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A method based on Legendre Pseudo-Spectral Approximations for Solving Inverse Problems of Parabolic Types Equations

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Abstract: This paper reports a new Legendre Gauss-Lobatto collocation (SL-GL-C) method to solve numerically two inverse problems of parabolic partial differential types equations subject to initial-boundary conditions. This problem is reformulated by eliminating the unknown functions using some special assumptions based on Legendre Gauss-Lobatto quadrature rule. The SL-GL-C is utilized to solve non-classical parabolic initial-boundary value problems. Accordingly, the inverse problem is reduced into a system of ordinary differential equations (SODEs) and afterwards, such system can be solved numerically using implicit Runge-Kutta (IRK) method of order four. For demonstrating the robust, effectiveness and stable approximations of the present method, several test examples are presented.

Keywords: Inverse problems; Nonlinear parabolic partial differential equations; Systems of ordinary differential equations; Pseudo-spectral scheme; Gauss-Lobatto quadrature.

1 Introduction

One of the best methods, in term of the accuracy, for obtaining the numerical solution of various kinds of differential equations is spectral method (see, [1]-[5]). Because of all types of spectral methods are global, they very convenient for approximating linear and nonlinear partial differential equations. A significant advantage of the spectral methods over the finite-difference and finite-element methods is that the high accuracy of spectral techniques. From the overview of approximation to the underline problem, the spectral method has been divided to three primary classifications namely Galerkin [6,7], tau [8,9] and collocation [10,11] methods.

The collocation method has a wide range of application, due to its ease of use and adaptable in various problems, including linear and nonlinear differential equations [12]–[16], integral equations [17,18], integro-differential equations [19,20], fractional differential equations [21,22] and variational problems [1]. According to exponential rate of convergence obtained by collocation method, it is very useful in providing highly accurate solutions. On the other hand, the collocation method has became increasingly popular for solving partial differential equations during the last decades [23,24].

Parabolic partial differential equations describe a wide range of problems in various fields of science including heat diffusion [25], ocean acoustic propagation [26], population dynamics [27], dynamics of nuclear reactors [28], adsorption of pollutants in soil and the diffusion of neutrons. The parabolic partial differential problem is concerned with calculation of unknown solution while the initial and boundary conditions are given. Otherwise in inverse parabolic partial differential problem with over specified condition, the determination of unknown solution and unknown source term are required. The inverse problems have been widely used in modelling of physical [29,30] and engineering [31,32,33,34] problems. Most often, the analytical solution for inverse problem is difficult to obtain. Several numerical methods had been introduced to obtain the solutions of inverse problems, see for example [35,36,37,38,39,40].

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Our main motivation in this paper is to develop a spectral approximation of a class of inverse problems of parabolic partial differential types equations, using shifted Legendre collocation method in conjunction with Legendre Gauss-Lobatto quadrature and the implicit Runge Kutta method. We construct a spectral approximation in spatial discretization to solve the inverse problems of parabolic partial differential types equations. This scheme has the advantage of reducing the inverse problems of parabolic partial differential types equation into SODEs, which greatly simplifying the problem. We then use implicit Runge Kutta algorithm for solving this SODEs. Finally, we implement the algorithm to solve several reasonable examples in order to demonstrate the method is accurate and efficient compared with alternative methods.

The rest of this article is organized as follows. In the next section, we present some preliminaries and properties of Legendre polynomials. In Section 3, by using spectral collocation method, we construct and develop an algorithm for the solution of the inverse problems of parabolic partial differential types equations with Dirichlet conditions. In Section 4, some illustrative numerical experiments are given and some comparisons are made between our method and other methods. The paper ends with some conclusions and observations in Section 5.

2 Some properties of shifted Legendre polynomials

The well-known Legendre polynomials $P_i(x)$ are defined on the interval $(-1,1)$. Firstly, some properties about the standard Legendre polynomials have been recalled in this section. The Legendre polynomials $P_k(x)$ $(k=0,1\dots,)$ satisfy the following Rodrigues formula

$$P_k(x)=\frac{(-1)^k}{2^k k!}D^k((1-x^2)^k), \tag{2.1}$$

we recall also that $P_k(x)$ is a polynomial of degree k and therefore $P_k^{(q)}(x)$ (the qth derivative of $P_k(x)$) will given by

$$P_k^{(q)}(x)=\sum_{i=0(k+i=even)}^{k-q} C_q(k,i)P_i(x), \tag{2.2}$$

where

$$C_q(k,i)=\frac{2^{q-1}(2i+1)\Gamma(\frac{q+k-i}{2})\Gamma(\frac{q+k+i+1}{2})}{\Gamma(q)\Gamma(\frac{2-q+k-i}{2})\Gamma(\frac{3-q+k+i}{2})}.$$

The Legendre polynomials satisfy the following relations $P_0(x)=1,\quad P_1(x)=x,\quad P_{k+2}(x)=\frac{2k+3}{k+2}xP_{k+1}(x)-\frac{k+1}{k+2}P_k(x)$, and the orthogonality relation

$$(P_k(x),P_l(x))_w=\int_{-1}^{1}P_k(x)P_l(x)\,w(x)=h_k\delta_{lk}, \tag{2.3}$$

where $w(x)=1$, $h_k=\frac{2}{2k+1}$. The Legendre-Gauss-Lobatto quadrature has been used to evaluate the previous integrals accurately. For any $\phi\in S_{2N-1}[-1,1]$, we have that

$$\int_{-1}^{1}\phi(x)dx=\sum_{j=0}^{N}\varpi_{N,j}\phi(x_{N,j}). \tag{2.4}$$

We introduce the following discrete inner product

$$(u,v)_w=\sum_{j=0}^{N}u(x_{N,j})\,v(x_{N,j})\,\varpi_{N,j}. \tag{2.5}$$

For Legndre Gauss-Lobatto, we find that [1]

$$x_{N,0}=-1,\quad x_{N,N}=1,\quad x_{N,j}\,(j=1,\cdots,N-1)$$
are the zeros of $(P_N(x))'$,

