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On (φ,ψ)-generalized weak contractions in quasi-normed spaces

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Abstract: We propose the definition of quasi-n-normed spaces and prove some new results on fixed points theory related to weak contractions in this framework. We prove the existence and uniqueness of fixed point for (φ,ψ)-generalized weak contractions and (φ,ψ)-generalized weak C-contractions in quasi n-normed spaces. The obtained results extend some known theorems for nonlinear contractive functions on quasi n-normed spaces. In addition, we demonstrate an application of obtained results to Integral Equation.

Keywords: Cauchy sequence, Fixed point, Generalized weak C-contraction, Nonlinear contraction, Quasi n-normed space, 2-normed space

1 Introduction

The study of obtained functions from the generalization of the norm has been the focus of many mathematicians over the years. In 1963, the mathematician Gähler [1] introduced the concept of 2-metric space and presented its topological structure in his work. Many researchers have studied 2-metric spaces and fixed points theory [2], [3]. Later, Gähler extended his work to 2-normed spaces [4], and then to n-normed spaces [5]. These spaces have been the object of study for many authors [6,**?**,**?**,**?**,**?**,**?**]

In 2001, Gunawan and Mashadi [12] studied the n-normed spaces, their completeness, Cauchy sequences and proved a fixed-point theorem. Inspired by their work, several mathematicians assured significant fixed-point results in 2-Banach and n-normed spaces [13,**?**,**?**,**?**,**?**].

The concept of 2-normed spaces was extended to quasi 2-normed spaces [18] analogously as b-metric spaces [19]. The fixed-point theory in quasi-2-normed space and n-normed space has been a focus of research for authors [20], where they have proven the existence and uniqueness of a fixed point for several contractive functions and shown its applicable side [21].

In this paper, we give and prove some new results on the existence and uniqueness of a fixed point for (φ,ψ)-generalized weak contractive and (φ,ψ)-generalized weak C-contractive, respectively, on quasi n-normed spaces. Some analogies are obtained from the main theorems, which generalize some known results in quasi$-n$-normed spaces. Examples illustrate the highlights of this work. In addition, an application of the main result to Integral Equations is given to show the applicable side of this framework.

2 Preliminaries

Definition 1.*Let E be a linear space with* $\dim E \geq 2$ *and* $\mathbb{R}^+$ *the set of nonnegative real numbers. The function* $\|\cdot,\cdot\| : E^2 \to \mathbb{R}^+$ *is called 2-norm, if it satisfies the following conditions:*

1. $\|x,y\| = 0$ *if and only if the vectors* $\{x,y\}$ *are dependent in E;*
2.For every $(x,y) \in E^2$, $\|x,y\| = \|y,x\|$ *;*
3.For every $(\alpha,x,y) \in \mathbb{R}\times E^2$, $\|\alpha x,y\| = |\alpha|\,\|x,y\|$*;*
4.For all $(x,y,z) \in E^3$, $\|x+y,z\| \leq \|x,z\| + \|y,z\|$.

The pair $(E,\|\cdot,\cdot\|)$ *is called quasi 2-normed space.*

Park defined the quasi 2-norm as follows:

Definition 2.*[2] Let E be a linear space with* $\dim E \geq 2$ *and* $\mathbb{R}^+$ *the set of nonnegative real numbers. If the function* $\|\cdot,\cdot\| : E^2 \to \mathbb{R}^+$ *satisfies the following conditions:*

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1. $\|x,y\|=0$ *if and only if the vectors* $\{x,y\}$ *are dependent in* E;
2. *For every* $(x,y)\in X^2$, $\|x,y\|=\|y,x\|$;
3. *For every* $(\alpha,x,y)\in\mathbb{R}\times X^2$, $\|\alpha x,y\|=|\alpha|\cdot\|x,y\|$;
4. *There exists* $s\geq 1$, *such that for all* $(x,y,z)\in E^3$, $\|x+y,z\|\leq s(\|x,z\|+\|y,z\|)$.

It is called is a quasi 2-norm. The pair $(E,\|\cdot,\cdot\|)$ *is called quasi 2-normed space.*

Gunawan extended the concept of 2-normed space to n-normed space as below:

Definition 3. *[12] Let* E *be a real linear space with* $\dim E=d\geq n$ *(d is allowed to be infinite) and* $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}^+$ *be a function which satisfies the following conditions:*

1. $\|e_1,e_2,\dots,e_n\|=0$ *if and only if* $e_1,e_2,\dots,e_n\in E$ *are linearly dependent;*
2. $\|e_1,e_2,\dots,e_n\|=\|e_{j_1},e_{j_2},\dots,e_{j_n}\|$, *for every permutation* $(j_1,j_2,\dots,j_n)$ *of* $(1,2,\dots,n)$;
3. $\|\alpha e_1,e_2,\dots,e_n\|=|\alpha|\,\|e_1,e_2,l\dots,e_n\|$;
4. $\|x+y,e_1,e_2,\dots,e_{n-1}\|\leq\|x,e_1,e_2,\dots,e_{n-1}\|+\|y,e_1,e_2,\dots,e_{n-1}\|$;

for all $\alpha\in\mathbb{R}$ *and* $x,y,e_1,e_2,\cdots,e_n\in E$.
The function $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}^+$ *is called n-norm and the pair* $(E,\ \|\cdot,\dots,\cdot\|)$ *is called n-normed space.*

Example 1. [12] Let $E=\mathbb{R}^n$, $(e_1,e_2,\dots,e_n)\in E^n$ where $e_j=(x_{1j},x_{2j},\dots,x_{n+1j})$ for $j\in\{1,2,\dots,n\}$. The function $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}$

$$\|e_1,e_2,\dots,e_n\|=\left|\begin{pmatrix}x_{11}&\cdots&x_{1n}\\ \vdots&\ddots&\vdots\\ x_{n,1}&\cdots&x_{n,n}\end{pmatrix}\right|$$

is n-norm and $(E,\|\cdot,\dots,\cdot\|)$ is n-normed space.

Below, we define the quasi n-normed space as follows.

Definition 4. *Let* E *be a linear space with* $\dim E\ =d\geq n$ *(d is allowed to be infinite). The function* $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}^+$ *is called quasi n-norm, if it satisfies the following conditions:*

1. $\|e_1,e_2,\dots,e_n\|=0$ *if and only if the vectors* $\{e_1,e_2,\dots,e_n\}$ *are dependent in* E;
2. *For every* $(e_1,e_2,\dots,e_n)\in E^n$, $\|e_1,e_2,\dots,e_n\|$ *is invariant related to the permutations of* $\{e_1,e_2,\dots,e_n\}$
3. *For every* $(\alpha,e_1,e_2,\dots,e_n)\in\mathbb{R}\times E^n$, $\|\alpha e_1,e_2,\dots,e_n\|=|\alpha|\,\|e_1,e_2,\dots,e_n\|$;
4. *There exists* $s\geq 1$, *such that for all* $(x,y,e_1,e_2,\dots,e_{n-1})\in E^{n+1}$, *the following inequality holds:*

$$\|x+y,e_1,e_2,\dots,e_{n-1}\|\leq s(\|x,e_1,e_2,\dots,e_{n-1}\| + \|y,e_1,e_2,\dots,e_{n-1}\|).$$

The couple $(E,\|\cdot,\cdots,\cdot\|)$ *is called quasi n-normed space.*

Example 2. Let $E=\mathbb{R}^{n+1}$, $(e_1,e_2,\dots,e_n)\in E^n$ where $e_j=(x_{1j},x_{2j},\dots,x_{n+1j})$ for $j\in\{1,2,\dots,n\}$ and $s\geq 1$. Define the matrix $X=\begin{pmatrix}x_{11}&\cdots&x_{1n}\\ \vdots&\ddots&\vdots\\ x_{n+1,1}&\cdots&x_{n+1,n}\end{pmatrix}$.
We take the function $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}^+$,

$$\|e_1,\dots,e_n\|=s\cdot\left|\det\left(x_{i_0,j}\right)_{n\times n}\right|+\sum_{i\neq i_0}^{n+1}\left|\det(x_{i,j})_{n\times n}\right|,$$

where $\left|\det\left(x_{i_0,j}\right)_{n\times n}\right|=\min\{\left|\det(x_{i,j})_{n\times n}\right|\}$ and $(x_{i,j})_{n\times n}$ is the matrix of order n obtained from matrix X removing the ith row.
Using the properties of the determinants and absolute value, it is easy to prove that the function $\|\cdot,\dots,\cdot\|:E^n\to\mathbb{R}^+$, is a quasi $n-$norm and the couple $(E,\|\cdot,\dots,\cdot\|)$ is quasi n-normed space.

