \frac{t^k}{k!}. \quad (2)$$

*Remark.*The first few circular Euler-type polynomials are as follows:

$\tilde{S}_0(x) = 1$
$\tilde{S}_1(x) = x$
$\tilde{S}_2(x) = x^2 + 1$
$\tilde{S}_3(x) = x^3 + 2x$
$\tilde{S}_4(x) = x^4 + 2x^2 + 5$
$\tilde{S}_5(x) = x^5 + 16x$
$\tilde{S}_6(x) = x^6 - 5x^4 + 31x^2 + 61$
$\tilde{S}_7(x) = x^7 - 14x^5 + 56x^3 + 272x.$
$\tilde{S}_8(x) = x^8 - 28x^6 + 126x^4 + 692x^2 + 1385$
$\tilde{S}_9(x) = x^9 - 48x^7 + 336x^5 + 1280x^3 + 7936x$
$\tilde{S}_{10}(x) = x^{10} - 75x^8 + 882x^6 + 1490x^4 + 25261x^2 + 50521$

*Remark.*Therefore, we can write a table of circular Euler-type numbers:

$$\tilde{S}_0(0) = 1 \qquad \tilde{S}_2(0) = 1$$
$$\tilde{S}_4(0) = 5 \qquad \tilde{S}_6(0) = 61$$
$$\tilde{S}_8(0) = 1385 \qquad \tilde{S}_{10}(0) = 50521$$

$$\tilde{S}_{2k+1}(0) = 0\,, \qquad \forall\, k \geq 0.$$

Note that the Euler numbers (in absolute value) appear, since we have:

$$\tilde{S}_{2k}(0) = |\tilde{E}_{2k}|\,. \tag{3}$$

Taking into account the relations between the hyperbolic and circular functions:

$$\sinh(ix) = i \sin x\,, \qquad \cosh(ix) = \cos x\,,$$

the following relation between the polynomials $\tilde{S}_k(x)$ and $\tilde{E}_k(x)$ follows:

$$\tilde{S}_k(ix) = i^k \tilde{E}_k(x)\,. \tag{4}$$

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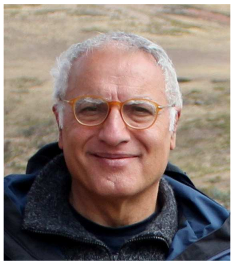
G. Bretti is a researcher at the National Researh Council (I.A.C.) in Rome, Italy. She works by many years in Numerical Analysis, Special functions, Matrix functions, Integral and discrete transforms, Ordinary and partial differential equations, Asymptotic analysis, Eigenvalue problems.

P. Natalini is professor and researcher at the RomaTre University. He works by many years in Numerical Analysis, Special functions, Matrix functions, Integral and discrete transforms, Ordinary and partial differential equations, Asymptotic analysis, Eigenvalue problems.



P. E. Ricci is professor and researcher at the UniNettuno International Telematic University in Rome, being retired from Rome University Sapienza by more than nine years. He works by many years in Numerical Analysis, Special functions, Matrix functions, Integral and discrete transforms, Ordinary and partial differential equations, Asymptotic analysis, Eigenvalue problems.