$$\varpi_{N,j}=\frac{2}{N(N+1)(P_N(x_{N,j}))^2}, \tag{2.6}$$

where $x_{N,j}$ $(0\le j\le N)$ and $\varpi_{N,j}$ $(0\le j\le N)$ are used as usual the nodes and the corresponding Christoffel numbers in the interval $[-1,1]$, respectively. In order to use these polynomials on the interval $x\in(0,L_1)$ we defined the so-called shifted Legendre polynomials by introducing the change of variable $x=\dfrac{2x}{L_1}-1$. Let the shifted Legendre polynomials $P_i(\dfrac{2x}{L_1}-1)$ be denoted by $P_{L_1,i}(x)$. Then $P_{L_1,i}(x)$ can be obtained with the aid of the following recurrence formula:

$$(i+1)P_{L_1,i+1}(x)=(2i+1)(\tfrac{2x}{L_1}-1)P_{L_1,i}(x)-iP_{L_1,i-1}(x),\quad i=1,2,\cdots. \tag{2.7}$$

The analytic form of the shifted Legendre polynomials $P_{L_1,i}(x)$ of degree i is given by

$$P_{L_1,i}(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!\,(k!)^2\,L_1^k}\,x^k, \tag{2.8}$$

the orthogonality condition may be given by

$$\int_0^{L_1}P_{L_1,j}(x)P_{L_1,k}(x)w_{L_1}(x)dx=h_k\,\delta_{jk}, \tag{2.9}$$

where $w_{L_1}(x)=1$ and $\hbar_k=\dfrac{L_1}{2k+1}$.

A function $u(x)$, square integrable in $(0,L_1)$, may be expressed in terms of shifted Legendre polynomials as

$$u(x)=\sum_{j=0}^{\infty}c_jP_{L_1,j}(x),$$

where the coefficients c_j are given by

$$c_j=\frac{1}{\hbar_j}\int_0^{L_1}u(x)P_{L_1,j}(x)w_{L_1}(x)dx,\quad j=0,1,2,\cdots. \tag{2.10}$$

In practice, only the first $(N+1)$-terms shifted Legendre polynomials are considered. Hence $u(x)$ can be expressed in the form

$$u_N(x) \simeq \sum_{j=0}^{N} c_j P_{L_1,j}(x). \tag{2.11}$$

3 Shifted Legendre pseudo-spectral scheme

We propose a pseudo-spectral algorithm based on shifted Legendre pseudo-spectral method in conjunction Gauss-Lobatto quadrature rule to integrate the spatial variable for the $(1+1)$ inverse problems of parabolic partial differential types equations. The problem is then transformed into a SODEs in temporal discretization. The IRK scheme is employed to integrate the resulting SODEs.

3.1 Extra condition in an integral form

Consider the following $(1+1)$ parabolic PDEs with unknown source term of the form

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \lambda(t) z(x,t) + f(x,t), \qquad (x,t) \in \mathrm{I} \times \mathrm{T}, \tag{3.1}$$

with the boundary conditions

$$z(0,t) = c_1(t), \quad z(L_1,t) = c_2(t), \quad t \in \mathrm{T}, \tag{3.2}$$

and the initial state

$$z(x,0) = c_3(x), \quad x \in \mathrm{I}, \tag{3.3}$$

where $\mathrm{I} \equiv [0,L_1]$, $\mathrm{T} \equiv (0,L_2]$, $c_1(t)$, $c_2(t)$, $c_3(x)$ and $f(x,t)$ are given functions, while $z(x,t)$ $\lambda(t)$ are unknown functions. To determine the unknown source term function $\lambda(t)$, we introduce the extra condition

$$\int_0^{\eta(t)} \Theta(x) z(x,t) dx = c_4(t), \quad t \in \mathrm{T}. \tag{3.4}$$

Let us first use the following transformations

$$u(x,t) = \chi(t) z(x,t), \quad \chi(t) = e^{-\int_0^t \lambda(t) dt}, \tag{3.5}$$

enable one to rewrite (3.1)-(3.4) in the form

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \chi(t) f(x,t), \qquad (x,t) \in \mathrm{I} \times \mathrm{T}, \tag{3.6}$$

with the boundary conditions

$$u(0,t) = c_1(t)\chi(t), \quad u(L_1,t) = c_2(t)\chi(t), \quad t \in \mathrm{T}, \tag{3.7}$$

the initial state

$$u(x,0) = c_3(x), \quad x \in \mathrm{I}, \tag{3.8}$$

and the extra condition

$$\int_0^{\eta(t)} \Theta(x) u(x,t) dx = c_4(t)\chi(t), \quad t \in \mathrm{T}. \tag{3.9}$$

The solution of partial differential equation may be approximated by a finite expansion of orthogonal polynomial for some collocation points for the space variable. After computing partial derivatives, for the space variable, at these collocation points, we set these expansions in the differential equation to obtain SODEs in time discretization. The interested reader can see also, [41,42]. To this end, the polynomial approximation of degree N to $u(x,t)$ may be expressed in terms of the Legendre series $\{P_{L_1,i}(x)\}$ in the form

$$u(x,t) = \sum_{i=0}^{N} a_i(t) P_{L_1,i}(x). \tag{3.10}$$

It follows from the information included in the previous section, that

$$a_i(t) = \frac{1}{\hbar_i} \sum_{j=0}^{N} P_{L_1,i}(\zeta_{N,j}) \varpi_{N,j}^{L_1} u_j(t), \tag{3.11}$$

where

$$\zeta_{N,j} = \frac{L_1(x_{N,j}+1)}{2}, \quad \varpi_{N,j}^{L_1} = \frac{L_1}{2}\varpi_{N,j}, \quad u_j(t) = u(\zeta_{N,j},t).$$

We can further rewrite (3.10) as

$$u(x,t) = \sum_{j=0}^{N} \left(\sum_{i=0}^{N} \frac{1}{\hbar_i} P_{L_1,i}(\zeta_{N,j}) P_{L_1,i}(x) \varpi_{N,j}^{L_1} \right) u_j(t). \tag{3.12}$$