Remark. A quasi n-normed space may not be n-normed space. Indeed, if we take the quasi n-normed space $(E,\|\cdot,\dots,\cdot\|)$ given in Example 2 and $x=(-2,0,0\dots0)$, $y=(7,7,7,\dots,7)$, $e_2=(7,5,7,\dots,7)$, $e_3=(7,7,5,\dots,7)$, $\dots,e_n=(7,7,\dots,5,7)$, we have:

$$\|x+y,e_2,e_3,\dots,e_n\|=7s2^{n-1}+n(7n-2)2^{n-1},$$

$$\|x,e_2,e_3,\dots,e_n\|=s2^n+n(7n-9)2^{n-1},$$

$$\|y,e_2,e_3,\dots,e_n\|=7n\cdot2^{n-1}$$

and

$$\|x+y,e_2,e_3,\dots,e_n\|\nleq s(\|x,e_2,e_3,\dots,e_n\| + \|y,e_2,e_3,\dots,e_n\|)$$

for every $s>1$. As a result the pair $(E,\|\cdot,\dots,\cdot\|)$ is not n-normed space.

Example 3. Let $E=C_{[0,1]}=\{f:[0,1]\to\mathbb{R}, f$ is continuous and $s\geq 1$.
Define $\|\cdot,\dots,\cdot\|_\infty:E^n\to\mathbb{R}^+$ as follows:

$$\|f_1,\dots,f_n\|_\infty=\begin{cases}s\sup\limits_{t\in[0,1]}\prod\limits_{i=1}^{n}|f_1(t)|, & f_1,\dots,f_n \text{ are linearly indipendent}\\ 0, & \text{otherwise}\end{cases}$$

The space $(E,\|\cdot,\dots,\cdot\|_\infty)$ is an infinite dimensional quasi n-Banach space with $s\geq 1$.

Definition 5. *Let* $(E,\|\cdot,\dots,\cdot\|)$ *be a quasi n-normed space. The sequence* $\{x_k\}_{k\in\mathbb{N}}$ *in* E *is called convergent to* $x_0\in E$, *if for every* $\varepsilon>0$, *there exists* $p\in\mathbb{N}$, *such that for every* $k\in\mathbb{N},k>p$, $\|x_k-x_0,e_2,\dots,e_n\|<\varepsilon$, *for each* $e_2,\dots,e_n\in E$ *or* $\lim_{k\to+\infty}\|x_k-x_0,e_2,\dots,e_n\|=0$.

Definition 6. *A sequence $\{x_k\}_{k\in\mathbb{N}}$ in a quasi n-normed space $(E,\|\cdot,\ldots,\cdot\|)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $p \in \mathbb{N}$, such that for every $k,l \in \mathbb{N}, k.l > p$, $\|x_k - x_l, e_2,\ldots,e_n\| < \varepsilon$, for each $e_2,\ldots,e_k \in E$. (It is denoted $\lim_{k,l\to+\infty}\|x_k - x_l, e_2,\ldots,e_n\| = 0$.*

Definition 7. *The quasi n-normed space $(E,\|\cdot,\ldots,\cdot\|)$ is called complete if every Cauchy sequence in E is convergent in E. It is called quasi n-Banach space.*

Below, we recall the concept of (φ,ψ)-weak contraction and its generalizations.
Dutta and Choudhury in 2008 defined the nonlinear contraction known as (φ,ψ)-weak contraction in metric space as follows:

Definition 8. *[22] Let (X,d) be metric space and $T : X \to X$ be a map. The map T is called (φ,ψ)-weak contraction if it satisfies the inequality:*

$$\psi(d(Tx,Ty)) \le \psi(d(x,y)) - \varphi(d(x,y)) \qquad (1)$$

for every $(x,y) \in X^2$, where $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are monotone nondecreasing and continuous functions with $\varphi(t) = \psi(t) = 0$ iff $t = 0$.

Later, Doric in 2009 [23] improved this contraction by replacing $d(x,y)$ with $M(x,y) = \max\{d(x,y),\ d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(Tx,y)]\}$ in (1) and taking the function φ lower semi-continuous.
Recently, Xue generalized the above-mentioned contractions as follows:

Definition 9. *[24] Let (X,d) a metric space and $T : X \to X$ be a map. The map T is called (φ,ψ)-generalized weak contraction if for every $(x,y) \in X^2$, it satisfies the inequality*

$$\psi(d(Tx,Ty)) \le \psi(M(x,y)) - \varphi(M(x,y)) \qquad (2)$$

where $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ are two functions which satisfy the conditions:

1. *$\varphi(t) = \psi(t) = 0$ iff $t = 0$;*
2. *$\liminf_{\tau\to t}\psi(\tau) > \limsup_{\tau\to t}\psi(\tau) - \liminf_{\tau\to t}\varphi(\tau)$.*

3 Main results

Motivated from the above results, we consider the (φ,ψ)-generalized weak contraction in a quasi n-normed space as follows:

Definition 10. *Let $(E,\ \|\cdot,\ldots,\cdot\|)$ be a quasi n-Banach space with constant $s \ge 1$ and $T : E \to E$. The function T is called (φ,ψ)-nonlinear generalized weak contraction if it satisfies the inequality*

$$\psi(\|Tx - Ty, e_2,\ldots,e_n\|) \le \psi(M_0(x,y)) - \varphi(M_0(x,y)) \qquad (3)$$

for each $(x,y) \in E^2$ and $e_2,\ldots,e_n \in E$, where $\psi,\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the following conditions:

1. *$\varphi(t) = \psi(t) = 0$ iff $t = 0$;*
2. *ψ is a nondecreasing function;*
3. *$\lim_{\tau\to t}\inf\psi(\tau) > \lim_{\tau\to t}\sup\psi(\tau) - \lim_{\tau\to t}\inf\varphi(\tau)$.*

and

$$M_0(x,y) = \max\Big\{\|x-y,e_2,\ldots,e_n\|,\ \|x-Tx,e_2,\ldots,e_n\|, \|y-Ty,e_2,\ldots,e_n\|,\ \frac{\|y-Tx,e_2,\ldots,e_n\| + \|x-Ty,e_2,\ldots,e_n\|}{2s}\Big\}$$

for $e_2,\ldots,e_n \in E$.

Theorem 1. *Let $(E,\ \|\cdot,\ldots,\cdot\|)$ be a quasi n-Banach space with constant $s \ge 1$ and let $T : E \to E$ be (φ,ψ)- nonlinear generalized contraction. Then, the function T has a unique fixed point in E.*

Proof. Let $x_0 \in E$ be an arbitrary point in E. Define the sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $x_k = Tx_{k-1} = T^k x_0$, $k = 1,2,\ldots$
If there exists any $r \in \mathbb{N}$ such that $x_r = x_{r-1}$, then $Tx_{r-1} = x_{r-1}$, and x_{r-1} is a fixed point of map T.
Suppose that for each $k \in \mathbb{N}$, $x_k \ne x_{k-1}$.
For $k \in \mathbb{N}$ and $e_2,\ldots,e_n \in E$, we have

$$\psi(\|x_k - x_{k+1},e_2,\ldots,e_n\|) \le \psi(M_0(x_{k-1},x_k)) - \varphi(M_0(x_{k-1},x_k))$$

where

$$\begin{aligned}M_0(x_{k-1},x_k) &= \max\Big\{\frac{1}{s}\|x_{k-1}-x_k,e_2,\ldots,e_n\|,\ \|x_{k-1}-x_k,e_2,\ldots,e_n\|,\ \|x_k-x_{k+1},e_2,\ldots,e_n\|,\\ &\qquad \frac{\|x_k-x_k,e_2,\ldots,e_n\|}{2s} + \frac{\|x_{k-1}-x_{k+1},e_2,\ldots,e_n\|}{2s}\Big\}\\ &= \max\Big\{\|x_{k-1}-x_k,e_2,\ldots,e_n\|,\ \|x_k-x_{k+1},e_2,\ldots,e_n\|,\ \frac{\|x_{k-1}-x_{k+1},e_2,\ldots,e_n\|}{2s}\Big\}\\ &= \max\Big\{\|x_{k-1}-x_k,e_2,\ldots,e_n\|,\ \|x_k-x_{k+1},e_2,\ldots,e_n\|,\ \frac{\|x_{k-1}-x_{k+1},e_2,\ldots,e_n\|}{2s}\Big\}\end{aligned}$$

Let us consider the following cases.

