Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/130202

Solution of Fractional SIR Epidemic Model Using Residual Power Series Method

Shatha Hasan[1], *Ayat Al-Zoubi*[1], *Asad Freihet*[1], *Mohammed Al-Smadi*[1,*] *and Shaher Momani*[2,3]

[1] Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816, Jordan
[2] Department of Mathematics, Faculty of Science, The University of Jordan, Amman 11942, Jordan
[3] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah, 21589, Kingdom of Saudi Arabia

Received: 22 Dec. 2018, Revised: 2 Jan. 2019, Accepted: 19 Jan. 2019
Published online: 1 Mar. 2019

Abstract: The SIR model with unknown parameters is an important issue for scientists in the study of epidemiology and medical care for the injured people. In this work, an efficient technique based on the generalized Taylor series, called the residual power series method, is applied to solve the SIR epidemic model of fractional order. The fractional derivative is described in the Caputo sense. The use of the residual power series method enables us to get an analytic solution of the SIR model in the form of a convergent power series in addition to the approximate solution. To show the efficiency of the proposed technique, we apply it to the fractional SIR model and compare the results with the fourth-order Runge-Kutta method. The numerical and graphical results show that the residual power series method can be considered as an alternative technique for solving many real-life problems involving differential equations of any order.

Keywords: Fractional derivative, Caputo concept, Residual power series method, Fractional SIR epidemic model.

1 Introduction

Building mathematical models for real-world phenomena and developing effective techniques to address them is one of the most critical issues in applied mathematics, physics, engineering, biology, and other fields of science. The spread of infectious diseases and epidemic has always been a threat to public health, which causes serious problems not only for the survival of human beings but also for the economics and the social development of human society. Recently, many diseases and epidemics have emerged and spread in the poor and overcrowded cities, and refugee camps. This alerts to the real danger that should be controlled. Indeed, infectious diseases have had a long history, and great progress has been achieved, especially during the 20th century.

Physicians rely on the field of applied mathematics for studying the development of diseases and epidemics, and how people are affected and treated. In general, dealing with epidemics requires many mathematical stages, including model creation, model analysis, solving differential equations, and statistical analysis. One of the most popular models in mathematical epidemiology is the Kermack-McKendrick Susceptible-Infected-Recovered (SIR) compartmental epidemic model which was studied by Kermack and McKendrick in 1927 [1,2,3]. In this model, the population is divided into three groups (compartments); a susceptible, labeled $S(t)$, in which all individuals are susceptible if they contact with a disease but are at risk of being infected at time t; an infected compartment, labeled $I(t)$,in which all individuals are infected by the disease and can transmit it to others at time t; and a removed compartment, labeled $R(t)$, represent the number of removed or recovered individuals from the disease at time t. The SIR model is described in the following non-linear system of first-order ordinary differential equations:

$$\begin{cases} \dfrac{dS}{dt} = -p_1 S(t)I(t), \\ \dfrac{dI}{dt} = p_1 S(t)I(t) - p_2 I(t), \\ \dfrac{dR}{dt} = p_2 I(t), \end{cases} \tag{1}$$

subject to $S(0) = S_0$, $I(0) = I_0$, and $R(0) = R_0$. Here, p_1 and p_2 are positive constants, which are called the infection rate and the removal rate, respectively. Using

* Corresponding author e-mail: mhm.smadi@bau.edu.jo

this model, we don't consider a mild, short-lived epidemic, e.g. influenza, in a closed (no immigration or emigration) population. Moreover, given the time scale of influenza epidemics, we will not consider demographic turnover (birth or death), and all infections are assumed to end with recovery. The size of the population $(S+I+R)$ is therefore constant and equal to the initial population size. The history and solutions to epidemic models were studied [2,3,4,5].

Due to the importance of the SIR model, many studies have been attempted to solve it and other epidemic models. For example, in [6], the authors solved the SIR model by the homotopy analysis method. Later, in [7], exact solution of the SIR model had been proposed. The homotopy perturbation method and the differential transform method were applied to this model in [8] and [9], respectively. Lyapunov function for a variety of SIR models in epidemiology was constructed in [10].

On the other hand, the last decades witness fundamental developments in the fractional calculus which includes integrals and derivatives of arbitrary order. Many real problems in different areas were translated into mathematical models via fractional differential equations because of the ability of fractional calculus to keep not only the behavior of the original physical system, but also all of their historical states [11,12,13,14]. For example, half-order derivatives and integrals were shown to be more useful in formulation some electrochemical problems than classical ones [15,16,17,18,19]. A large amount of studies and applications for the fractional calculus in fluid dynamics, visceolasticity, physics, entropy theory and engineering can be found in [20,21, 22,23,24].Therefore, most of the differential equations of integer order were generalized to fractional order. The purpose of the present work is to study the solution behavior of the SIR model of fractional order in the following form:

$$\begin{cases} D_0^{\alpha_1}S(t) = -p_1S(t)I(t), \\ D_0^{\alpha_2}I(t) = p_1S(t)I(t) - p_2I(t), \\ D_0^{\alpha_3}R(t) = p_2I(t), \end{cases} \tag{2}$$

subject to

$$S(0) = S_0, I(0) = I_0, R(0) = R_0, \tag{3}$$

where $D_0^{\alpha_1}S(t)$, $D_0^{\alpha_2}I(t)$ and $D_0^{\alpha_3}R(t)$ are the Caputo derivatives of orders α_1, α_2 and α_3 for $S(t)$, $I(t)$ and $R(t)$, respectively, and $0 < \alpha_i \leq 1$, $i = 1,2,3$.

Unfortunately, there are no methods in literature that produce exact solutions for non-linear differential equations of fractional order. So, many techniques have been developed to obtain approximate solutions. Among the methods that were used to get approximate solutions for the SIR model of fractional order are the homotopy analysis method [25], the homotopy analysis method for fractional SEIR epidemic model [26] and the multi-step generalized differential transform method [27]. The RPSM was initially developed to get numerical solutions of fuzzy differential equations of fractional order [28]. It provides a power series solution with rapid convergence and can be applied to linear and nonlinear differential equations [29,30,31,32,33,34,35].

In this work, we extend the application of the residual power series (RPS) method to approximate the numerical solution of the fractional SIR model as well as we compare the numerical results with the fourth-order Runge-Kutta method. The pattern of this paper is as follows: in the next section, we review some definitions and theorems of fractional calculus and fractional power series. In section 3, the application of the RPS method to the fractional SIR model is discussed. In section 4, the solution for the SIR model is presented in graphs and tables. A short conclusion is presented in section 5.