The first-order spatial partial derivative at a specific collocation node $\zeta_{N,n}$ can be obtained from (3.12) as

$$u_x(\zeta_{N,n},t) = \sum_{i=0}^{N} \mu_{n,i} u_i(t), \quad 0 \le n \le N, \tag{3.13}$$

where

$$\mu_{n,i} = \sum_{j=0}^{N} \frac{1}{\hbar_j} P_{L_1,j}(\zeta_{N,i}) \partial_x (P_{L_1,j}(\zeta_{N,n})) \varpi_{N,i}^{L_1}. \tag{3.14}$$

This result can be extended to compute the second-order spatial partial derivative at a specific collocation node $\zeta_{N,n}$ as

$$u_{xx}(\zeta_{N,n},t) = \sum_{i=0}^{N} \gamma_{n,i} u_i(,t), \quad 0 \le n \le N, \tag{3.15}$$

where

$$\gamma_{n,i} = \sum_{j=0}^{N} \frac{1}{\hbar_j} P_{L_1,j}(\zeta_{N,i}) \partial_{xx}(P_{L_1,j}(\zeta_{N,n})) \varpi_{N,i}^{L_1}. \tag{3.16}$$

Moreover, the collocation treatment of the extra condition (3.9) immediately, gives

$$\int_0^{\eta(t)} \Theta(x) \sum_{j=0}^{N} \Big(\sum_{i=0}^{N} \frac{1}{\hbar_i} P_{L_1,i}(\zeta_{N,j}) P_{L_1,i}(x) \varpi_{N,j}^{L_1} \Big) u_j(t) dx = c_4(t)\chi(t), \quad t \in \mathrm{I}, \tag{3.17}$$

yields

$$\chi(t) = \frac{\sum_{j=0}^{N} I_j(t) u_j(t)}{c_4(t)}, \quad I_j(t) = \sum_{i=0}^{N} \Big(\frac{\varpi_{N,j}^{L_1}}{\hbar_i} P_{L_1,i}(\zeta_{N,j}) \int_0^{\eta(t)} \Theta(x) P_{L_1,i}(x) dx \Big). \tag{3.18}$$

Consequently, the boundary conditions (3.7) can be reformulated as

$$u_0(t) = \frac{c_1(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t), \quad u_N(t) = \frac{c_2(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t), \quad t \in \mathrm{T}, \tag{3.19}$$

and due to (3.19), we find

$$\begin{aligned} u_0(t) &= \frac{(c_4(t) - c_2(t) I_N(t)) c_1(t) \sum_{r=1}^{n-1} I_r(t) u_r(t) - c_1(t) c_2(t) I_N(t) \sum_{r=1}^{n-1} I_r(t) u_r(t)}{(c_4(t) - c_1(t) I_0(t))(c_4(t) - c_2(t) I_N(t)) - c_1(t) c_2(t) I_0(t) I_N(t)}, \\ u_N(t) &= \frac{c_2(t) \Big((c_4(t) - c_2(t) I_N(t)) c_1(t) \sum_{r=1}^{n-1} I_r(t) u_r(t) - c_1(t) c_2(t) I_N(t) \sum_{r=1}^{n-1} I_r(t) u_r(t) \Big)}{c_1(t) \Big((c_4(t) - c_1(t) I_0(t))(c_4(t) - c_2(t) I_N(t)) - c_1(t) c_2(t) I_0(t) I_N(t) \Big)}. \end{aligned} \tag{3.20}$$

In the context of Legendre pseudo-spectral approximation, putting (3.13) and (3.15) in (3.6) gives

$$\dot{u}_n(t) = \sum_{i=0}^{N} \gamma_{n,i} u_i(t) + \gamma_{n,0} u_0(t) + \gamma_{n,N} u_N(t) + \frac{f_n(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t), \quad 1 \le n \le N-1, \tag{3.21}$$

where $u_0(t)$ and $u_N(t)$ may be given in (3.20). Furthermore, the preceding equation provides a system of $(N-1)$ ODEs in time, namely

$$\dot{u}_n(t) = \sum_{i=0}^{N} \gamma_{n,i} u_i(t) + \gamma_{n,0} u_0(t) + \gamma_{n,N} u_N(t) + \frac{f_n(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t), \tag{3.22}$$

subject to

$$u_n(t) = c_3(\zeta_{N,n}), \quad n = 1, \cdots, N-1, \tag{3.23}$$

Or in matrix notation as:

$$\begin{pmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \\ \cdots \\ \cdots \\ \cdots \\ \dot{u}_{N-2}(t) \\ \dot{u}_{N-1}(t) \end{pmatrix} = \begin{pmatrix} \sum_{i=0}^{N} \gamma_{1,i} u_i(t) + \gamma_{1,0} u_0(t) + \gamma_{1,N} u_N(t) + \frac{f_1(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t) \\ \sum_{i=0}^{N} \gamma_{2,i} u_i(t) + \gamma_{2,0} u_0(t) + \gamma_{2,N} u_N(t) + \frac{f_2(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t) \\ \cdots \\ \cdots \\ \cdots \\ \sum_{i=0}^{N} \gamma_{N-2,i} u_i(t) + \gamma_{N-2,0} u_0(t) + \gamma_{N-2,N} u_N(t) + \frac{f_{N-2}(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t) \\ \sum_{i=0}^{N} \gamma_{N-1,i} u_i(t) + \gamma_{N-1,0} u_0(t) + \gamma_{N-1,N} u_N(t) + \frac{f_{N-1}(t)}{c_4(t)} \sum_{j=0}^{N} I_j(t) u_j(t) \end{pmatrix}, \tag{3.24}$$

$$\begin{pmatrix} u_1(0) \\ u_2(0) \\ \cdots \\ \cdots \\ \cdots \\ u_{N-2}(0) \\ u_{N-1}(0) \end{pmatrix} = \begin{pmatrix} c_3(\zeta_{N,1}) \\ c_3(\zeta_{N,2}) \\ \cdots \\ \cdots \\ \cdots \\ c_3(\zeta_{N,N-2}) \\ c_3(\zeta_{N,N-1}) \end{pmatrix}. \tag{3.25}$$