Case 1: If $M_0(x_{k-1},x_k)=\|x_{k-1}-x_k,e_2,\dots,e_n\|$ then

$$\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)\le\psi(\|x_{k-1}-x_k,e_2,\dots,e_n\|)$$
$$-\varphi(\|x_{k-1}-x_k,e_2,\dots,e_n\|)$$
$$<\psi(\|x_{k-1}-x_k,e_2,\dots,e_n\|).$$

Consequently, the inequality

$$\|x_k-x_{k+1},e_2,\dots,e_n\|<\|x_{k-1}-x_k,e_2,\dots,e_n\|$$

is true.

Case 2: If $M_0(x_{k-1},x_k)=\|x_k-x_{k+1},e_2,\dots,e_n\|$, then

$$\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)\le\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)$$
$$-\varphi(\|x_k-x_{k+1},e_2,\dots,e_n\|)$$
$$<\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)$$

which is a contradiction. Consequently, this case does not hold.

Case 3: If $M_0(x_{k-1},x_k)=\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}$, then

$$\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)$$
$$\le\psi\left(\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$
$$-\varphi\left(\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$
$$<\psi\left(\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$

So, we have

$$\|x_k-x_{k+1},e_2,\dots,e_n\|<\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}$$
$$\le\frac{s(\|x_{k-1}-x_k,e_2,\dots,e_n\|}{2s}$$
$$\frac{+\|x_k-x_{k+1},e_2,\dots,e_n\|)}{2s}$$
$$=\frac{\|x_{k-1}-x_k,e_2,\dots,e_n\|+\|x_k-x_{k+1},e_2,\dots,e_n\|)}{2}$$

and

$$\|x_k-x_{k+1},e_2,\dots,e_n\|<\|x_{k-1}-x_k,e_2,\dots,e_n\|$$

Considering the above cases, we have proved that the sequence

$$\{\|x_k-x_{k+1},e_2,\dots,e_n\|\}_{k\in\mathbb{N}}=\{\lambda_k\}_{k\in\mathbb{N}}$$

is monotone decreasing and bounded below from zero. Consequently, it converges to its infimum $\lambda\ge 0$, $\lim_{k\to\infty}\lambda_k=\lambda$.

If we replace in (3), the value of $M_0(x,y)$ according to Case 1 and Case 3, respectively, we have:

For $M_0(x,y)=\|x_{k-1}-x_k,e_2,\dots,e_n\|$, the following inequalities

$$\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)\le\psi(\|x_{k-1}-x_k,e_2,\dots,e_n\|)$$
$$-\varphi(\|x_{k-1}-x_k,e_2,\dots,e_n\|)$$

and

$$\psi(\lambda_k)\le\psi(\lambda_k)-\varphi(\lambda_k)$$

hold.

Taking the limit of both sides when $k\to\infty$, we have

$$\psi(\lambda)\le\psi(\lambda)-\varphi(\lambda)$$

If $M_0(x,y)=\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}$, we have

$$\psi(\|x_k-x_{k+1},e_2,\dots,e_n\|)$$
$$\le\psi\left(\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$
$$-\varphi\left(\frac{\|x_{k-1}-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$
$$\le\psi\left(\frac{s(\|x_{k-1}-x_k,e_2,\dots,e_n\|}{2s}\right.$$
$$\left.\frac{+\|x_k-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$
$$-\varphi\left(\frac{s(\|x_{k-1}-x_k,e_2,\dots,e_n\|}{2s}\right.$$
$$\left.\frac{+\|x_k-x_{k+1},e_2,\dots,e_n\|}{2s}\right)$$

and

$$\psi(\lambda_k)\le\psi\left(\frac{\lambda_{k-1}+\lambda_k}{2}\right)-\varphi\left(\frac{\lambda_{k-1}+\lambda_k}{2}\right)$$

As a result, taking the limit of both sides we have when $k\to\infty$, we have

$$\psi(\lambda)\le\psi(\lambda)-\varphi(\lambda)$$

Consequently, $\varphi(\lambda)=0,\lambda=0$ and

$$\lim_{k\to\infty}\|x_k-x_{k+1},e_2,\dots,e_n\|=0$$

Now, we claim that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy.

Suppose that the sequence $\{x_k\}_{k\in\mathbb{N}}$ is not Cauchy. So, there exist $\varepsilon>0$, such that for each $p\in\mathbb{N}$, there exist $k(p)$, $l(p)$ where $k(p)$ is the smallest index for which

$$k(p)>l(p)>p \text{ and } \left\|x_{l(p)}-x_{k(p)},e_2,\dots,e_n\right\|\ge\varepsilon$$

It is clear that $\left\|x_{l(p)}-x_{k(p)-1},e_2,\dots,e_n\right\|<\varepsilon$. From the third condition of quasi n-norm, it yields

$$\varepsilon\le\left\|x_{l(p)}-x_{k(p)},e_2,\dots,e_n\right\|$$
$$\le s\left(\left\|x_{l(p)}-x_{k(p)-1},e_2,\dots,e_n\right\|\right.$$
$$\left.+\left\|x_{k(p)-1}-x_{k(p)},e_2,\dots,e_n\right\|\right)$$
$$<s\left(\varepsilon+\left\|x_{k(p)-1}-x_{k(p)},e_2,\dots,e_n\right\|\right)$$

Taking the limit when $p \to +\infty$ in the above inequality, we have

$$\varepsilon \leq \lim_{p\to+\infty} \left\|x_{l(p)} - x_{k(p)}, e_2, \dots, e_n\right\| \leq s\varepsilon. \quad (4)$$

Furthermore,

$$\begin{aligned}&\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \dots, e_n\right\| \\ &\leq s(\left\|x_{l(p)-1} - x_{k(p)}, e_2, \dots, e_n\right\| \\ &+ \left\|x_{k(p)} - x_{k(p)-1}, e_2, \dots, e_n\right\|)\end{aligned}$$

and

$$\lim_{p\to+\infty} \left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \dots, e_n\right\| \leq s\varepsilon \quad (5)$$

Next, using (4) and (5), we evaluate the $\lim_{p\to+\infty} M_0\left(x_{l(p)-1}, x_{k(p)-1}\right)$.
We see that

$$\begin{aligned}&\varepsilon \leq M_0\left(x_{l(p)-1}, x_{k(p)-1}\right) \\ &= \max\left\{\frac{1}{s}\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \dots, e_n\right\|,\right. \\ &\quad \left\|x_{l(p)-1} - x_{l(p)}, e_2, \dots, e_n\right\|, \\ &\quad \left\|x_{k(p)-1} - x_{k(p)}, e_2, \dots, e_n\right\|, \\ &\quad \frac{s(\left\|x_{l(p)-1} - x_{k(p)}, e_2, \dots, e_n\right\|}{2s} \\ &\quad \left.\frac{+\left\|x_{l(p)} - x_{k(p)-1}, e_2, \dots, e_n\right\|)}{2s}\right\}\end{aligned}$$

and

$$\varepsilon \leq \lim_{p\to+\infty} M_0\left(x_{l(p)-1}, x_{k(p)-1}\right) \leq \max\left\{\varepsilon, 0, 0, \frac{\varepsilon+\varepsilon}{2}\right\} = \varepsilon$$