2 Preliminaries

Many definitions for derivatives of non-integer order can be found in literature, for example, Riemann-Liouville, Riesz, Grünwald-Letnikovand, and Caputo derivatives. Many researchers prefer to use the Caputo definition in modeling real-life problems because in the Caputo sense, the derivative of any constant is zero, and the initial conditions of fractional differential equations take on the classical form, similar to those for integer order. Next, the definition of Caputo derivative is given. For details, see [36,37,38]. In this section, we introduce some necessary definitions of fractional calculus.

Definition 2.1.*The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f(t)$ is defined by*

$$(J_{a+}^{\alpha}f)(t) = \frac{1}{\Gamma(\alpha)}) \int_a^t \frac{f(z)}{(t-z)^{1-\alpha}} dz, t > a.$$

For $\alpha = 0$, $(J_{a+}^{\alpha}f)(t)$ is the identity operator. That is, $(J_{a+}^{0}f)(t) = f(t)$.

Definition 2.2.*The Caputo fractional derivative of order $\alpha > 0$ is given by*

$$D_a^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(z)}{(t-z)^{\alpha-n+1}} dz,$$

$t > a$, $n-1 < \alpha \leqslant n$, $n \in N$.

The Caputo's derivative satisfies the following properties: $(J_{a+}^{\alpha}D_{a+}^{\alpha}f)(t) = f(t) - \Sigma_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$ and $(D_{a+}^{\alpha}J_{a+}^{\alpha}f)(t) = f(t)$. Another important property that gives the derivative of functions of the form $f(t) = (t-a)^{\beta-1}$is
For α, β $>$ 0, we have

$(D^{\alpha}_{a+}(t-a)^{\beta-1}(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} for \beta > n$, and $(D^{\alpha}_{a+}(t-a)^{k})(x) = 0$ for $k = 0,1,2,\ldots,n-1$.

Next, some definitions and results related to fractional power series in the sense of Caputo derivative are given. For detailed instructions, see [36].

Definition 2.3. *A fractional power series (FPS) about $t = t_0$ is defined as*

$$\Sigma_{m=0}^{\infty} c_m (t-t_0)^{m\alpha} = c_0 + c_1(t-t_0)^{\alpha} + c_2(t-t_0)^{2\alpha} + \cdots;$$

$n-1 < \alpha \leq n, n \in N, t \leq t_0$, where the constants c_m, $m = 0,1,2,\cdots$, are called the coefficients of the power series.

Theorem 2.1. *Let f have a FPS representation at $t = t_0$ of the form*

$$f(t) = \Sigma_{m=0}^{\infty} c_m (t-t_0)^{m\alpha};$$

$t_0 \leqslant t < t_0+\rho$. It was found that if $D^{m\alpha}_{t_0} f(t), m = 0,1,2,\cdots$ are continuous on $(t_0, t_0+\rho)$, then $c_m = \frac{D^{m\alpha}_{t_0} f(t_0)}{\Gamma(1+m\alpha)}$. Where Γ is the gamma function, $D^{m\alpha}_{t_0} = D^{\alpha}_{t_0} D^{\alpha}_{t_0} \cdots D^{\alpha}_{t_0}$ (m-times), and ρ is the radius of convergence.

A simple example of the FPS expansion about $t = 0$ is the hyperbolic sine and cosine functions of order α. They are given by the formulas:

$$sinh(x^{\alpha},\alpha) = \Sigma_{m=0}^{\infty} \frac{x^{(2m+1)\alpha}}{\Gamma(1+(2m+1)\alpha)},$$

and

$$cosh(x^{\alpha},\alpha) = \Sigma_{m=0}^{\infty} \frac{x^{(2m)\alpha}}{\Gamma(1+(2m)\alpha)},$$

$\alpha > 0, 0 \leqslant x$.

3 The RPS method for solving the SIR epidemic model of fractional order

To construct a solution for the non-linear fractional SIR model described in system 2 and 3, we do the following steps:

Step1: Suppose that $S(t), I(t)$, and $R(t)$ have the FPS about $t_0 = 0$ as follows:

$$\begin{cases} S(t) = \Sigma_{k=0}^{\infty} \frac{a_k}{\Gamma(1+k\alpha_1)} t^{k\alpha_1}, \\ I(t) = \Sigma_{k=0}^{\infty} \frac{b_k}{\Gamma(1+k\alpha_2)} t^{k\alpha_2}, \\ R(t) = \Sigma_{k=0}^{\infty} \frac{c_k}{\Gamma(1+k\alpha_3)} t^{k\alpha_3}, \end{cases} \quad (4)$$

where $0 \leqslant t < \rho$ for some $\rho > 0$.

Then, we can denote the $n-th$ truncated series of $S(t), I(t)$, and $R(t)$, respectively, by $S_n(t), I_n(t)$, and $R_n(t)$ whose definitions are

$$\begin{cases} S_n(t) = \Sigma_{k=0}^{n} \frac{a_k}{\Gamma(1+k\alpha_1)} t^{k\alpha_1}, \\ I_n(t) = \Sigma_{k=0}^{n} \frac{b_k}{\Gamma(1+k\alpha_2)} t^{k\alpha_2}, \\ R_n(t) = \Sigma_{k=0}^{n} \frac{c_k}{\Gamma(1+k\alpha_3)} t^{k\alpha_3}. \end{cases} \quad (5)$$

For $n = 0$, by using the initial conditions in Eq. (3), we have $S_0(t) = a_0 = S_0(0) = S_0$, $I_0(t) = b_0 = I_0(0) = I_0$, and $R_0(t) = c_0 = R_0(0) = R_0$. So, the $n-th$ truncated series in Eq. (5) can be written in the form:

$$\begin{cases} S_n(t) = S_0 + \Sigma_{k=1}^{n} \frac{a_k}{\Gamma(1+k\alpha_1)} t^{k\alpha_1}, \\ I_n(t) = I_0 + \Sigma_{k=1}^{n} \frac{b_k}{\Gamma(1+k\alpha_2)} t^{k\alpha_2}, \\ R_n(t) = R_0 + \Sigma_{k=1}^{n} \frac{c_k}{\Gamma(1+k\alpha_3)} t^{k\alpha_3}. \end{cases} \quad (6)$$