3.2 Extra condition at x^*

Consider the following $(1+1)$ parabolic PDEs with unknown source term of the form

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \lambda(t) z(x,t) + f(x,t), \qquad (x,t) \in \mathrm{I} \times \mathrm{T}, \tag{3.26}$$

with the boundary conditions

$$z(0,t) = c_1(t), \quad z(L_1,t) = c_2(t), \quad t \in \mathrm{T}, \tag{3.27}$$

and the initial state

$$z(x,0) = c_3(x), \quad x \in \mathrm{I}, \tag{3.28}$$

where $\mathrm{I} \equiv [0, L_1]$, $\mathrm{T} \equiv (0, L_2]$, $c_1(t)$, $c_2(t)$, $c_3(y)$ and $f(x,t)$ are given functions, while $z(x,t)$ $\lambda(t)$ are unknown functions. To determine the unknown source term function $\lambda(t)$, we introduce the extra condition

$$z(x^*, t) = c_4(t), \quad t \in \mathrm{T}. \tag{3.29}$$

Considering the following transformations

$$u(x,t) = \chi(t) z(x,t), \quad \chi(t) = e^{-\int_0^t \lambda(t) dt}, \tag{3.30}$$

thus (3.26)-(3.30) can be transformed into

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \chi(t) f(x,t), \qquad (x,t) \in \mathrm{I} \times \mathrm{T}, \tag{3.31}$$

with the boundary conditions

$$u(0,t) = c_1(t)\chi(t), \quad u(L_1,t) = c_2(t)\chi(t), \quad t \in \mathrm{T}, \tag{3.32}$$

and the initial state

$$u(x,0) = c_3(x), \quad x \in \mathrm{I}, \tag{3.33}$$

the extra condition

$$u(x^*, t) = c_4(t)\chi(t), \quad t \in \mathrm{T}. \tag{3.34}$$

Here, the collocation treatment of the extra condition (3.34) immediately, gives

$$\sum_{j=0}^{N} \Big(\sum_{i=0}^{N} \frac{1}{\hbar_i} P_{L_1,i}(\zeta_{N,j}) P_{L_1,i}(x^*) \varpi_{N,j}^{L_1} \Big) u_j(t) = c_4(t)\chi(t), \quad t \in \mathrm{I}, \tag{3.35}$$

yields

$$\chi(t)=\frac{\sum_{j=0}^{N}\kappa_j u_j(t)}{c_4(t)},\quad \kappa_j=\sum_{i=0}^{N}\left(\frac{\varpi_{N,j}^{L_1}}{\hbar_i}P_{L_1,i}(\zeta_{N,j})P_{L_1,i}(x^*)\right), \tag{3.36}$$

consequently, the boundary conditions (3.32) can be reformulated as

$$u_0(t)=\frac{c_1(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t),\quad u_N(t)=\frac{c_2(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t),\quad t\in \mathrm{T}, \tag{3.37}$$

and due to (3.37), we find

$$\begin{aligned} u_0(t)=&\frac{(c_4(t)-c_2(t)\kappa_N)c_1(t)\sum_{r=1}^{n-1}\kappa_r u_r(t)-c_1(t)c_2(t)\kappa_N\sum_{r=1}^{n-1}\kappa_r u_r(t)}{(c_4(t)-c_1(t)\kappa_0)(c_4(t)-c_2(t)\kappa_N)-c_1(t)c_2(t)\kappa_0\kappa_N},\\ u_N(t)=&\frac{c_2(t)\Big((c_4(t)-c_2(t)\kappa_N)c_1(t)\sum_{r=1}^{n-1}\kappa_r u_r(t)-c_1(t)c_2(t)\kappa_N\sum_{r=1}^{n-1}\kappa_r u_r(t)\Big)}{c_1(t)\Big((c_4(t)-c_1(t)\kappa_0)(c_4(t)-c_2(t)\kappa_N)-c_1(t)c_2(t)\kappa_0\kappa_N\Big)}. \end{aligned} \tag{3.38}$$

Based on the information included in this subsection and the recent one, we obtain the following SODEs

$$\dot{u}_n(t)=\sum_{i=0}^{N}\gamma_{n,i}u_i(t)+\gamma_{n,0}u_0(t)+\gamma_{n,N}u_N(t)+\frac{f_n(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t),\quad 1\le n\le N-1, \tag{3.39}$$

where $u_0(t)$ and $u_N(t)$ may be given in (3.38). Furthermore, the preceding equation provides a system of $(N-1)$ ODEs in time, namely

$$\dot{u}_n(t)=\sum_{i=0}^{N}\gamma_{n,i}u_i(t)+\gamma_{n,0}u_0(t)+\gamma_{n,N}u_N(t)+\frac{f_n(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t), \tag{3.40}$$

subject to

$$u_n(t)=c_3(\zeta_{N,n}),\quad n=1,\cdots,N-1, \tag{3.41}$$

Or in matrix notation as:

$$\begin{pmatrix}\dot{u}_1(t)\\ \dot{u}_2(t)\\ \cdots\\ \cdots\\ \cdots\\ \dot{u}_{N-2}(t)\\ \dot{u}_{N-1}(t)\end{pmatrix}=\begin{pmatrix}\sum_{i=0}^{N}\gamma_{1,i}u_i(t)+\gamma_{1,0}u_0(t)+\gamma_{1,N}u_N(t)+\frac{f_1(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t)\\ \sum_{i=0}^{N}\gamma_{2,i}u_i(t)+\gamma_{2,0}u_0(t)+\gamma_{2,N}u_N(t)+\frac{f_2(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t)\\ \cdots\\ \cdots\\ \cdots\\ \sum_{i=0}^{N}\gamma_{N-2,i}u_i(t)+\gamma_{N-2,0}u_0(t)+\gamma_{N-2,N}u_N(t)+\frac{f_{N-2}(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j(t)u_j(t)\\ \sum_{i=0}^{N}\gamma_{N-1,i}u_i(t)+\gamma_{N-1,0}u_0(t)+\gamma_{N-1,N}u_N(t)+\frac{f_{N-1}(t)}{c_4(t)}\sum_{j=0}^{N}\kappa_j u_j(t)\end{pmatrix}, \tag{3.42}$$