Considering the contraction, we have

$$\begin{aligned}&\psi\left(\left\|x_{l(p)} - x_{k(p)}, e_2, \dots, e_n\right\|\right) \leq \psi\left(M_0(x_{l(p)-1}, x_{k(p)-1}\right) \\ &-\varphi\left(M_0(x_{l(p)-1}, x_{k(p)-1})\right)\end{aligned}$$

and

$$\begin{aligned}&\inf_{i\geq p} \psi\left(\left\|x_{l(p)} - x_{k(p)}, e_2, \dots, e_n\right\|\right) \\ &+ \inf_{i\geq p} \varphi\left(M_0\left(x_{l(p)-1}, x_{k(p)-1}\right)\right) \\ &\leq \sup_{i\geq p} \psi\left(M_0\left(x_{l(p)-1}, x_{k(p)-1}\right)\right)\end{aligned}$$

Consequently, it yields

$$\liminf_{t\to\varepsilon} \psi(t) + \liminf_{t\to\varepsilon} \varphi(t) \leq \limsup_{t\to\varepsilon} \psi(t)$$

and

$$\psi(\varepsilon) + \varphi(\varepsilon) \leq \psi(\varepsilon)$$

which is true if only if $\varepsilon = 0$, which is a contradiction. So, $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence and since the quasi n-Banach space $(E, \|\cdot, \dots, \cdot\|)$ is complete, the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to a point $x^* \in E$,

$$\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} T^k x_0 = x^*$$

Next, we prove that x^* is a fixed point of function T.
Using the contraction inequality, we have

$$\begin{aligned}&\psi\left(\|Tx^* - x_k, e_2, \dots, e_n\|\right) \quad (6) \\ &\leq \psi\left(M_0\left(x^*, x_k\right)\right) - \varphi\left(M_0\left(x^*, x_k\right)\right)\end{aligned}$$

where

$$\begin{aligned}&M_0\left(x^*, x_k\right) = \max\left\{\frac{1}{s}\|x^* - x_k, e_2, \dots, e_n\|,\right. \\ &\quad \|x^* - Tx^*, e_2, \dots, e_n\|, \|x_k - x_{k+1}, e_2, \dots, e_n\|, \\ &\quad \left.\frac{\|Tx^* - x_k, e_2, \dots, e_n\| + \|x^* - x_{k+1}, e_2, \dots, e_n\|}{2s}\right\}\end{aligned}$$

Taking

$$\begin{aligned}&\|Tx^* - x_k, e_2, \dots, e_n\| + \|x^* - x_{k+1}, e_2, \dots, e_n\| \\ &\leq s(\|Tx^* - x^*, e_2, \dots, e_n\| + \|x^* - x_k, e_2, \dots, e_n\| \\ &\quad + \|x^* - x_k, e_2, \dots, e_n\| + \|x_k - x_{k+1}, e_2, \dots, e_n\|) \\ &= s(\|Tx^* - x^*, e_2, \dots, e_n\| + 2\|x^* - x_k, e_2, \dots, e_n\| \\ &\quad + \|x_k - x_{k+1}, e_2, \dots, e_n\|)\end{aligned}$$

then it yields

$$\begin{aligned}&M_0(x^*, x_k) = \max\left\{\frac{1}{s}\|x^* - x_k, e_2, \dots, e_n\|,\right. \\ &\quad \|x^* - Tx^*, e_2, \dots, e_n\|, \|x_k - x_{k+1}, e_2, \dots, e_n\|, \\ &\quad \|x^* - x_k, e_2, \dots, e_n\| \\ &\quad \left.+\frac{\|Tx^* - x^*, e_2, \dots, e_n\| + \|x_k - x_{k+1}, e_2, \dots, e_n\|}{2}\right\}\end{aligned}$$

We see that

$$\begin{aligned}&\lim_{k\to+\infty} M_0\left(x^*, x_k\right) = \max\left\{0, \|x^* - Tx^*, e_2, \dots, e_n\|, 0,\right. \\ &\quad \left.\frac{\|Tx^* - x^*, e_2, \dots, e_n\|}{2}\right\} = \|x^* - Tx^*, e_2, \dots, e_n\|\end{aligned}$$

From inequality (6), we have:

$$\begin{aligned}&\inf_k \psi\left(\|Tx^* - x_k, e_2, \dots, e_n\|\right) + \inf_k \varphi\left(M_0\left(x^*, x_k\right)\right) \\ &\leq \sup_k \psi\left(M_0\left(x^*, x_k\right)\right).\end{aligned}$$

Taking the limit in the above inequality

$$\begin{aligned}&\lim_{t\to\|Tx^*-x^*, e_2, \dots, e_n\|} \inf_k \psi(t) + \lim_{t\to\|Tx^*-x^*, e_2, \dots, e_n\|} \inf_k \varphi(t) \\ &\leq \lim_{t\to\|Tx^*-x^*, e_2, \dots, e_n\|} \sup_k \psi\left(M_0(x^*, x_k)\right)\end{aligned}$$

we have $\varphi(\|x^*-Tx^*,e_2,\dots,e_n\|)=0$ and $\|x^*-Tx^*,e_2,\dots,e_n\|=0$ for each $e_2,\dots,e_n\in E$. Consequently, $x^*=Tx^*$and x^* is a fixed point of T.

Finally, we show the uniqueness of the fixed point x^* of T. Suppose that there exists another fixed point y^* of T, $y^*=Ty^*$.
Using the inequality

$$\psi(\|Tx^*-Ty^*,e_2,\dots,e_n\|) \le \psi(M_0(x^*,y^*))-\varphi(M_0(x^*,y^*))$$

where

$$M_0(x^*,y^*)=\|x^*-y^*,e_2,\dots,e_n\|,$$

we have:

$$\psi(\|x^*-y^*,e_2,\dots,e_n\|)\le\psi(\|x^*-y^*,e_2,\dots,e_n\|) -\varphi(\|x^*-y^*,e_2,\dots,e_n\|)$$

From this, it yields $\|x^*-y^*,e_2,\dots,e_n\|=0$ for every $e_2,\dots,e_n\in E$ and $x^*=y^*$.

*Example 4.*Let $E=\mathbb{R}^d$, where $n<d<\infty$. Define $\|\cdot,\dots,\cdot\|:E^n\to[0,+\infty)$ such that

$$\|e_1,e_2,\dots,e_n\|=\begin{cases} s\prod_{i=1}^{n}|e_i|, & e_1,e_2,\dots,e_n \text{ linearly independent}\\ 0, & e_1,e_2,\dots,e_n \text{ linearly dependent}\end{cases}$$

The couple $(E,\ \|x_1,x_2,\dots,x_n\|)$ is a complete n-normed space.

Taking $s=\frac{3}{2}$,$T:E\to E$, $T(x)=T(x_1,\dots x_d)=\frac{1}{10}(\sin x_1,\sin x_2,\dots,\sin x_d)$, for $x_i\in\mathbb{R},i\in\{1,2,\dots d\}$, $\psi:\mathbb{R}^+\to\mathbb{R}^+,\psi(t)=\frac{t\cdot\ln(t^2+1)}{2}$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$, $\varphi(t)=\frac{\sqrt{t}}{4}$, we show that the function T satisfies the conditions of Theorem 1.
The first three conditions are clear.