Step2: Define the residual functions for the model in Eq. (2) as:

$$\begin{cases} Res_S(t) = D_0^{\alpha_1} S(t) + p_1 S(t) I(t), \\ Res_I(t) = D_0^{\alpha_2} I(t) - p_1 S(t) I(t) + p_2 I(t), \\ Res_R(t) = D_0^{\alpha_3} R(t) - p_2 I(t). \end{cases} \quad (7)$$

Hence, the $n-th$ residual functions of $S(t), I(t)$, and $R(t)$, respectively, are

$$\begin{cases} Res_{S,n}(t) = D_0^{\alpha_1} S_n(t) + p_1 S_n(t) I_n(t), \\ Res_{I,n}(t) = D_0^{\alpha_2} I_n(t) - p_1 S_n(t) I_n(t) + p_2 I_n(t), \\ Res_{R,n}(t) = D_0^{\alpha_3} R_n(t) - p_2 I(t). \end{cases} \quad (8)$$

Obviously, $Res_S(t) = Res_I(t) = Res_R(t) = 0$, for all $t \geq 0$, $\lim_{n\to\infty} Res_{S,n}(t) = Res_S(t)$, $\lim_{n\to\infty} Res_{I,n}(t) = Res_I(t)$, and $\lim_{n\to\infty} Res_{R,n}(t) = Res_R(t)$. Since the Caputo derivative of any constant is zero, one can deduce that $D_0^{(k-1)\alpha_1} Res_S(0) = D_0^{(k-1)\alpha_1} Res_{S,k}(0)$, $D_0^{(k-1)\alpha_2} Res_I(0) = D_0^{(k-1)\alpha_2} Res_{I,k}(0)$, $D_0^{(k-1)\alpha_3} Res_R(0) = D_0^{(k-1)\alpha_3} Res_{R,k}(0)$, for $k = 1,\cdots,n$. See [29]. Actually, this idea is the basis of the RPS method as it appears in the following step.

Step3: To obtain the coefficients a_k, b_k, c_k, $k = 1,2,3,\cdots,n$, we substitute the $n-th$ truncated series of $S(t)$, $I(t)$, and $R(t)$ into Eq. (8), and then apply the Caputo fractional derivative operators $D_0^{(n-1)\alpha_1}$, $D_0^{(n-1)\alpha_2}$, and $D_0^{(n-1)\alpha_3}$ on $Res_{S,n}(t)$, $Res_{I,n}(t)$, and $Res_{R,n}(t)$, respectively. Consequently, we have the equations:

$$\begin{cases} D_0^{(n-1)\alpha_1} Res_{S,n}(0) = 0, \\ D_0^{(n-1)\alpha_2} Res_{I,n}(0) = 0, \\ D_0^{(n-1)\alpha_3} Res_{R,n}(0) = 0, \end{cases} \tag{9}$$

for $n = 1, 2, 3, \cdots$.

Step4: Solve the algebraic system (9) for the values of a_k, b_k, c_k, $k = 1, 2, 3, \cdots, n$ to get the $n-th$ residual power series approximate solution of system (2) and (3).

Step5: We repeat the procedure to obtain sufficient number of coefficients. Higher accuracy for the solution can be achieved by evaluating more terms in the series solution.

4 Application and Numerical Results

In this section, we give numerical results for the solution of the fractional SIR model in Eqs. (2) and (3) to demonstrate the performance and the efficiency of the RPS method in handling such epidemic models. Using our proposed method, we obtain an analytic solution for the SIR model in the form of a rapid convergent series. The approximate solutions are presented in graphics and tabulated values of $S(t)$, $I(t)$ and $R(t)$. We compare the results of the RPS method with the fourth order Runge-Kutta method. The computations are performed using Mathematica10 software package.

Consider the following fractional SIR model:

$$\begin{cases} D_0^{\alpha} S(t) = -0.001 S(t) I(t), \\ D_0^{\alpha} I(t) = 0.001 S(t) I(t) - 0.072 I(t), \\ D_0^{\alpha} R(t) = 0.072 I(t), \end{cases} \tag{10}$$

subject to $S(0) = 620$, $I(0) = 10$ and $R(0) = 70$. Here, α is the order of the fractional derivative described in the Caputo sense where $0 < \alpha \leqslant 1$.

Following the steps of the RPS method discussed in the previous section, the first truncated power series approximations have the forms

$$S_1(t) = 620 + \frac{a_1}{\Gamma(1+\alpha)} t^{\alpha},$$

$$I_1(t) = 10 + \frac{b_1}{\Gamma(1+\alpha)} t^{\alpha},$$

$$R_1(t) = 70 + \frac{c_1}{\Gamma(1+\alpha)} t^{\alpha}.$$

According Eq. (8), the first residual functions of $S(t)$, $I(t)$, and $R(t)$, respectively, are:

$Res_{S,1}(t) =$
$D_0^{\alpha}(620 + \frac{a_1 t^{\alpha}}{\Gamma(1+\alpha)}) + 0.001(620 + \frac{a_1 t^{\alpha}}{\Gamma(1+\alpha)})(10 + \frac{b_1 t^{\alpha}}{\Gamma(1+\alpha)})$
$= 6.2 + a_1 + \frac{0.01 t^{\alpha} a_1}{\Gamma(1+\alpha)} + \frac{0.62 t^{\alpha} b_1}{\Gamma(1+\alpha)} + \frac{0.001 t^{2\alpha} a_1 b_1}{\Gamma(1+\alpha)^2}$,

$Res_{I,1}(t) = D_0^{\alpha} I_n(t) - 0.001(620 + \frac{a_1 t^{\alpha}}{\Gamma(1+\alpha)})(10 + \frac{b_1 t^{\alpha}}{\Gamma(1+\alpha)}) + 0.072(10 + \frac{b_1 t^{\alpha}}{\Gamma(1+\alpha)})$
$= -4.464 - \frac{0.0072 t^{\alpha} a_1}{\Gamma(1+\alpha)} + b_1 - \frac{0.8928 t^{\alpha} b_1}{\Gamma(1+\alpha)} - \frac{0.00144 t^{2\alpha} a_1 b_1}{\Gamma(1+\alpha)^2} - \frac{0.04464 t^{2\alpha} b_1^2}{\Gamma(1+\alpha)^2} - \frac{0.000072 t^{3\alpha} a_1 b_1^2}{\Gamma(1+\alpha)^3}$,

$Res_{R,1}(t) = D_0^{\alpha}(70 + \frac{c_1 t^{\alpha}}{\Gamma(1+\alpha)}) - 0.072(10 + \frac{b_1 t^{\alpha}}{\Gamma(1+\alpha)}) = -0.72 - \frac{0.072 t^{\alpha} b_1}{\Gamma(1+\alpha)} + c_1$.

