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Linear Matrix Inequality Approach to Robust $\mathscr{H}_\infty$ Fuzzy Speed Control Design for Brushless DC Motor System

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Abstract: This paper examines the problem of designing a robust fuzzy speed controller of Brushless DC (BLDC) motors which is described by a Takagi-Sugeno (TS) fuzzy model. Based on a linear matrix inequality (LMI) approach, LMI-based sufficient conditions for BLDC motor to have an $\mathscr{H}_\infty$ performance are derived. The proposed approach can overcome the nonlinearity and disturbances problems, while a conventional controller may suffer in controlling the motor speed due to some disturbance and nonlinearity of the BLDC motor. Finally, the effectiveness of the designed approach is demonstrated that the proposed methodology is provided a high performance robust control system for the BLDC motor through the simulation results.

Keywords: BLDC motor, Fuzzy controller, Robust $\mathscr{H}_\infty$ control, LMI approach

1 Introduction

The brushless direct current (BLDC) motors are normally used in tuning a process and also have been extensively used in many applications such as air condition, blower fan within the respirator. Currently, BLDC motors provide advantages in efficiency, compact, reliability and performance [1,2,3,4]. However, it is the fact that BLDC motor system involves a nonlinear model and complex process including the parameters tuning A trial and error is very time consuming which it may fail to optimize the performance. These also bring some difficulties to analysis and control. So far, there have been many researchers seeking highly effective methods for such systems in order to overcome the uncertain nonlinear control problems such that neural network control system has a strong ability to solve the structure uncertainty but it requires more computing capacity and data storage space. For genetic algorithms and ant-colony algorithms, this techniques can help in improving performance but they also need longer computation time and larger storage capacity [5,6,7,8].

Although PID control design is widely used approaches for brushless DC motor system, it is very difficult to obtain the proper results by using PID controller due the fact that in most situations, the systems deal with the uncertainty of dynamic behavior of the system and also nonlinear variation of parameters during operation. In addition, PID control design may be sufferred from larger overshoot, oscillation and slower response [9,10]. According to the above reasons, the conventional PID controller maybe not a good controller since the PID controller sometimes shows an unsatisfied performance in the applications of BLDC motor systems that containing nonlinear variables [11].

Recently, the nonlinear $\mathscr{H}_\infty$ control problems have been extensively studied by many researchers; see [12,13,14,15]. $\mathscr{H}_\infty$-control method can involve with multi-input and multi-output problems as well as disturbance and model error problems. The nonlinear $\mathscr{H}_\infty$-control problem can be stated as follows: given a dynamic system with the exogenous input noise and the measured output, find a controller such that the $\mathscr{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than or equal to a prescribed value. Presently, there are two commonly used approaches for providing solutions to the nonlinear $\mathscr{H}_\infty$ control problems. The first approach is based on the nonlinear version of classical Bounded Real Lemma; see [14,15,16,17]. The second approach is based on the dissipativity theory and the theory of differential games; see [12,17,18]. Both approaches show that the solution of the nonlinear $\mathscr{H}_\infty$ control problem is fact

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related to the solvability of Hamilton-Jacobi inequalities (HJIs). So far to our best knowledge, there is no easily computation technique to solve those inequalities.

Very recently, the problems of $\mathscr{H}_\infty$ control theories have been investigated; see [19,20,22,23,24,25,26] where the desired controllers have been designed in terms of the solution to linear matrix inequalities (LMI). LMI techniques are valuable alternative to classical analytical method and can be solved as efficient interior-point optimization [27,28]. The main advantage of LMI formulation is the ability to combine various design objectives in a numerically tractable manner.

Over the past two decades, there has been rapidly growing interest in applications of fuzzy logic to control problem. Researches have been focused on its application to industrial processes and a number of successful results have been reported in the literature. In spite of these successes, there are many basic issues remain to be addressed. One of them is how to achieve a systematic design that guarantees closed-loop stability and performance. Recently, a great amount of effort has been devoted to describing a nonlinear system using a Takagi-Sugeno fuzzy model; see [29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45]. Lately, the design of fuzzy $\mathscr{H}_\infty$ for a class of nonlinear systems which can be represented by a Takagi-Sugeno (TS) fuzzy model has been considered by many researchers; see [25,26,32,44]. TS fuzzy model has been attracted by a number of researchers in great attentions to both theoretical researches and application techniques which they can be seen in a number of excellent and important results on a plenty of papers; see [19,20,25,26,32,44].