Considering $x,y,\ e_2,\dots,e_n\in E$, and

$$\|Tx-Ty,e_2,\dots,e_n\|=\frac{1}{10}\|\sin x_1-\sin y_1,\sin x_2-\sin y_2,\dots,\sin x_d-\sin y_d),e_2,\dots,e_n\|$$

$$=\frac{s}{10}\left(\sum_{j=1}^{d}(\sin x_j-\sin y_j)^2\right)^{\frac12}\prod_{i=1}^{n}|e_i|$$

$$\le\frac{s}{10}\left(\sum_{j=1}^{d}(x_j-y_j)^2\right)^{\frac12}\prod_{i=1}^{n}|e_i|$$

$$=\frac{1}{10}\|x-y,e_2,\dots,e_n\|,$$

$$\psi(\|Tx-Ty,e_2,\dots,e_n\|)+\varphi(M_0(x,y))$$

$$\le\psi\left(\frac{1}{10}\|x-y,e_2,\dots,e_n\|\right)+\frac{\sqrt{M_0(x,y)}}{4}=$$

$$\frac{\frac{1}{10}\|x-y,e_2,\dots,e_n\|\cdot\ln\left(\frac{1}{100}(\|x-y,e_2,\dots,e_n\|)^2+1\right)}{2}$$

$$+\frac{1}{4}\frac{M_0(x,y)\cdot\ln\left((M_0(x,y))^2+1\right)}{2}<$$

$$\frac{3}{20}\frac{\frac{2}{3}\|x-y,e_2,\dots,e_n\|\cdot\ln(\frac{4}{9}(\|x-y,e_2,\dots,e_n\|)^2+1)}{2}$$

$$+\frac{1}{4}\psi(M_0(x,y))$$

$$\le\frac{3}{20}\psi(M_0(x,y))+\frac{1}{4}\psi(M_0(x,y))=\frac{13}{20}\psi(M_0(x,y))$$

$$<\psi(M_0(x,y))$$

where

$$M_0(x,y)=\max\left\{\begin{array}{l}\frac{2}{3}\|x-y,e_2,\dots,e_n\|,\\ \|x-Tx,e_2,\dots,e_n\|,\\ \|y(t)-Ty(t),e_2,\dots,e_n\|,\\ \frac{\|y(t)-Tx(t),e_2,\dots,e_n\|+\|x(t)-Ty(t),e_2,\dots,e_n\|}{3}\end{array}\right\}$$

Since, $\psi(\|Tx(t)-Ty(t),e_2,\dots,e_n\|)\le\psi(M_0(x,y))-\varphi(M_0(x,y))$, we prove that the function T has a unique fixed point $x=0$.
In 2013, Saha and Ganguly recalled weakly C-contractive function in 2-normed space, as follows:

Definition 11.*[25] Let$(E,\ \|\cdot,\cdot\|)$ 2-normed space. A function $T:E\to E$ is called weakly C-contractive if for all $x,y\in E$,*

$$\|Tx-Ty\|\le\frac{\|x-Ty,a\|+\|y-Tx,a\|}{2} -\varphi(\|x-Ty,a\|,\|y-Tx,a\|$$

where $\varphi:\mathbb{R}^{+2}\to\mathbb{R}^+$ is a continuous map and $\varphi(0,0)=0$.

Below, we generalize weak C-contraction to (φ,ψ)-generalized weak C-contractions and prove some fixed-point results related to these weak contractions in quasi n-normed space.

Definition 12.*A function $\varphi:\mathbb{R}^{+^5}\to\mathbb{R}^+$ is called of C-type if it satisfies the following conditions:*

1.$\varphi(t_1,t_2,t_3,t_4,t_5)=0$ iff $t_1=t_2=t_3=t_4=t_5=0$;
2.φ is lower semi continuous.

*Example 5.*Let $\varphi:\mathbb{R}^{+5}\to\mathbb{R}^+$ be a nonnegative map and $\varphi(t_1,t_2,t_3,t_4,t_5)=t_1+t_2e^{t_2}+\log(1+t_3)+\max\{t_4,t_5\}$. It is clear that this map is of C-type.

Definition 13. *Let $(E, \|\cdot,\ldots,\cdot\|)$ be a quasi n-Banach space with constant $s \geq 1$ and $T : E \to E$. The function T is called (φ,ψ)- nonlinear generalized weak C-contraction if it satisfies the inequality*

$$\psi(\|Tx - Ty, e_2, \ldots, e_n\|) \leq \psi(M_0(x,y)) - \varphi(\|x - y, e_2, \ldots, e_n\|, \|x - Tx, e_2, \ldots, e_n\|, \|y - Ty, e_2, \ldots, e_n\|, \|y - Tx, e_2, \ldots, e_n\|, \|x - Ty, e_2, \ldots, e_n\|) \quad (7)$$

where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ and $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ which complete the following conditions:

1. *$\psi(t) = 0$ iff $t = 0$;*
2. *ψ is a nondecreasing function;*
3. *ψ is upper semi continuous function;*
4. *φ is C-type;*
5. *$\underline{\lim}_p \psi(t_p) > \overline{\lim}_p \psi(t_p) - \underline{\lim}_p \varphi(t_p, t_p, t_p, t_p, t_p)$;*

and

$$M_0(x,y) = \max \left\{ \begin{array}{l} \frac{1}{s}\|x - y, e_2, \ldots, e_n\|, \\ \|x - Tx, e_2, \ldots, e_n\|, \\ \|y - Ty, e_2, \ldots, e_n\|, \\ \frac{\|y - Tx, e_2, \ldots, e_n\| + \|x - Ty, e_2, \ldots, e_n\|}{2s} \end{array} \right\}$$

for $e_2, \ldots, e_n \in E$.

Theorem 2. *Let $(E, \|\cdot,\ldots,\cdot\|)$ be a quasi n-Banach space with constant $s \geq 1$ and let $T : E \to E$ be a (φ,ψ)-generalized weak C-contraction. Then the function T has a unique fixed point in E.*

Proof. Let $x_0 \in E$ be an arbitrary point in E. Define the sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $x_k = Tx_{k-1} = T^k x_0$, $k = 1, 2, \ldots$
If there exists any $r \in \mathbb{N}$ such that $x_r = x_{r-1}$, then x_{r-1} is a fixed point of map T.
Suppose that for each $k \in \mathbb{N}$, $x_k \neq x_{k-1}$.
For $k \in \mathbb{N}$ and $e_2, \ldots, e_n \in E$, we have

$$\psi(\|x_k - x_{k+1}, e_2, \ldots, e_n\|) \leq \psi(M_0(x_{k-1}, x_k)) - \varphi\left(\begin{array}{l} \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \\ \|x_k - x_{k+1}, e_2, \ldots, e_n\|, \|x_k - x_k, e_2, \ldots, e_n\|, \\ \|x_{k-1} - x_{k+1}, e_2, \ldots, e_n\| \end{array} \right)$$

$$= \psi(M_0(x_{k-1}, x_k)) - \varphi\left(\begin{array}{l} \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \\ \|x_k - x_{k+1}, e_2, \ldots, e_n\|, 0, \\ \|x_{k-1} - x_{k+1}, e_2, \ldots, e_n\| \end{array} \right)$$

where

$$M_0(x_{k-1}, x_k) = \max \left\{ \frac{1}{s}\|x_{k-1} - x_k, e_2, \ldots, e_n\|, \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \|x_k - x_{k+1}, e_2, \ldots, e_n\|, \frac{\|x_k - x_k, e_2, \ldots, e_n\| + \|x_{k-1} - x_{k+1}, e_2, \ldots, e_n\|}{2s} \right\}$$

$$= \max \left\{ \|x_{k-1} - x_k, e_2, \ldots, e_n\|, \|x_k - x_{k+1}, e_2, \ldots, e_n\|, \frac{\|x_{k-1} - x_{k+1}, e_2, \ldots, e_n\|}{2s} \right\}.$$