Equating $Res_{S,1}(0)$, $Res_{I,1}(0)$ and $Res_{R,1}(0)$ by zero gives the values of a_1, b_1, and c_1. So that

$$S_1(t) = 620 - \frac{6.2 t^{\alpha}}{\Gamma(1+\alpha)},$$

$$I_1(t) = 10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)},$$

$$R_1(t) = 70 + \frac{0.72 t^{\alpha}}{\Gamma(1+\alpha)}.$$

For $n = 2$, the second truncated power series approximations have the forms

$$S_2(t) = 620 - \frac{6.2 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{2\Gamma(1+\alpha)},$$

$$I_2(t) = 10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$R_2(t) = 70 + \frac{0.72 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{c_2 t^{2\alpha}}{\Gamma(1+2\alpha)},$$

and the second residual functions are

$Res_{S,2}(t) = D_0^{\alpha}(620 - \frac{6.2 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)}) + 0.001(620 - \frac{6.2 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)})(10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)})$,

$Res_{I,2}(t) =$
$D_0^{\alpha} I_n(t) - 0.001(620 - \frac{6.2 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{a_2 t^{2\alpha}}{\Gamma(1+2\alpha)})(10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)}) + 0.072(10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)})$,

$Res_{R,2}(t) = D_0^{\alpha}(70 + \frac{0.72 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{c_2 t^{2\alpha}}{\Gamma(1+2\alpha)}) - 0.072(10 + \frac{4.464 t^{\alpha}}{\Gamma(1+\alpha)} + \frac{b_2 t^{2\alpha}}{\Gamma(1+2\alpha)})$.

Applying the operator D_0^{α} to $Res_{S,2}(t)$, $Res_{I,2}(t)$, and $Res_{R,2}(t)$, we get

$D_0^{\alpha} Res_{S,2}(t) = -\frac{0.0553536 t^{\alpha} \alpha \Gamma(2\alpha)}{\Gamma(1+\alpha)^3} + 2.70568 + a_2 + \frac{0.013392 t^{2\alpha} \alpha \Gamma(3\alpha) a_2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} + \frac{0.01 t^{\alpha} a_2}{\Gamma(1+\alpha)} - \frac{0.0186 t^{2\alpha} \alpha \Gamma(3\alpha) b_2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} + \frac{0.62 t^{\alpha} b_2}{\Gamma(1+\alpha)} + \frac{0.004 t^{3\alpha} \alpha \Gamma(4\alpha) a_2 b_2}{\Gamma(1+2\alpha)^2 \Gamma(1+3\alpha)}$,

$D_0^{\alpha} Res_{I,2}(t) = -\frac{1.6994 t^{\alpha} \alpha \Gamma(2\alpha)}{\Gamma(1+\alpha)^3} - 3.94082 + \frac{0.0266866 t^{2\alpha} \alpha \Gamma(3\alpha)}{\Gamma(1+\alpha)^3 \Gamma(1+2\alpha)} - \frac{0.0192844 t^{2\alpha} \alpha \Gamma(3\alpha) a_2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} - \frac{0.0072 t^{\alpha} a_2}{\Gamma(1+\alpha)} - \frac{0.00573906 t^{3\alpha} \alpha \Gamma(4\alpha) a_2}{\Gamma(1+\alpha)^2 \Gamma(1+2\alpha) \Gamma(1+3\alpha)} + \frac{\alpha \Gamma(\alpha) b_2}{\Gamma(1+\alpha)} - \frac{1.168854 t^{2\alpha} \alpha \Gamma(3\alpha) b_2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} -$

$\frac{0.8928t^{\alpha}b_2}{\Gamma(1+\alpha)} + \frac{0.0159418t^{3\alpha}\alpha\Gamma(4\alpha)b_2}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)\Gamma(1+3\alpha)} - \frac{0.00576t^{3\alpha}\alpha\Gamma(4\alpha)a_2b_2}{\Gamma(1+2\alpha)^2\Gamma(1+3\alpha)} - \frac{0.00321408t^{4\alpha}\alpha\Gamma(5\alpha)a_2b_2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)} - \frac{0.17856t^{3\alpha}\alpha\Gamma(4\alpha)b_2^2}{\Gamma(1+2\alpha)^2\Gamma(1+3\alpha)} + \frac{0.002232t^{4\alpha}\alpha\Gamma(5\alpha)b_2^2}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2\Gamma(1+4\alpha)} - \frac{0.000432t^{5\alpha}\alpha\Gamma(6\alpha)a_2b_2^2}{\Gamma(1+2\alpha)^3\Gamma(1+5\alpha)}$,

$D_0^{\alpha}Res_{R,2}(t) = -0.321408 - \frac{0.072t^{\alpha}b_2}{\Gamma(1+\alpha)} + c_2.$

Again, using the fact that $D_0^{\alpha}Res_{S,2}(0) = D_0^{\alpha}Res_{I,2}(0) = D_0^{\alpha}Res_{R,2}(0) = 0$, we get the values of a_2, b_2, and c_2. Hence,

$$S_2(t) = 620 - \frac{6.2t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2.70568t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$I_2(t) = 10 + \frac{4.464t^{\alpha}}{\Gamma(1+\alpha)} + \frac{3.9408t^{2\alpha}}{\Gamma(1+2\alpha)},$$

$$R_2(t) = 70 + \frac{0.72t^{\alpha}}{\Gamma(1+\alpha)} + \frac{0.3214t^{2\alpha}}{\Gamma(1+2\alpha)}.$$

Continuing this process, we get the fourth approximations:

$S_4(t) = 620 - \frac{6.2t^{\alpha}}{\Gamma(1+\alpha)} - \frac{2.70568t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{(0.027677\Gamma(1+2\alpha)-2.4162\Gamma(1+\alpha)^2)t^{3\alpha}}{\Gamma(1+\alpha)^2+\Gamma(1+3\alpha)} + (\frac{0.0241617357001972\,37}{\Gamma(1+4\alpha)} - \frac{1.053639846743295\alpha\Gamma(2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{2.169355}{\Gamma(1+4\alpha)} - \frac{2.767629801837706\,3\Gamma(1+2\alpha)}{10000\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} + \frac{0.036512027491408\,94\Gamma(1+3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)})t^{4\alpha}$,