Fuzzy system theory enables us to utilize qualitative, linguistic information about a highly complex nonlinear system to construct an mathematical model for it. In this TS fuzzy model, local dynamics in different state space regions are represented by local linear systems. The overall model of the system is obtained by 'blending' of these linear models through nonlinear membership functions. Unlike conventional modelling where a single model is used to describe the global behaviour of a system, the fuzzy modelling is essentially a multi-model approach in which simple sub-models (linear models) are combined to describe the global behaviour of the systems.

Therefore, in this paper based on LMI approach, a robust $\mathscr{H}_\infty$ fuzzy speed controller for brushless DC motor is presented. We develop a fuzzy controller of BLDC motor such that the $\mathscr{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value. This paper is organized as follows. Section 2 shows the list of used symbols. Section 3 explains the mathematical modeling of BLDC motor. In Section 4, TS fuzzy model is presented for nonlinear systems. In Section 5, based on a LMI approach we develop a technique for designing a fuzzy $\mathscr{H}_\infty$ speed controller of BLDC motor that guarantees the $\mathscr{L}_2$-gain of the mapping from the exogenous input noise to the regulated output is less than a prescribed value. The validity of this approach is finally demonstrated through simulation results in Section 6. Finally in Section 7, the conclusion is given.

2 List of the used symbols

$V_a(t), V_b(t), V_c(t)$	stator voltages, V
$i_a(t), i_b(t), i_c(t)$	stator winding current, A
$e_a(t), e_b(t), e_c(t)$	stator back-emf, V
L	stator self inductance, H
R	stator resistance/phase, Ω
$\omega_r(t)$	electrical rotor speed, rad/s
θ_r	rotor position, rad
$v_q(t), v_d(t)$	qd axis voltages, V
$i_q(t), i_d(t)$	qd axis current, A
$L_q(t), L_d(t)$	qd axis inductance, H
λ_m	mutual air gap flux linkages, V·s
$T_{(i_q,i_d)}(t)$	electromagnetic torque, N·m
P	number of poles
$T_L(t)$	load torque, N·m
J	load inertia, kg·m^2
B	viscous friction, N·m/(rad/s)

3 Modeling of brushless DC motor

The coupled circuit equations of the three-phase BLDC motor are [11]

$$\begin{bmatrix} V_a(t) \\ V_b(t) \\ V_c(t) \end{bmatrix} = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix} \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} + \begin{bmatrix} L & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & L \end{bmatrix} \frac{d}{dt} \begin{bmatrix} i_a(t) \\ i_b(t) \\ i_c(t) \end{bmatrix} + \begin{bmatrix} e_a(t) \\ e_b(t) \\ e_c(t) \end{bmatrix}. \tag{1}$$

In addition, the relationship between the back-emfs and the function of rotor speed are as follows [4]

$$\begin{bmatrix} e_a(t) \\ e_b(t) \\ e_c(t) \end{bmatrix} = \omega_r(t)\lambda_m \begin{bmatrix} \sin(\theta_r) \\ \sin(\theta_r - \frac{2\pi}{3}) \\ \sin(\theta_r + \frac{2\pi}{3}) \end{bmatrix}. \tag{2}$$

A simple three-phase brushless motor shows in Figure 1. The *abc* variable via the Park's transform are applied such that the rotor frame *qd* variable is obtained. A set of voltage is obtained as [3]

$$\begin{aligned} v_q(t) &= Ri_q(t) + \tfrac{d}{dt}\lambda_q(t) - \omega_r(t)\lambda_d(t) \\ v_d(t) &= Ri_d(t) + \tfrac{d}{dt}\lambda_d(t) - \omega_r(t)\lambda_q(t) \\ \lambda_q(t) &= L_q i_q(t) \\ \lambda_d(t) &= L_d i_d(t) + \lambda_m. \end{aligned} \tag{3}$$

The electromagnetic torque of the motor is [3]

$$T_{(i_q,i_d)}(t) = \frac{3}{2}\big(P(\lambda_m i_q(t) + (L_q - L_d) i_q(t) i_d(t))\big). \tag{4}$$

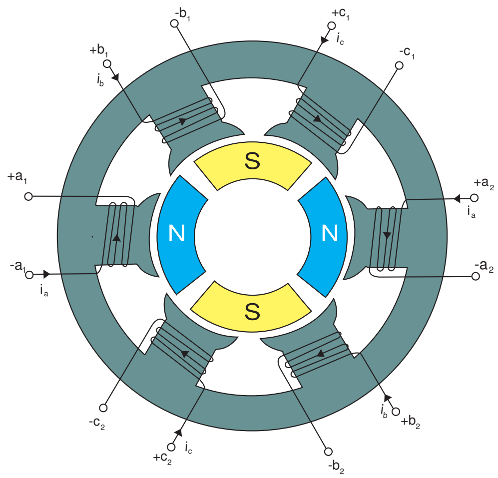

Fig. 1: Basic three-phase, 4 poles brushless motor.