Using the same method as in Theorem 1, the inequality

$$\|x_k - x_{k+1}, e_2, \ldots, e_n\| < \|x_{k-1} - x_k, e_2, \ldots, e_n\|$$

can be proved for every $e_2, \ldots, e_n \in E$.
As a result, the sequence $\{\|x_k - x_{k+1}, e_2, \ldots, e_n\|\}_{k\in\mathbb{N}} = \{\lambda_k\}_{k\in\mathbb{N}}$ is monotone decreasing and bounded below from zero. So, it converges to its infimum $\lambda \geq 0$, $\lim_{k\to\infty} \lambda_k = \lambda$.
Considering the inequality $\underline{\lim}_p \psi(\lambda_k) > \overline{\lim}_p \psi(\lambda_k) - \underline{\lim}_k \varphi(\lambda_k, \lambda_k, \lambda_k, \lambda_k, \lambda_k)$, we have $\psi(\lambda) \geq \psi(\lambda) - \varphi(\lambda, \lambda, \lambda, \lambda, \lambda)$ and $\varphi(\lambda, \lambda, \lambda, \lambda, \lambda) = 0$. So, we obtain $\lambda = 0$.
Consequently, $\lim_{k\to\infty} \|x_k - x_{k+1}, e_2, \ldots, e_n\| = 0$, for every $e_2, \ldots, e_n \in E$.
Next step is to prove that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence.
Suppose that $\{x_k\}_{k\in\mathbb{N}}$ is not a Cauchy sequence. Consequently, there exists $\varepsilon > 0$, such that for each $p \in \mathbb{N}$, there exists $k(p)$, $l(p)$ where $k(p)$ is the smallest index for which $k(p) > l(p) > p$ and

$$\left\|x_{l(p)} - x_{k(p)}, e_2, \ldots, e_n\right\| \geq \varepsilon \quad (8)$$

and

$$\left\|x_{l(p)} - x_{k(p)-1}, e_2, \ldots, e_n\right\| < \varepsilon \quad (9)$$

Using the same manner as in Theorem 1, we prove that $\lim_{p\to+\infty} M_0\left(x_{l(p)-1}, x_{k(p)-1}\right) = \varepsilon$.
Furthermore,

$$\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \ldots, e_n\right\| \leq s(\left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\| + \left\|x_{k(p)} - x_{k(p)-1}, e_2, \ldots, e_n\right\|)$$

Also, we see that

$$\begin{aligned} \varepsilon &\leq \left\|x_{l(p)} - x_{k(p)}, e_2, \ldots, e_n\right\| \\ &\leq s\left(\left\|x_{l(p)} - x_{l(p)-1}, e_2, \ldots, e_n\right\| + \left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|\right) \\ &< s\left\|x_{l(p)} - x_{l(p)-1}, e_2, \ldots, e_n\right\| + s^2(\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \ldots, e_n\right\| + \left\|x_{k(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|) \end{aligned}$$

Taking the limit above, the inequality (9) holds:

$$\frac{\varepsilon}{s^2} \le \lim_{p\to+\infty} \left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \ldots, e_n\right\| \quad (10)$$

Furthermore, using

$$\begin{aligned}\varepsilon &\le \left\|x_{l(p)} - x_{k(p)}, e_2, \ldots, e_n\right\| \\ &\le s\left(\left\|x_{l(p)} - x_{l(p)-1}, e_2, \ldots, e_n\right\| \right. \\ &\quad \left. + \left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|\right)\end{aligned}$$

we obtain

$$\frac{\varepsilon}{s} \le \lim_{p\to+\infty} \left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\| \quad (11)$$

Considering the C-contraction, we have

$$\begin{aligned}&\psi\left(\left\|x_{l(p)} - x_{k(p)}, e_2, \ldots, e_n\right\|\right) \\ &\quad +\varphi\left(\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \ldots, e_n\right\|,\right. \\ &\quad \left\|x_{l(p)-1} - x_{l(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left\|x_{k(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left.\left\|x_{l(p)} - x_{k(p)-1}, e_2, \ldots, e_n\right\|\right) \\ &\le \psi\left(M_0\left(x_{l(p)-1}, x_{k(p)-1}\right)\right)\end{aligned}$$

Taking limits of both sides and using the inequalities (8), (9), (10) and (11), there is acquired

$$\begin{aligned}&\psi(\varepsilon) + \varphi\left(\frac{\varepsilon}{s^2}, 0,\ 0, \frac{\varepsilon}{s}, \varepsilon\right) \\ &\le \underline{\lim}_p \psi\left(\left\|x_{l(p)} - x_{k(p)}, e_2, \ldots, e_n\right\|\right) \\ &+ \underline{\lim}_p \varphi\left(\left\|x_{l(p)-1} - x_{k(p)-1}, e_2, \ldots, e_n\right\|,\right. \\ &\quad \left\|x_{l(p)-1} - x_{l(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left\|x_{k(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left\|x_{l(p)-1} - x_{k(p)}, e_2, \ldots, e_n\right\|, \\ &\quad \left.\left\|x_{l(p)} - x_{k(p)-1}, e_2, \ldots, e_n\right\|\right) \\ &\le \overline{\lim}_p \psi\left(M_0\left(x_{l(p)-1}, x_{k(p)-1}\right)\right) \le \psi(\varepsilon)\end{aligned}$$

Consequently, we have

$$\psi(\varepsilon) + \varphi\left(\frac{\varepsilon}{s^2}, 0,\ 0, \frac{\varepsilon}{s}, \varepsilon\right) \le \psi(\varepsilon)$$

This inequality holds only if

$$\varphi\left(\frac{\varepsilon}{s^2}, 0,\ 0, \frac{\varepsilon}{s}, \varepsilon\right) = 0$$

and $\varepsilon = 0$, which is a contradiction.

So, $\{x_k\}_{k\in\mathbb{N}}$ is Cauchy sequence.
Since $(E, \|\cdot,\ldots,\cdot\|)$ is complete, the sequence $\{x_k\}_{k\in\mathbb{N}}$ converges to a point $x^* \in E$,

$$\lim_{k\to+\infty} x_k = \lim_{k\to+\infty} T^k x_0 = x^*$$

Now we prove that $Tx^* = x^*$.
Taking the C-contraction inequality

$$\begin{aligned}&\psi\left(\|Tx^* - x_{k+1}, e_2, \ldots, e_n\|\right) \le \psi\left(M_0(x^*, x_k)\right) \\ &\quad -\varphi\left(\|x^* - x_k, e_2, \ldots, e_n\|\right. \\ &\quad \|x^* - Tx^*, e_2, \ldots, e_n\|, \|x_k - x_{k+1}, e_2, \ldots, e_n\|, \\ &\quad \left.\|Tx^* - x_k, e_2, \ldots, e_n\|,\ \|x^* - x_{k+1}, e_2, \ldots, e_n\|\right)\end{aligned}$$

and

$$\begin{aligned}M_0(x^*, x_k) = \max&\left\{\frac{1}{s}\|x^* - x_k, e_2, \ldots, e_n\|,\right. \\ &\|x^* - Tx^*, e_2, \ldots, e_n\|, \|x_k - x_{k+1}, e_2, \ldots, e_n\|, \\ &\left.\frac{\|Tx^* - x_k, e_2, \ldots, e_n\| + \|x^* - x_{k+1}, e_2, \ldots, e_n\|}{2s}\right\}\end{aligned}$$

We see that

$$\begin{aligned}\lim_{k\to+\infty} M_0(x^*, x_k) &= \max\left\{0, \|x^* - Tx^*, e_2, \ldots, e_n\|, 0,\right. \\ &\left.\frac{\|Tx^* - x^*, e_2, \ldots, e_n\|}{2}\right\} = \|x^* - Tx^*, e_2, \ldots, e_n\|\end{aligned}$$

and

$$\begin{aligned}&\psi\left(\|x^* - Tx^*, e_2, \ldots, e_n\|\right) \le \psi\left(\|x^* - Tx^*, e_2, \ldots, e_n\|\right) \\ &\quad -\varphi\left(\|x^* - Tx^*, e_2, \ldots, e_n\|, \|x^* - Tx^*, e_2, \ldots, e_n\|,\right. \\ &\quad \left.0, \|x^* - Tx^*, e_2, \ldots, e_n\|, \|x^* - Tx^*, e_2, \ldots, e_n\|\right)\end{aligned}$$

From which $\|x^* - Tx^*, e_2, \ldots, e_n\| = 0$ for all $e_2, \ldots, e_n \in E$ and $x^* = Tx^*$.
Next, we show the uniqueness of the fixed point x^* of function T.
Suppose that there exists another fixed point y^* of function T, $y^* = Ty^*$. We have

$$\begin{aligned}&\psi\left(\|Tx^* - Ty^*, e_2, \ldots, e_n\|\right) \le \psi\left(M_0(x^*, y^*)\right) \\ &\quad -\varphi\left(\|x^* - y^*, e_2, \ldots, e_n\|\ \|x^* - Tx^*, e_2, \ldots, e_n\|,\right. \\ &\quad \|y^* - Ty^*, e_2, \ldots, e_n\|, \|Tx^* - y^*, e_2, \ldots, e_n\|, \\ &\quad \left.\|x^* - Ty^*, e_2, \ldots, e_n\|\right)\end{aligned}$$

and

$$\begin{aligned}&\psi\left(\|x^* - y^*, e_2, \ldots, e_n\|\right) \le \psi\left(\|x^* - y^*, e_2, \ldots, e_n\|\right) \\ &\quad -\varphi(\|x^* - y^*, e_2, \ldots, e_n\|, 0, 0, \|x^* - y^*, e_2, \ldots, e_n\|, \\ &\quad \|x^* - y^*, e_2, \ldots, e_n\|)\end{aligned}$$

From this, it yields $\|x^* - y^*, e_2, \ldots, e_n\| = 0$ for every $e_2, \ldots, e_n \in E$ and $x^* = y^*$.