$I_4(t) = 10 + \frac{4.464t^{\alpha}}{\Gamma(1+\alpha)} + \frac{3.9408t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}\alpha\Gamma(2\alpha)(-1.6994-\frac{6.9978\Gamma(1+\alpha)^2}{\Gamma(1+2\alpha)})}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} + (\frac{4.55405\alpha\Gamma(3\alpha)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+4\alpha)} - \frac{0.0267\alpha^3\Gamma(3\alpha)}{\Gamma(1+\alpha)^3\Gamma(1+4\alpha)} + \frac{0.0004\alpha^3\Gamma(2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} + \frac{37.48}{12\Gamma(1+4\alpha)} + \frac{1.51723\Gamma(2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)} - \frac{0.104}{6\Gamma(1+4\alpha)})t^{4\alpha}$,

$R_4(t) = 70 + \frac{0.72t^{\alpha}}{\Gamma(1+\alpha)} + \frac{0.32141t^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{0.28374t^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{3.023\Gamma(1+\alpha)^2+0.7341\Gamma(1+\alpha)}{12\Gamma(1+\alpha)^2\Gamma(1+4\alpha)}t^{4\alpha}$.

In order to show the accuracy of the RPS method for approximating the solution of the SIR model, a numerical comparison between the $15th$-RPS solution and the fourth-order Runge-Kutta (4RK) solution for $\alpha = 1$ is given in Tables 1, 2 and 3. It is convenient to have a notation for the error of the approximation. Accordingly, in Table 1, we compute the absolute error $Abs_S(t)$ and the relative error $Rel_S(t)$ for $S(t)$, where the absolute error is defined using the formula $Abs_S(t_i)$=$|4RKS(t_i) - RPSS(t_i)|$ and the relative error is defined using the formula $Rel_S(t_i)$=$|\frac{(RKS(t_i)-RPSS(t_i))}{RKS(t_i)}|, i = 0, 1, 2, \cdots, 10$. Similarly to compute the absolute and relative errors for $I(t)$ and $R(t)$ as in Tables 2 and 3, respectively. From these Tables, it can be seen that the accuracy obtained using the RPS technique is advanced by using only a few approximation terms. Also, it can be observed that the error estimate is more accurate at the beginning of the independent values of the interval $[0,1]$. In addition, it can be concluded that higher accuracy can be achieved by evaluating more components of the RPS-solution. Anyhow, the results reported in these tables confirm the effectiveness of the RPS technique.

While Figures 1, 2, and 3 show the behavior of the RPS-solution and 4RK solution at the fractional order $\alpha = 1$, and $n = 15$ with step size $h = 0.5$ over the interval $[0,1]$. From these graphical results, it is clear that the approximations obtained by the RPS method are very efficient and the efficiency can be achieved using relatively small number of terms; 15 terms in our example. However, the efficiency can be dramatically increased by increasing the number of terms in the power series. These graphs also show that the presented method can predict the behavior of the compartments $S(t)$, $I(t)$ and $R(t)$ accurately for the region under consideration, where the behavior of such approximations are in good agreement with each other.

However, for $\alpha = 1$, the solution of the SIR model can be approximated using the RPS method by these polynomials:

$S_{15}(t) = 620 - 6.2t - 1.6678t^2 - 0.2870228t^3 - 0.0337364506t^4 - 2.43675566216 \times 10^{-3}t^5 - 2.96979291 \times 10^{-6}t^6 + 2.99448399 \times 10^{-5}t^7 + 5.229584121432554 \times 10^{-6}t^8 + 5.35791399 \times 10^{-7}t^9 + 2.830270564 \times 10^{-8}t^{10} - 1.701299185 \times 10^{-9}t^{11} - 6.43806345 \times 10^{-10}t^12 - 9.0619602856 \times 10^{-11}t^{13} - 7.767239656 \times 10^{-12}t^{14} - 2.28152532 \times 10^{-13}t^{15}$,

$I_{15}(t) = 10 + 5.48t + 1.47052t^2 + 0.25173032t^3 + 0.029205t^4 + 0.0020161993t^5 - 2.1224598359888 \times 10^{-5}t^6 - 2.972652977 \times 10^{-5}t^7 - 4.9620454 \times 10^{-6}t^8 - 4.96095036 \times 10^{-7}t^9 - 2.473082137 \times 10^{-8}t^{10} + 1.8631736525 \times 10^{-9}t^{11} + 6.32627306 \times 10^{-10}t^{12} + 8.7115821 \times 10^{-11}t^{13} + 7.31921543 \times 10^{-12}t^{14} + 1.930202976 \times 10^{-13}t^{15}$,

$R_{15}(t) = 70 + 0.72t + 0.19728t^2 + 0.03529248t^3 + 0.00453114576t^4 + 0.000420556389696t^5 + 2.4194391269568 \times 10^{-5}t^6 - 2.1831015456 \times 10^{-7}t^7 - 2.675387679 \times 10^{-7}t^8 - 3.969636283 \times 10^{-8}t^9 - 3.57188426 \times 10^{-9}t^{10} - 1.61874467179 \times 10^{-10}t^{11} + 1.1179 \times 10^{-11}t^{12} + 3.503782 \times 10^{-12}t^{13} + 4.4802422 \times 10^{-13}t^{14} + 3.5132234 \times 10^{-14}t^{15}$.

Moreover, to show the effect of the fractional derivative to the SIR model, the graphs of the RPS-approximate solutions of susceptible, infected and recovered population for different values of α such that $\alpha_i = 0.1i$, $i = 2, 3, \cdots, 10$ are established in Figures 4, 5 and 6 with step size $h = 0.2$ over the interval $[0,1]$. From these graphs, it is obvious that fractional order derivative provides a greater degree of freedom as compared to integer-order derivative. By taking non-integer values of fractional parameter, remarkable responses of the compartments of the proposed model are obtained. However, it is clear that the curves of various compartments $S(t)$, $I(t)$ and $R(t)$ of the fractional SIR

Table 1: Approximate solution of $S(t)$ using RK and RPS methods.