Combining with equation of motion, one has the system equation in terms of the *qd* variables.

$$\begin{aligned} \dot{\omega}_r(t) &= -\tfrac{B}{J}\omega_r(t) + \tfrac{3}{2}\tfrac{P}{J}\lambda_m i_q(t) - \tfrac{P}{J}T_L(t) \\ \dot{i}_q(t) &= -\tfrac{\lambda_m}{L_q}\omega_r(t) - \tfrac{R}{L_q}i_q(t) - \tfrac{L_d}{L_q}\omega_r(t)i_d(t) + \tfrac{1}{L_q}v_q(t) \\ \dot{i}_d(t) &= \tfrac{L_q}{L_d}\omega_r(t)i_q(t) + \tfrac{R}{L_d}i_d(t) + \tfrac{1}{L_d}v_d(t) \end{aligned} \tag{5}$$

(5) can be rewritten as follows

$$\begin{bmatrix} \dot{\omega}_r(t) \\ \dot{i}_q(t) \\ \dot{i}_d(t) \end{bmatrix} = \begin{bmatrix} -\frac{B}{J} & \frac{3}{2}\frac{P}{J}\lambda_m & 0 \\ -\frac{\lambda_m}{L_q} & -\frac{R}{L_q} & -\frac{L_d}{L_q}\omega_r(t)i_d(t) \\ \frac{L_q}{L_d}\omega_r(t)i_q(t) & 0 & -\frac{R}{L_d} \end{bmatrix} \begin{bmatrix} \omega_r(t) \\ i_q(t) \\ i_d(t)) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{1}{L_q} & 0 \\ 0 & \frac{1}{L_d} \end{bmatrix} \begin{bmatrix} v_q(t) \\ v_d(t) \end{bmatrix} + \begin{bmatrix} -\frac{P}{J}T_L(t) \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

4 Nonlinear fuzzy model

First, we generalize the TS fuzzy system to represent a TS fuzzy system with parametric uncertainties. In this paper, we examine a TS fuzzy system with parametric uncertainties as follows:

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=1}^{r}\mu_i(\nu(t))\Big[[A_i+\Delta A_i]x(t) + B_w w(t) \\ &\quad +[B_i+\Delta B_i]u(t)\Big], \quad x(0)=0 \\ z(t) &= \textstyle\sum_{i=1}^{r}\mu_i(\nu(t))\Big[[C_i+\Delta C_i]x(t) + [D_i+\Delta D_i]u(t)\Big] \end{aligned} \tag{7}$$

where $\nu(t) = [\nu_1(t) \ \cdots \ \nu_\vartheta(t)]$ is the premise variable vector that may depend on states in many cases, $\mu_i(\nu(t))$ denotes the normalized time-varying fuzzy weighting functions for each rule (i.e., $\mu_i(\nu(t)) \geq 0$ and $\sum_{i=1}^{r}\mu_i(\nu(t)) = 1$), ϑ is the number of fuzzy sets, $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input, $w(t) \in \Re^p$ is the disturbance which belongs to $\mathscr{L}_2[0,\infty)$, $z(t) \in \Re^s$ is the controlled output, the matrices A_i, B_w, B_i, C_i, and D_i are of appropriate dimensions, and r is the number of IF-THEN rules. The matrices ΔA_i, ΔB_i, ΔC_i, and ΔD_i represent the uncertainties in the system and satisfy the following assumption.

Assumption 1

$$\Delta A_i = F(x(t),t)H_{1_i}, \ \ \Delta B_i = F(x(t),t)H_{2_i},$$

$$\Delta C_i = F(x(t),t)H_{3_i}, \ \ and \ \ \Delta D_i = F(x(t),t)H_{4_i}$$

where H_{j_i}, $j = 1,2,\cdots,4$ are known matrix functions which characterize the structure of the uncertainties. Furthermore, the following inequality holds:

$$\|F(x(t),t)\| \leq \rho \tag{8}$$

for any known positive constant ρ.