*Remark.*If we take $\psi(t) = t$ in Theorem 2 there exists a unique fixed point for a function $T : E \to E$ that satisfies the contraction

$$\|Tx - Ty, e_2, \ldots, e_n\| \le M_0(x,y) - \varphi(\|x-y, e_2, \ldots, e_n\|,$$

$$\|x - Tx, e_2, \ldots, e_n\|, \|y - Ty, e_2, \ldots, e_n\|,$$
$$\|y - Tx, e_2, \ldots, e_n\|, \|x - Ty, e_2, \ldots, e_n\|) \quad (12)$$

in a quasi n-normed space$(E, \|\cdot, \ldots, \cdot\|)$ with $s \ge 1$.

*Example 6.*Considering P_k the set of real polynomials of degree less or equal to k with coefficients from $[0,1]$. Taking the usual addition and multiplication with scalar, the triple $(P_k, +, \cdot)$ is an infinite dimensional vector space. Let $\{x_1, \ldots, x_{kn}\}$ be a set of points in $[0,1]$.

The function $\|\cdot, \ldots, \cdot\| : P_k^n \to [0, +\infty[$,

$$\|f_1, f_2, \ldots, f_n\| = \begin{cases} s\sum_{i=1}^{kn} |f_1(x_i)\ldots f_n(x_i)|, & f_1, \ldots, f_n \text{ linearly in}-\text{dependent} \\ 0, & f_1, \ldots, f_n \text{ linearly dependent} \end{cases}$$

for $s \ge 1$ is a quasi n-norm and the pair $(E = P_k, \|\cdot \ldots, \cdot\|)$ is a quasi n-normed space.

$s = \frac{5}{2}$. Taking $T : E \to E$, $Tx = \frac{1}{4}x$, where x is from E,

$\psi : \mathbb{R}^+ \to \mathbb{R}^+, \psi(t) = 4te^t$, and $\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{2}{5}t_1 + t_2 + t_3 + \frac{t_2+t_3}{5}$, we show that the function T satisfies the conditions of Theorem 2.

The first three conditions are clear.

Now we see $\|Tx - Ty, e_2, \ldots, e_n\| = \|\frac{x}{4} - \frac{y}{4}, e_2, \ldots, e_n\| = \frac{1}{4}\|x - y, e_2, \ldots, e_n\|$.

In addition,

$$M_0(x,y) = \max\left\{\frac{2}{5}\|x-y, e_2, \ldots, e_n\|, \right.$$
$$\|x - Tx, e_2, \ldots, e_n\|, \|y - Ty, e_2, \ldots, e_n\|,$$
$$\left.\frac{\|y - Tx, e_2, \ldots, e_n\| + \|x - Ty, e_2, \ldots, e_n\|}{5}\right\}$$

Since the inequality $t < e^t$ for every $t \ge 0$, we have that

$$\psi(M_0(x,y)) = 4M_0(x,y)\, e^{M_0(x,y)}$$
$$\ge \|x-y, e_2, \ldots, e_n\|\, e^{\frac{1}{4}\|x-y, e_2, \ldots, e_n\|}$$
$$+\frac{2}{5}\|x-y, e_2, \ldots, e_n\| + \|x - Tx, e_2, \ldots, e_n\|$$
$$+\|y - Ty, e_2, \ldots, e_n\|$$
$$+\frac{\|y - Tx, e_2, \ldots, e_n\| + \|x - Ty, e_2, \ldots, e_n\|}{5}$$
$$= \psi(\|Tx - Ty, e_2, \ldots, e_n\|)$$
$$+\varphi(\|x-y, e_2, \ldots, e_n\|, \|x - Tx, e_2, \ldots, e_n\|,$$
$$\|y - Ty, e_2, \ldots, e_n\|, \|y - Tx, e_2, \ldots, e_n\|,$$
$$\|x - Ty, e_2, \ldots, e_n\|)$$

Consequently,

$$\psi(\|Tx - Ty, e_2, \ldots, e_n\|) \le \psi(M_0(x,y))$$
$$-\varphi(\|x-y, e_2, \ldots, e_n\|, \|x - Tx, e_2, \ldots, e_n\|,$$
$$\|y - Ty, e_2, \ldots, e_n\|, \|y - Tx, e_2, \ldots, e_n\|,$$
$$\|x - Ty, e_2, \ldots, e_n\|$$

and we are in condition of Theorem 2. As a result, the function T has a unique fixed point in E, $x = 0$.

4 Corollaries

Corollary 1.*Let $(E, \|\cdot, \ldots, \cdot\|)$ be quasi n-Banach space with constant $s \ge 1$ and let $T : E \to E$ φ-weak contraction in E. Then T has a unique fixed point in E.*

Proof. Let us consider the φ-weak contraction

$$\|Tx - Ty, e_2, \ldots, e_n\| \le M_0(x,y) - \varphi(M_0(x,y))$$

If we take $\psi(t) = t$, the conditions of Theorem 1 are satisfied and T has a unique fixed point in E.

*Remark.*Corollary 1 is an extension of result of [26] [26] in quasi n-normed space.

Corollary 2.*Let $(E, \|\cdot, \ldots, \cdot\|)$ be a quasi n-Banach space with constant $s \ge 1$ and let $T : E \to E$ be a map. If there exists a nonnegative real number α, where $\alpha < 1$, such that for all $x, y \in X$,*

$$\|Tx - Ty, e_2, \ldots, e_n\| \le \alpha \cdot M_0(x,y)$$

then T has a unique fixed point in E.

Proof. Let us consider $T:E\to E$ be a map such that there exists a nonnegative real number

$$\alpha<1, \|Tx-Ty,e_2,\dots,e_n\|\le\alpha\cdot M_0(x,y)$$

Taking $\varphi(t)=(1-\alpha)t$ in the contraction of Corollary 1, we have that T has a unique fixed point in E.

*Remark.*The above result is an extension of result of [27] in quasi n-Banach space.

*Example 7.*Considering $(E,\|\cdot,\dots,\cdot\|_\infty)$ the quasi n-Banach space given in Example 6 with $s=\frac{3}{2}$.
Taking $T:E\to E$, $Tx=\frac{x}{5}$, $\alpha=\frac{1}{2}$, the function T satisfies the condition of Corollary 2.
Consequently, $\|Tx-Ty,e_2,\dots,e_n\|\le\alpha\cdot M_0(x,y)$, and the function T has a unique fixed point in E, $x=0$.

Corollary 3.*Let $(E,\|\cdot,\dots,\cdot\|)$ be a quasi n-Banach space with constant $s\ge1$ and $T:X\to X$. If there exists a nonnegative real number α, where $\alpha<1$, such that for all $x,y\in X$,*

$$\|Tx-Ty,e_2,e_3,\cdots,e_n\|\le\alpha\cdot\max\{\frac{1}{s}\|x-y,e_2,\dots,e_n\|,$$
$$\|x-Tx,e_2,\dots,e_n\|,\|y-Ty,e_2,\dots,e_n\|\}$$

then T has a unique fixed point in X.