t_i	4RK $S(t_i)$, $\alpha=1$	RPS $S(t_i), \alpha=1$
0	620	620
0.1	619.3630315796735	619.3630315791873
0.2	618.6909370609654	618.6909370597241
0.3	617.9818692109469	617.9818692025717
0.4	617.2338939798807	617.2338939757545
0.5	616.4449876950995	616.4449876822653
0.6	615.6130341588159	615.6130341421871
0.7	614.7358219653072	614.7358219528321
0.8	613.811041871202	613.8110418536481
0.9	612.8362842538178	612.8362842258337
1.	611.809036780519	611.8090367600192
t_i	$Abs_S(t_i)$	$Rel_S(t_i)$
0	0	0
0.1	4.8612492×10^{-10}	7.8487881×10^{-13}
0.2	1.2413466×10^{-9}	2.0064082×10^{-12}
0.3	8.3751956×10^{-9}	1.3552494×10^{-11}
0.4	4.1261501×10^{-9}	6.6849052×10^{-12}
0.5	1.2834221×10^{-8}	2.0819734×10^{-11}
0.6	1.6628860×10^{-8}	2.7011871×10^{-11}
0.7	1.2475084×10^{-8}	2.0293407×10^{-11}
0.8	1.7553816×10^{-8}	2.8598078×10^{-11}
0.9	2.7984129×10^{-8}	4.5663303×10^{-11}
1.	2.0499783×10^{-8}	3.3506833×10^{-11}

Table 2: Approximate solution of $I(t)$ using RK and RPS methods.

t_i	4RK $I(t_i)$, $\alpha=1$	RPS $I(t_i), \alpha=1$
0	10	10
0.1	10.562959870557034	10.56295987098823
0.2	11.156882013379226	11.15688201447968
0.3	11.783384951388577	11.783384958664044
0.4	12.444162099479442	12.444162103140075
0.5	13.140984032348298	13.1409840435319
0.6	13.87570081089476	13.875700825391162
0.7	14.650244093149224	14.650244104180128
0.8	15.466629169899582	15.466629185349186
0.9	16.32695689112441	16.326956915534016
1.	17.23341537452655	17.233415392703552
t_i	$Abs_I(t_i)$	$Rel_I(t_i)$
0	0	0
0.1	4.3119641×10^{-10}	4.0821552×10^{-11}
0.2	1.10045306×10^{-9}	9.8634463×10^{-11}
0.3	7.27546734×10^{-9}	6.1743441×10^{-10}
0.4	3.66063269×10^{-9}	2.9416466×10^{-10}
0.5	1.11836016×10^{-8}	8.5104750×10^{-10}
0.6	1.44964023×10^{-8}	1.0447330×10^{-9}
0.7	1.10309042×10^{-8}	7.5295020×10^{-10}
0.8	1.54496043×10^{-8}	9.9889925×10^{-10}
0.9	2.44096050×10^{-8}	1.49504927×10^{-9}
1.	1.81770012×10^{-8}	1.05475327×10^{-9}

model approach those of classical SIR model as the fractional order approaches the integer order.

Table 3: Approximate solution of $R(t)$ using RK and RPS methods.

t_i	4RK $R(t_i)$, $\alpha=1$	RPS $R(t_i), \alpha=1$
0	70	70
0.1	70.07400854976943	70.07400854982431
0.2	70.15218092565529	70.1521809257962
0.3	70.23474583766449	70.23474583876428
0.4	70.32194392063991	70.32194392110532
0.5	70.41402827255223	70.41402827420283
0.6	70.51126503028927	70.51126503242176
0.7	70.6139339415436	70.61393394298769
0.8	70.72232895889846	70.72232896100263
0.9	70.83675885505782	70.83675885863217
1.	70.95754784495439	70.95754784727708
t_i	$Abs_R(t_i)$	$Rel_R(t_i)$
0	0	0
0.1	5.48823209×10^{-11}	7.83205101×10^{-13}
0.2	1.40914835×10^{-10}	2.00870213×10^{-12}
0.3	1.099792257×10^{-9}	1.56588060×10^{-11}
0.4	$4.654054919\times10^{-10}$	6.61821141×10^{-12}
0.5	1.650604986×10^{-9}	2.34414225×10^{-11}
0.6	2.132495069×10^{-9}	3.02433245×10^{-11}
0.7	1.444078634×10^{-9}	2.04503354×10^{-11}
0.8	2.104172836×10^{-9}	2.97525954×10^{-11}
0.9	3.574342600×10^{-9}	5.04588671×10^{-11}
1.	2.322693149×10^{-9}	3.27335600×10^{-11}

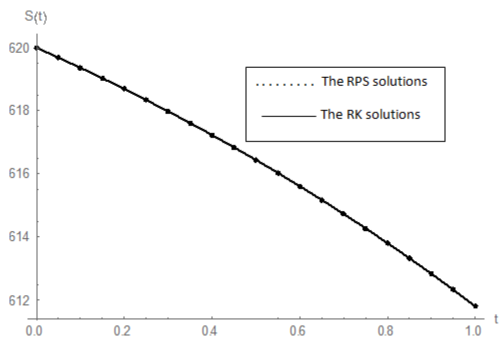

Fig. 1: Comparison between the RK and the RPS solutions of $S(t)$ for $\alpha=1$.

5 Conclusions

In this work, we have applied the RPS method to solve one of the important epidemic models; the SIR model of fractional order. We have observed that the proposed method yields accurate approximations by comparing its results by those of the fourth order Runge-Kutta method. It has the objective of approximating the solution of any non-linear differential equation by a rapid convergent series. To see the effect of the fractional derivative to the SIR model, we have solved it for different values of α and observed that the curves of the solutions of the fractional SIR approach those of classical SIR models.

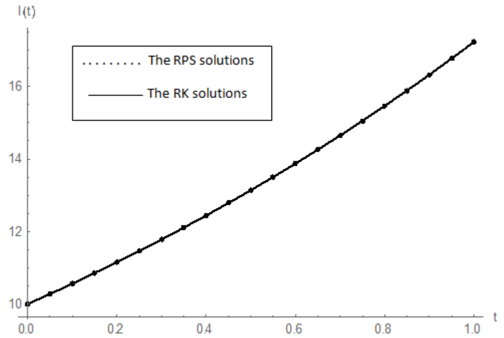

Fig. 2: Comparison between the RK and the RPS solutions of $I(t)$ for $\alpha = 1$.

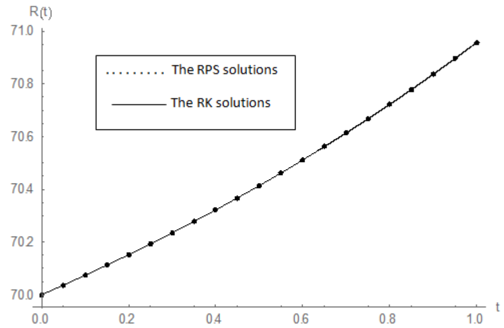

Fig. 3: Comparison between the RK and the RPS solutions of $R(t)$ for $\alpha = 1$.