$$\begin{pmatrix}u_1(0)\\ u_2(0)\\ \cdots\\ \cdots\\ \cdots\\ u_{N-2}(0)\\ u_{N-1}(0)\end{pmatrix}=\begin{pmatrix}c_3(\zeta_{N,1})\\ c_3(\zeta_{N,2})\\ \cdots\\ \cdots\\ \cdots\\ c_3(\zeta_{N,N-2})\\ c_3(\zeta_{N,N-1})\end{pmatrix}. \tag{3.43}$$

The SODEs (3.24) and (3.42) may be solved by IRK Scheme. The Runge-Kutta method can be expressed as one of powerful numerical integrations tools used for initial value SODEs of first order [43,44,45]. The IRK method represented a subclass of the well-known family of Runge-Kutta methods (see, [46]), and has many applications in the efficient numerical solution of system of initial and boundary value ordinary differential equations. These methods are suitable for stiff problems (where the global accuracy of the numerical solutions is determined by the stability rather than by the local error and implicit methods are more appropriate for it. In fact, we used the IRK to solve numerically the resulted SODEs. The interested reader is referred to [43,44,45], for more details about how to solve such systems and some special cases of them.

4 Applications and numerical results

In this section, we give some numerical results obtained by using the algorithms presented in the previous section. Comparisons of our results with those obtained by other methods reveal that our methods are very effective and convenient.

The difference between the measured value of approximate solution and its actual value (absolute error), given by

$$E(x,t)=|u(x,t)-u_N(x,t)|, \tag{4.1}$$

where x, t $u(x,t)$ and $u_N(x,t)$ are the space variable, time, exact and the numerical solution, respectively. Moreover, the maximum absolute errors (MAEs) is given by

$$L_\infty=\mathrm{Max}\{E(\boldsymbol{x},t): \text{over all domain}\}. \tag{4.2}$$

Also, we can define the norm infinity as

$$L_\infty^\lambda=\mathrm{Max}\{|\lambda(t)-\lambda_N(t)|: \text{over all time domain}\}. \tag{4.3}$$

Example 1*Our first example deals with the [47]*

$$\frac{\partial z(x,t)}{\partial t}=\frac{\partial^2 z(x,t)}{\partial x^2}+\lambda(t)z(x,t)+e^t\left(-t^2x+(\pi-t)(\pi+t)\cos(\pi x)\right),\ (x,t)\in[0,1]\times[0,1], \tag{4.4}$$

subject to the initial condition

$$u(x,0)=x+\cos(\pi x),\quad x\in[0,1], \tag{4.5}$$

the boundary conditions

$$u(0,t)=e^t,\quad u(1,t)=0,\quad t\in[0,1], \tag{4.6}$$

and the extra condition

$$\int_0^1(1+x^2)u(x,t)dx=e^t\left(\frac{3}{4}-\frac{2}{\pi^2}\right),\quad t\in[0,1], \tag{4.7}$$

Table 2: MAEs for problem (4.4).

	$N=4$	$N=8$	$N=12$	$N=20$
L_∞	1.49493×10^{-2}	2.12893×10^{-6}	4.58358×10^{-11}	3.9968×10^{-15}
L_∞^λ	8.81203×10^{-3}	2.5027×10^{-7}	5.88205×10^{-8}	9.53015×10^{-13}

Table 1: Comparison between the absolute errors of $u(x,t)$ for problem (4.4).

x	t	[47] at $N=50$	$N=12$	$N=20$
0.1	1	1.2889×10^{-7}	3.46869×10^{-11}	1.77636×10^{-15}
0.2		2.6637×10^{-7}	4.6593×10^{-11}	3.10862×10^{-15}
0.3		3.3278×10^{-7}	4.63749×10^{-11}	2.66454×10^{-15}
0.4		3.6539×10^{-7}	3.10243×10^{-11}	1.11022×10^{-15}
0.5		3.6196×10^{-7}	6.66134×10^{-16}	0
0.6		3.2816×10^{-7}	3.10241×10^{-11}	3.33067×10^{-16}
0.7		2.6980×10^{-7}	4.63748×10^{-11}	1.22125×10^{-15}
0.8		1.9168×10^{-7}	4.65951×10^{-11}	8.56953×10^{-16}
0.9		9.6142×10^{-8}	3.46844×10^{-11}	3.05311×10^{-16}

Table 3: Comparison between the absolute errors of $\lambda(t)$ for problem (4.4).

t	[47] at $N=50$	$N=8$	$N=20$
0.1	2.8274×10^{-6}	1.13693×10^{-10}	3.17524×10^{-13}
0.2	2.5925×10^{-6}	6.73483×10^{-12}	1.29452×10^{-13}
0.3	2.5241×10^{-6}	5.63261×10^{-12}	3.17524×10^{-14}
0.4	2.4637×10^{-6}	5.81424×10^{-12}	1.51656×10^{-13}
0.5	2.5298×10^{-6}	6.03961×10^{-12}	9.53015×10^{-13}
0.6	2.6315×10^{-6}	6.39955×10^{-12}	3.13083×10^{-14}
0.7	2.8398×10^{-6}	6.84208×10^{-12}	3.37064×10^{-13}
0.8	3.0592×10^{-6}	7.40097×10^{-12}	5.99965×10^{-13}
0.9	3.3091×10^{-6}	8.06932×10^{-12}	3.89688×10^{-13}
1.0	8.3675×10^{-6}	8.87801×10^{-12}	3.75255×10^{-13}

the exact solution and unknown term may be given by

$$u(x,t) = e^t(\cos(\pi x) + x), \quad \lambda(t) = 1 + t^2. \tag{4.8}$$

The solution of this problem is obtained by applying SL-GL-C method. In Table 1, we compare our numerical results obtained by the present method at two choice of N with those obtained by compact finite difference scheme [47]. From Table 1, we see that we can achieve a good approximation with the exact solution by using shifted Legendre polynomials and our method is more accurate than compact finite difference scheme [47]. Moreover, MAEs for four different choice of N are shown in Table 2. Finally we compare in Table 3, the absolute error of the unknown source term $\lambda(t)$ obtained by the present method at two choice of N with those obtained by compact finite difference scheme [47].