Proof. We note that the following inequality holds.

$$\|Tx-Ty,e_2,e_3,\cdots,e_n\|\le\alpha\cdot\max\left\{\frac{1}{s}\|x-y,e_2,\dots,e_n\|,\right.$$
$$\left.\|x-Tx,e_2,\dots,e_n\|,\ \|y-Ty,e_2,\dots,e_n\|\right\}$$
$$\le\alpha\cdot M_0(x,y).$$

Consequently, the function T has a unique fixed point.
Remark. Corollary 3 generalizes the Sehgal's result [28] in a quasi n-Banach space.

Corollary 4.*Let $(E,\|\cdot,\dots,\cdot\|)$ be a quasi n-Banach space with constant $s\ge1$ and let $T:E\to E$ that satisfies the weak C-contraction*

$$\|Tx-Ty,e_2,\dots,e_n\|$$
$$\le\frac{\|y-Tx,e_2,\dots,e_n\|+\|x-Ty,e_2,\dots,e_n\|}{2s}$$
$$-\varphi\begin{pmatrix}\|x-y,e_2,\dots,e_n\|,\|x-Tx,e_2,\dots,e_n\|,\\ \|y-Ty,e_2,\dots,e_n\|,\|y-Tx,e_2,\dots,e_n\|,\\ \|x-Ty,e_2,\dots,e_n\|\end{pmatrix}$$

where $\varphi:R^{+5}\to R^{+}$is C-type. Then the function T has a unique fixed point in E.

Proof. Using the contraction inequality and the fact

$$\frac{\|y-Tx,e_2,\dots,e_n\|+\|x-Ty,e_2,\dots,e_n\|}{2s}\le M_0(x,y)$$

we take:

$$\|Tx-Ty,e_2,\dots,e_n\|$$
$$\le\frac{\|y-Tx,e_2,\dots,e_n\|+\|x-Ty,e_2,\dots,e_n\|}{2s}$$
$$-\varphi\begin{pmatrix}\|x-y,e_2,\dots,e_n\|,\ \|x-Tx,e_2,\dots,e_n\|,\\ \|y-Ty,e_2,\dots,e_n\|,\|y-Tx,e_2,\dots,e_n\|,\\ \|x-Ty,e_2,\dots,e_n\|\end{pmatrix}$$
$$\le M_0(x,y)-\varphi(\|x-y,e_2,\dots,e_n\|,\|x-Tx,e_2,\dots,e_n\|,$$
$$\|y-Ty,e_2,\dots,e_n\|,\|y-Tx,e_2,\dots,e_n\|,$$
$$\|x-Ty,e_2,\dots,e_n\|)$$

Consequently, the function T has a unique fixed point.
Remark 4.9 Corollary 4 generalizes Theorem 6 of [25] in quasi n-normed space.

5 An application to Integral Equations

The applications of Fixed-Point Theory to Integral equations have been on focus of many researchers [3], [29]. In this section, we apply the result of Theorem 2 to prove the existence and uniqueness of solution under some conditions for integral equation

$$x(t)=h(t)+\int_0^1F(t,\tau)r(\tau,x(\tau))d\tau\qquad\text{in } C_{[0,1]}$$

Let $(E,\|\cdot,\dots,\cdot\|_\infty)$ be the complete quasi n-normed space where

$$E=C_{[0,1]}=\{f:[0,1]\to\mathbb{R}, f\text{ is real continuous function}\}$$

and

$$\|f_1,\dots,f_n\|_\infty=\begin{cases}s\cdot\sup\limits_{t\in[0,1]}\prod\limits_{i=1}^{n}|f_i(t)|, & f_1,\dots,f_n\text{ are}\\ & \text{linearly indipendent}\\ 0, & \text{otherwise}\end{cases}$$

Theorem 3.*The integral equation*

$$x(t)=h(t)+\int_0^1K(t,\tau)r(\tau,x(\tau))d\tau$$

where $x\in C_{[0,1]}$ and $h:[0,1]\to\mathbb{R}$ is a continuous function, $K:[0,1]\times\mathbb{R}\to[0,+\infty)$ and $r:[0,1]\times\mathbb{R}\to\mathbb{R}$ are continuous functions which satisfy the following conditions:

$$\int_0^1K(t,\tau)d\tau\le1$$

and

$$|r(\tau,x(\tau))-r(\tau,y(\tau))| \leq \frac{1}{2s}|x(\tau)-y(\tau)|, \quad \forall\tau\in[0,1]$$

has a unique solution in $C_{[0,1]}$.

Proof. Define the mapping $T : C_{[0,1]} \to C_{[0,1]}$ given by $Tx(t) = h(t) + \int_0^T K(t,\tau)r(\tau,x(\tau))d\tau$.
Below, we show that the mapping T satisfies the conditions of Theorem 2.
Firstly, we see that:

$$\begin{aligned}|Tx(t)-Ty(t)| &= \left|\int_0^T K(t,\tau)(r(\tau,x(\tau))-r(\tau,y(\tau)))d\tau\right| \\ &\leq \int_0^T K(t,\tau)|r(\tau,x(\tau))-r(\tau,y(\tau))|d\tau \\ &\leq \int_0^T K(t,\tau)\frac{1}{2s}|x(\tau)-y(\tau)|d\tau \\ &\leq \frac{1}{2s}|x(t)-y(t)|\end{aligned}$$

Consequently, for $e_i(t)\in C_{[0,1]}$, $i=2,3,\ldots,n$

$$\sup_{t\in[0,T]}|Tx(t)-Ty(t)|\cdot\prod_{i=2}^{n}|e_i(t)|$$

$$\leq \frac{1}{2s}\sup_{t\in[0,T]}|x(t)-y(t)|\cdot\prod_{i=2}^{n}|e_i(t)|$$

As a result, the following inequalities hold.

$$\begin{aligned}&\|Tx-Ty,e_2,e_3,\ldots,e_n\|_\infty\ e^{\|Tx-Ty,e_2,e_3,\ldots,e_n\|_\infty} \\ &+\frac{\|x-Ty,e_2,e_3,\ldots,e_n\|_\infty+\|Tx-y,e_2,e_3,\ldots,e_n\|_\infty}{2s} \\ &\leq \frac{1}{2s}\|x-y,e_2,e_3,\ldots,e_n\|_\infty\ e^{\frac{1}{2s}\|x-y,e_2,e_3,\ldots,e_n\|_\infty} \\ &+\frac{\|x-Ty,e_2,e_3,\ldots,e_n\|_\infty+\|Tx-y,e_2,e_3,\ldots,e_n\|_\infty}{2s} \\ &\leq \frac{1}{2}M_0(x,y)e^{M_0(x,y)}+\frac{1}{2}M_0(x,y)\leq M_0(x,y)e^{M_0(x,y)}\end{aligned}$$

This shows that the mapping T satisfies the conditions of Theorem 2 for $\psi(t) = te^t$ and $\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{t_4+t_5}{2s}$, and it has a unique fixed point in $C_{[0,1]}$, which guaranties the existence and the uniqueness of solution for $x(t) = h(t) + \int_0^1 F(t,\tau)r(\tau,x(\tau))d\tau$ in $C_{[0,1]}$.

6 Conclusions

In this paper there are defined quasi n-normed space as a generalization of n- normed space. There are given some examples on finite vector spaces and infinite vector spaces. Some topological facts for quasi n-normed spaces are given. Furthermore, there are proved fixed point results for generalized weak contractions in a quasi n-normed space. The highlights of the paper are Theorem 1 and Theorem 2 which show the existence and uniqueness of a fixed point for (φ,ψ)-generalized weak contraction and (φ,ψ)-generalized weak C-contraction, respectively. As a result, from these theorems there are obtained some corollaries which extend and generalize the result of [25,**?**,**?**,**?**] in a quasi n-normed space. Furthermore, all theorems and corollaries are true in n-normed space, quasi 2-normed space and 2-normed space. Some examples are given to show applicative side of obtained results. As an application of Theorem 2, there is given Theorem 3, which assure existence and uniqueness of a solution for a type of integral equation.

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