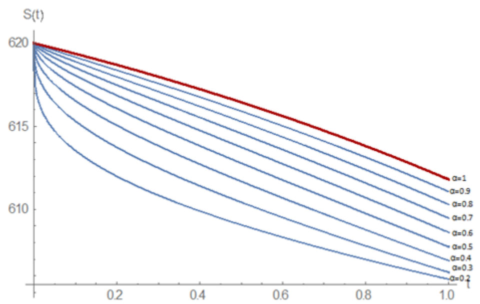

Fig. 4: The RPS solution of S(t) for different values of α.

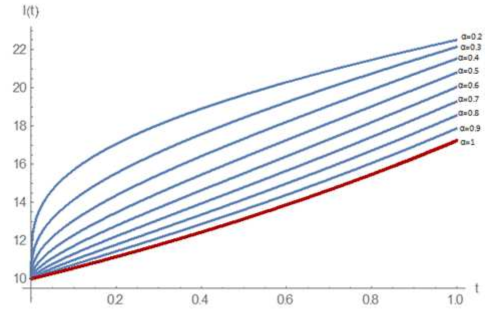

Fig. 5: The RPS solution of I(t) for different values of α.

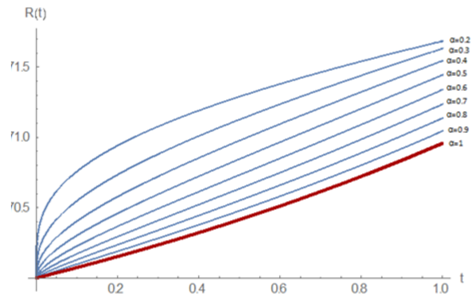

Fig. 6: The RPS solution of R(t) for different values of α.

References

[1] R.M. Anderson and R.M. May, Infectious Diseases of Humans: Dynamics and Control, Oxford University Press, Oxford, 1992.

[2] M.J.Keeling and P.Rohani, Modeling Infectious Diseases in Humans and Animals, Princeton University Press, Princeton, New Jersey, 2008.

[3] W.O. Kermack and A.G. McKendrick, A contribution to the mathematical theory of epidemics, Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences, Vol. 115, pp. 700–721, (1927).

[4] A. Yildirim and H. Koçak, An analytical approach to transmission dynamics of infectious diseases with waning immunity, J. Mech. Med. Biol., Vol. 11, No. 4, pp. 929-940, (2011).

[5] A. Yildirim and Y. Cherruault, Analytical approximate solution of a SIR epidemic model with constant vaccination strategy by homotopy perturbation method, Kybernetes, Vol. 38, No. 9, pp. 1566-1575, (2009).

[6] H. Khan, R.N. Mohapatra, K. Vajravelu and S.J. Liao, The explicit series solution of SIR and SIS epidemic models, Applied Mathematics and Computation, Vol. 215, No. 2, pp. 653-669, (2009).

[7] G. Shabbir, H. Khan and M.A. Sadiq, A note on exact solution of SIR and SIS epidemic models, arXiv:1012.5035, pp. 1-6, (2010).

[8] S. Abubakar, N.I. Akinwande, O.R. Jimoh, F.A. Oguntolu and O.D. Ogwumu, Approximate solution of SIR infectious disease model using homotopy perturbation kethod (HPM), The Pacific Journal of Science and Technology, Vol. 14, No. 2, pp. 163-169, (2013).

[9] M.Z. Ahmad, D. Alsarayreh, A. Alsarayreh and I. Qaralleh, Differential Transformation Method (DTM) for Solving SIS and SI Epidemic Model, Sains Malaysiana, Vol. 46, No. 10, pp. 2007–2017, (2017).

[10] S.M. O'Regan, T.C. Kelly, A. Korobeinikov, M.J. O'Callaghan and A.V. Pokrovskii, Lyapunov functions for SIR and SIRS epidemic models, Applied Mathematics Letters, Vol. 23, pp. 446–448, (2010).

[11] Z. Altawallbeh, M. Al-Smadi, I. Komashynska and A. Ateiwi, Numerical solutions of fractional systems of two-