Fig. 1 shows the Graph of numerical solution $\tilde{u}(x,t)$ obtained at $N=20$. We draw the absolute error graph for the unknown solution $u(x,t)$ at $N=20$ for problem (4.4) in Fig. 2. Moreover, in Fig. 3 we see the agreement of the curves of numerical and exact value of unknown source term $\lambda(t)$ at $N=20$ for the first example.

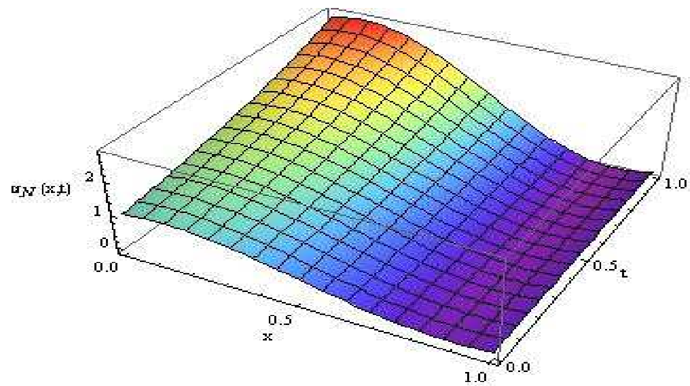

Fig. 1: Graph of numerical solution u_N for problem (4.4) at $N=20$.

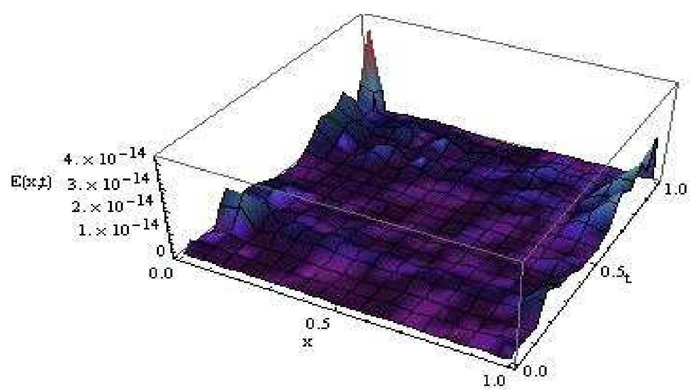

Fig. 2: Graph of absolute error functions at $N=20$ for problem (4.4).

Example 2 *We next consider the following initial-boundary value inverse problem [48]*

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \lambda(t)\,z(x,t) + e^t\left(2tx + \left(\pi^2+2t\right)\cos(\pi x)\right), \qquad (x,t)\in[0,1]\times[0,T], \tag{4.9}$$

subject to the initial condition

$$u(x,0) = x + \cos(\pi x), \quad x\in[0,1], \tag{4.10}$$

the boundary conditions

$$u(0,t) = e^t, \quad u(1,t) = 0, \quad t\in[0,T], \tag{4.11}$$

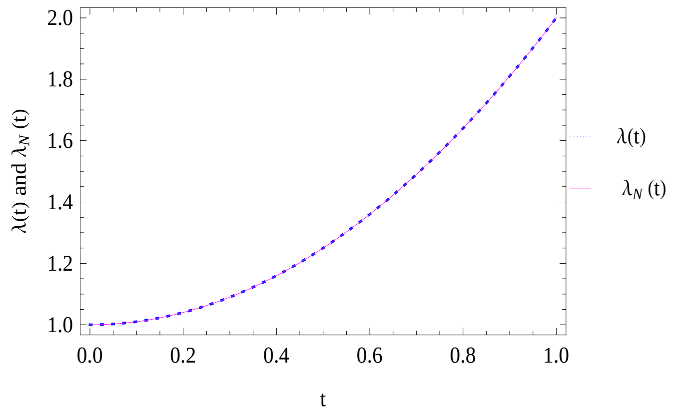

Fig. 3: Cruves of numerical and exact value of unknown source term $\lambda(t)$ at $N = 20$ for problem (4.4).

and the extra condition

$$\int_0^{0.5(1+\sqrt{t})} u(x,t)dx = e^t\left(\frac{(1+\sqrt{t})^2}{8} + \frac{\sin(0.5\pi(1+\sqrt{t}))}{\pi}\right), \quad t \in [0,T], \tag{4.12}$$

the exact solution and unknown term may be given by

$$u(x,t) = e^t(\cos(\pi x) + x), \quad \lambda(t) = 1 - 2t, \quad (x,t) \in [0,1]\times[0,T], \tag{4.13}$$

For several values of x, we introduce in Table 3 a comparison between our method and finite difference method [48]. Table 5 listed the MAE of numerical solution using SL-GL-C method. While in Fig. 4, the curves of exact and numerical solution of problem (4.9) at $t = 0.1,\ 0.5$ and 1.0 have been sketched. The t-direction curve of absolute error of problem (4.9), at $x = 0.5$ and $N = 20$ is plotted in Fig. 5. Fig. 6 displayed the behavior of the absolute errors curve in x-directions with specific value of t.