point BVPs by using the iterative reproducing kernel algorithm, Ukrainian Mathematical Journal, Vol. 70, No. 5, pp. 687-701, (2018).
[12] M. Al-Smadi and O. Abu Arqub, Computational algorithm for solving fredholm time-fractional partial integrodifferential equations of dirichlet functions type with error estimates, Applied Mathematics and Computation, Vol. 342, pp. 280-294, (2019).
[13] O. Abu Arqub and M. Al-Smadi, Atangana-Baleanu fractional approach to the solutions of Bagley-Torvik and Painlevé equations in Hilbert space, Chaos Solitons & Fractals, Vol. 117, pp. 161-167, (2018).
[14] O. Abu Arqub, Z. Odibat and M. Al-Smadi, Numerical solutions of time-fractional partial integrodifferential equations of Robin functions types in Hilbert space with error bounds and error estimates, Nonlinear Dynamics, Vol. 94, No. 3, pp. 1819-1834, (2018).
[15] M. Al-Smadi, Simplified iterative reproducing kernel method for handling time-fractional BVPs with error estimation, Ain Shams Engineering Journal, Vol. 9, No. 4, pp. 2517-2525, (2018).
[16] O. Abu Arqub and M. Al-Smadi, Numerical algorithm for solving time-fractional partial integrodifferential equations subject to initial and Dirichlet boundary conditions, Numerical Methods for Partial Differential Equations, Vol. 34, No. 5, pp. 1577-1597, (2018).
[17] K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
[18] K. Moaddy, A. Freihat, M. Al-Smadi, E. Abuteen and I. Hashim, Numerical investigation for handling fractional-order Rabinovich–Fabrikant model using the multistep approach, Soft Computing, Vol. 22, No. 3, pp. 773-782, (2018).
[19] J. He, Some applications of nonlinear fractional differential equations and their approximations. Sci. Technol. Soc., Vol. 15, pp. 86–90, (1999).
[20] M. Al-Smadi, A. Freihat, H. Khalil, S. Momani and R.A. Khan, Numerical multistep approach for solving fractional partial differential equations, International Journal of Computational Methods, Vol. 14, Art. ID 1750029, pp. 1-15, (2017). https://doi.org/10.1142/S0219876217500293.
[21] I. Pudlubny, Fractional Differential Equations, San Diego, Academic Press 1999.
[22] G. Gumah, K. Moaddy, M. Al-Smadi and I. Hashim, Solutions to Uncertain Volterra Integral Equations by Fitted Reproducing Kernel Hilbert Space Method, Journal of Function Spaces, Vol. 2016, Art. ID 2920463, pp. 1-11, (2016). http://dx.doi.org/10.1155/2016/2920463
[23] M. Al-Smadi, A. Freihat, M. Abu Hammad, S. Momani and O. Abu Arqub, Analytical approximations of partial differential equations of fractional order with multistep approach, Journal of Computational and Theoretical Nanoscience, Vol. 13, No. 11, pp. 7793-7801, (2016).
[24] S. Momani, O. Abu Arqub, A. Freihat and M. Al-Smadi, Analytical approximations for Fokker-Planck equations of fractional order in multistep schemes, Applied and Computational Mathematics, Vol. 15, No. 3, pp. 319-330, (2016).
[25] O. Abu Arqub and A. El-Ajou, Solution of the fractional epidemic model by homotopy analysis method, Journal of King Saud University-Science, Vol. 25, pp. 73-81, (2013).
[26] M. Al-Smadi and G. Gumah, On the homotopy analysis method for fractional SEIR epidemic model, Research Journal of Applied Sciences, Engineering and Technology, Vol. 7, No. 18, pp. 3809-3820, (2014).
[27] A. Freihat and A. Handam, Solution of the SIR models of epidemics using MSGDTM, Applications and Applied Mathematics, Vol. 9, No. 2, pp. 622-636, (2014).
[28] O. Abu Arqub, Series solution of fuzzy differential equations under strongly generalized differentiability. J. Adv. Res. Appl. Math., Vol. 5, pp. 31–52, (2013).
[29] A. El-Ajou, O. Abu Arqub and M. Al-Smadi, A general form of the generalized Taylor's formula with some applications, Applied Mathematics and Computation, Vol. 256, pp. 851-859, (2015).
[30] K. Moaddy, M. Al-Smadi and I. Hashim, A Novel Representation of the exact solution for differential algebraic equations system using residual power-series method, Discrete Dynamics in Nature and Society, vol. 2015, Art. ID 205207, pp. 1-12, (2015). http://dx.doi.org/10.1155/2015/205207
[31] L. Wang and X. Chen, Approximate Analytical Solutions of Time Fractional Whitham–Broer–Kaup Equations by a Residual Power Series Method, Entropy, Vol. 17, pp. 6519-6533, (2015).
[32] M. Al-Smadi, O. Abu Arqub, N. Shawagfeh and S. Momani, Numerical investigations for systems of second-order periodic boundary value problems using reproducing kernel method, Applied Mathematics and Computation, Vol. 291, pp. 137-148, (2016).
[33] G.N. Gumah, M.F.M. Naser, M. Al-Smadi and S.K. Al-Omari, Application of reproducing kernel Hilbert space method for solving second-order fuzzy Volterra integro-differential equations, Advances in Difference Equations, Vol. 2018: 475 (2018). doi:10.1186/s13662-018-1937-8
[34] R. Abu-Gdairi, M. Al-Smadi and G. Gumah, An Expansion Iterative Technique for Handling Fractional Differential Equations Using Fractional Power Series Scheme, Journal of Mathematics and Statistics, Vol. 11, No. 2, pp. 29-38, (2015).
[35] E. Abuteen, A. Freihat, M. Al-Smadi, H. Khalil and R.A. Khan, Approximate series solution of nonlinear, fractional Klein-Gordon equations using fractional reduced differential transform method, Journal of Mathematics and Statistics Vol. 12, No. 1, pp. 23-33, (2016).
[36] S. Hasan, M. Al-Smadi, A. Freihet and S. Momani, Two computational approaches for solving a fractional obstacle system in Hilbert space, Advances in Difference Equations, Vol. 2019, 55, (2019). https://doi.org/10.1186/s13662-019-1996-5
[37] O. Abu Arqub, A. EI-Ajou, Z. Al Zhour and S. Momani, Multiple solutions of nonlinear boundary value problems of fractional order: A new analytic iterative technique, Entropy, Vol. 16, pp. 471-493, (2014).
[38] A. Kilbas, H. Srivastava and J. Trujillo, Theory and Applications of Fractional Differential Equations, (1st ed.). New York, Elsevier Science Inc. 2006.

Shatha Hasan graduated from the university of Jordan in 2006 with a B.S. in Mathematics; and an M.S. degree in Mathematics in 2009. In 2016, she received her Ph.D. from the University of Jordan in applied mathematics.She then began work at the university of Jordan in 2017 as an assistant professor of applied mathematics. Her research interests are in the areas of applied mathematics specially in differential equations of fractional order.

Ayat I. Al-Zoubi graduated from Yarmouk University in 2010 with a B.S in statistics and M.S degree in statistics in 2013. Then she worked at Yarmouk University as a part-time lecturer. After that she joined the staff of working at Al-Balqa Applied University in 2015 as a full-time lecturer and still up to this date. Her research interests focus on applied mathematics, especially statistics and differential equations.

Asad Freihet received his Ph.D. from the University of Jordan, Amman in 2013. He then began work as assistant professor of applied mathematics in Al-balqa applied university. His research interests include mathematical analytical methods, numerical analysis, dynamical systems, and fuzzy and fractional differential equations.

Mohammed Al-Smadi received his Ph.D. from the University of Jordan, Amman in 2011. He then began work as assistant professor of applied mathematics in Qassim University. Currently, Dr. Al-Smadi is working as associate professor of applied mathematics at Al-Balqa Applied University (since Sep. 2017). His research interests include fractional calculus, numerical analysis, applied mathematics, and fuzzy calculus.

Shaher Momani received his Ph.D from the university of Wales (UK) in 1991. He then began work at Mutah university in 1991 as assistant professor of applied mathematics and promoted to full Professor in 2006. He left Mutah university to the university of Jordan in 2009 until now. His research interests focus on the numerical solution of fractional differential equations in fluid mechanics, non-Newtonian fluid mechanics and numerical analysis. Prof. Momani has written over 250 research papers and warded several national and international prizes. Also, he was classified as one of the top ten scientists in the world in fractional differential equations according to ISI web of knowledge.