Example 3*In the last example, we consider the following [47]*

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \lambda(t)z(x,t) + e^{-t^2}\left(\pi^2 - (1+t)^2\right)(\cos(\pi x) + \sin(\pi x)), \ (x,t) \in [0,1]\times[0,1], \tag{4.14}$$

subject to the initial condition

$$u(x,0) = \cos(\pi x) + \sin(\pi x), \quad x \in [0,1], \tag{4.15}$$

the boundary conditions

$$u(0,t) = e^{-t^2}, \quad u(1,t) = -e^{-t^2}, \quad t \in [0,1], \tag{4.16}$$

and the extra condition

$$u(0.25,t) = \sqrt{2}e^{-t^2}, \quad t \in [0,1], \tag{4.17}$$

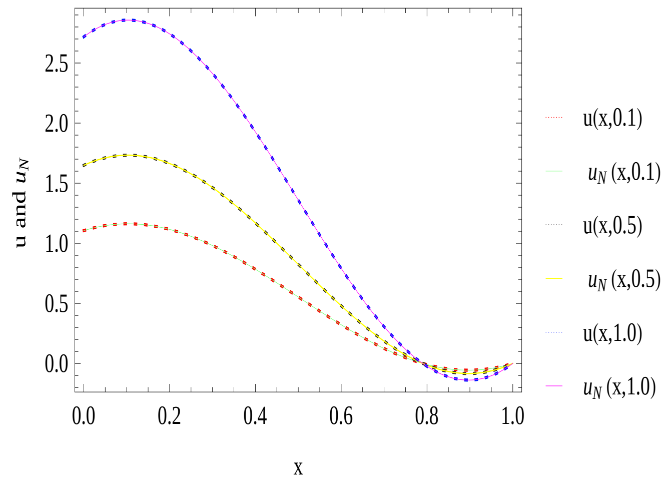

Fig. 4: The curves of exact and numerical solutions for problem (4.9) at $N = 20$.

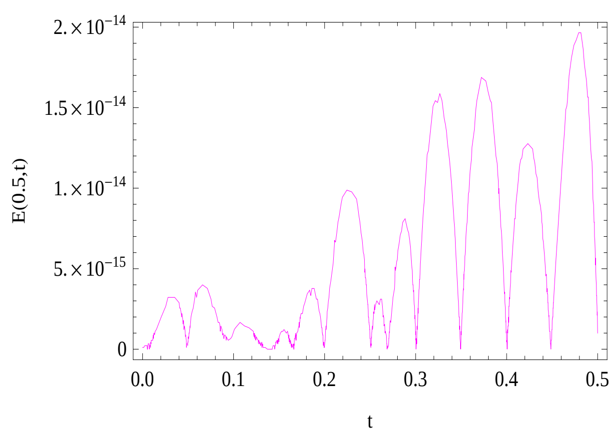

Fig. 5: t-direction absolute error curve at $x = 0.5$ and $N = 20$ for problem (4.9).

the exact solution and unknown term may be given by

$$u(x,t) = e^{-t^2}(\cos(\pi x) + \sin(\pi x)), \quad \lambda(t) = 1 + t^2, \ (x,t) \in [0,1]\times[0,1]. \tag{4.18}$$

Table 6 demonstrate that the results of MAEs of $\lambda(t)$ acquired by the proposed method are very accurate at $N = 8,\ 10,\ 12,$ and 20. While, a comparison is listed in Table 7 between the absolute error of $\lambda(t)$ for problem (4.14) which obtained in [47] and the results obtained in this paper.

We draw the numerical solution of problem (4.14), where $N = 20$ in Fig. 7. While, the x and t-direction curves of numerical $u_N(x,t)$ and exact $u(x,t)$ solutions of

Table 4: Comparison between the absolute errors of $u(x,t)$ for problem (4.9) at $T = 0.5$.

x	[48] at $N = 50$	$N = 10$	$N = 20$	x	[48] at $N = 50$	$N = 10$	$N = 20$
0.05	3.00×10^{-3}	4.06×10^{-9}	1.11×10^{-15}	0.55	2.50×10^{-3}	4.15×10^{-10}	1.55×10^{-15}
0.1	3.10×10^{-3}	1.56×10^{-9}	1.99×10^{-15}	0.6	2.40×10^{-3}	7.01×10^{-9}	1.44×10^{-15}
0.15	3.20×10^{-3}	6.72×10^{-9}	1.33×10^{-15}	0.65	2.50×10^{-3}	6.93×10^{-9}	1.33×10^{-15}
0.2	3.10×10^{-3}	3.22×10^{-9}	1.55×10^{-15}	0.7	2.60×10^{-3}	5.28×10^{-10}	1.55×10^{-15}
0.25	3.00×10^{-3}	5.34×10^{-9}	1.33×10^{-15}	0.75	2.70×10^{-3}	7.13×10^{-9}	1.27×10^{-15}
0.3	2.80×10^{-3}	7.01×10^{-9}	8.88×10^{-16}	0.8	2.70×10^{-3}	5.45×10^{-9}	1.41×10^{-15}
0.35	2.90×10^{-3}	$7.10 \times 10-10$	5.55×10^{-16}	0.85	2.80×10^{-3}	3.12×10^{-9}	5.15×10^{-16}
0.4	2.80×10^{-3}	7.04×10^{-9}	6.66×10^{-16}	0.9	2.90×10^{-3}	6.63×10^{-9}	9.99×10^{-16}
0.45	2.70×10^{-3}	7.12×10^{-9}	8.88×10^{-16}	0.95	2.90×10^{-3}	1.59×10^{-9}	7.77×10^{-16}
0.5	2.60×10^{-3}	4.10×10^{-9}	6.73×10^{-16}				

Table 5: MAEs for problem (4.9)at $T = 0.5$.

	$N = 4$	$N = 8$	$N = 10$	$N = 20$
L_∞	9.99014×10^{-3}	1.36809×10^{-6}	7.13045×10^{-9}	1.9984×10^{-15}

Table 6: MAEs for problem (4.14).

	$N = 8$	$N = 10$	$N = 12$	$N = 20$
L_∞^λ	1.07116×10^{-4}	4.28578×10^{-5}	3.37391×10^{-7}	1.1704×10^{-12}

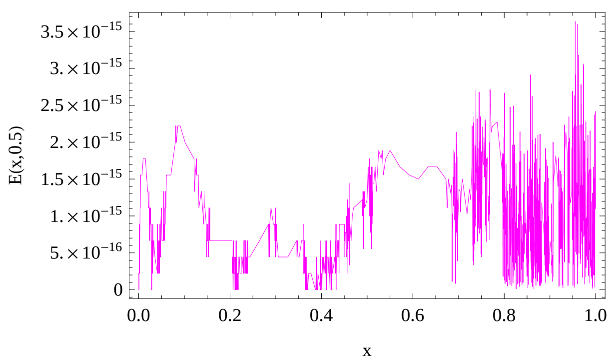

Fig. 6: x-direction absolute error curve at $t = 0.5$ and $N = 20$ for problem (4.9).

Table 7: Comparison between the absolute errors of $\lambda(t)$ for problem (4.14).

t	[47] at $N = 40$	$N = 12$	$N = 20$
0.1	4.1325×10^{-9}	3.71773×10^{-9}	9.28813×10^{-13}
0.2	4.1342×10^{-9}	3.82944×10^{-9}	4.91163×10^{-13}
0.3	4.1374×10^{-9}	3.85055×10^{-9}	5.11147×10^{-13}
0.4	4.1247×10^{-9}	3.8628×10^{-9}	7.24532×10^{-13}
0.5	4.1371×10^{-9}	3.87524×10^{-9}	5.4845×10^{-13}
0.6	4.1256×10^{-9}	3.88865×10^{-9}	4.28102×10^{-13}
0.7	4.1451×10^{-9}	3.90345×10^{-9}	1.07692×10^{-13}
0.8	4.1089×10^{-9}	3.91985×10^{-9}	3.98792×10^{-13}
0.9	4.1589×10^{-9}	3.93766×10^{-9}	1.17084×10^{-12}

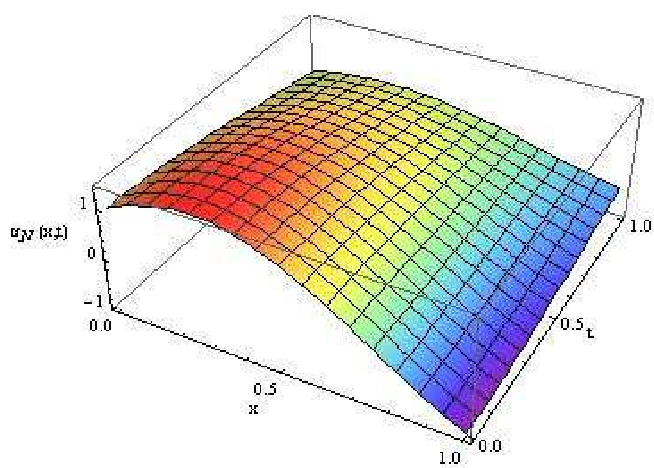

Fig. 7: Graph of numerical solution u_N for problem (4.14) at $N = 20$.

problem (4.14), where $N = 20$ are sketched in Figs. 8 and 9, respectively.

5 Conclusions

The shifted Legendre Gauss-Lobatto pseudo-spectral method was investigated successfully in spatial discretizations to get accurate approximate solutions of inverse problems of parabolic types equations. All of these problems were transformed to SODEs in time which greatly simplifying the problems. The IRK scheme is then applied to the resulting systems. From the numerical experiments, the obtained results were demonstrated the effectiveness and highly accuracy of

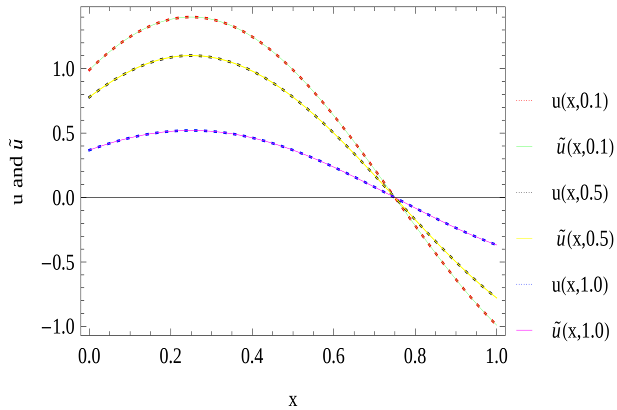

Fig. 8: x-direction curves of exact and numerical solutions for problem (4.9) at $N = 20$.

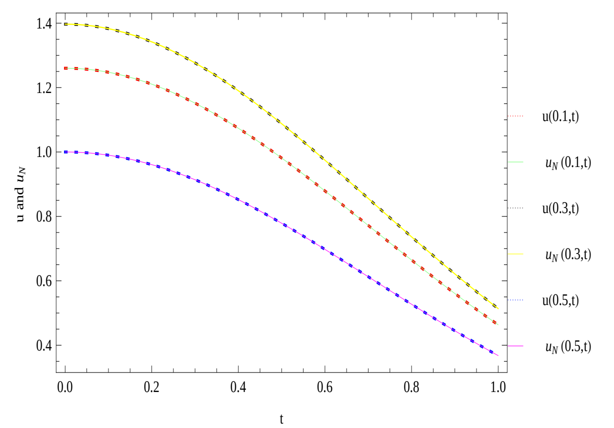

Fig. 9: t-direction curves of exact and numerical solutions for problem (4.9) at $N = 20$.

Legendre Gauss-Lobatto pseudo-spectral method for solving the mentioned problems.

The technique can be extended to more sophisticated problems. In principle, this method may be extended to related problems in mathematical physics. It is possible to use other orthogonal polynomials, say Chebyshev polynomials, or Jacobi polynomials to solve the mentioned problems in this article. Furthermore, the proposed spectral method might be developed by considering the Legendre pseudo-spectral approximation in both temporal and spatial discretizations. We should note that, as a numerical method, we are restricted to solving problems over a finite domain. Also, the pseudo-spectral approximation might be employed based on generalized Laguerre or modified generalized Laguerre polynomials to solve similar problems in a semi-infinite spatial intervals.

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