Next, let us recall the following definition.

Definition 1.*Suppose γ is a given positive number. A system (7) is said to have an $\mathscr{L}_2$-gain less than or equal to γ if*

$$\int_0^{T_f} z^T(t)z(t)dt \leq \gamma^2\left[\int_0^{T_f} w^T(t)w(t)dt\right], \ x(0)=0 \tag{9}$$

for all $T_f \geq 0$ and $w(t) \in \mathscr{L}_2[0,T_f]$.

Note that for the symmetric block matrices, we use $(*)$ as an ellipsis for terms that are induced by symmetry.

5 Main results

A robust $\mathscr{H}_\infty$ fuzzy state-feedback controller is readily established of the form

$$u(t) = \textstyle\sum_{j=1}^{r}\mu_j K_j x(t) \tag{10}$$

where K_j is the controller gain such that the inequality (9) holds. The state space form of the fuzzy system model (7) with the controller (10) is given by

$$\begin{aligned} \dot{x}(t) &= \textstyle\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big[[(A_i+B_iK_j) \\ &\quad +(\Delta A_i+\Delta B_iK_j)]x(t) + B_w w(t)\Big], \ x(0)=0. \end{aligned} \tag{11}$$

The following theorem provides sufficient conditions for the existence of a robust $\mathscr{H}_\infty$ fuzzy state-feedback controller. These sufficient conditions can be derived by the Lyapunov approach.

Theorem 1. *Consider the system (7). Given a prescribed $\mathscr{H}_\infty$ performance $\gamma > 0$ and a positive constant δ and α, if there exist a matrix $P = P^T$ and matrices Y_j, $j = 1,2,\cdots,r$, satisfying the following linear matrix inequalities:*

$$P > 0 \tag{12}$$

$$\Xi_{ii} < 0, \quad i = 1,2,\cdots,r \tag{13}$$

$$\Xi_{ij} + \Xi_{ji} < 0, \quad i < j \leq r \tag{14}$$

where

$$\Xi_{ij} = \begin{pmatrix} \Psi_{1_{ij}} & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T \\ \delta I & -\gamma I & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T \\ 0 & 0 & -\gamma I & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T \\ \delta I & 0 & 0 & -\gamma I & (*)^T & (*)^T & (*)^T & (*)^T & (*)^T \\ B_w & 0 & 0 & 0 & -\gamma I & (*)^T & (*)^T & (*)^T & (*)^T \\ \Psi_{2_{ij}} & 0 & 0 & 0 & 0 & -\gamma I & (*)^T & (*)^T & (*)^T \\ \Psi_{3_{ij}} & 0 & 0 & 0 & 0 & 0 & -\gamma I & (*)^T & (*)^T \\ \Psi_{4_{ij}} & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma I & (*)^T \\ \Psi_{5_{ij}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\gamma I \end{pmatrix} \tag{15}$$

with

$$\begin{aligned} \Psi_{1_{ij}} &= A_i P + P A_i^T + B_i Y_j + Y_j^T B_i^T \\ \Psi_{2_{ij}} &= \tfrac{\gamma\rho}{\delta} H_{1_i}^T P \\ \Psi_{3_{ij}} &= \tfrac{\gamma\rho}{\delta} H_{2_i}^T Y_j \\ \Psi_{4_{ij}} &= \sqrt{\alpha}\rho H_{3_i}^T P + \sqrt{\alpha}\rho H_{4_i}^T Y_j \\ \Psi_{5_{ij}} &= \sqrt{\alpha} C_i^T P + \sqrt{\alpha} D_i^T Y_j \end{aligned}$$

then the inequality (9) holds. Furthermore, a suitable choice of the fuzzy controller is

$$u(t) = \textstyle\sum_{j=1}^r \mu_j K_j x(t) \tag{16}$$

where

$$K_j = Y_j P^{-1}. \tag{17}$$

Proof: Using Assumption 1, the closed-loop fuzzy system (11) can be expressed as follows:

$$\dot{x}(t) = \textstyle\sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \Big([A_i + B_i K_j] x(t) + \tilde{B}_w \tilde{w}(t) \Big) \tag{18}$$

where

$$\tilde{B}_w = \begin{bmatrix} \delta I & 0 & \delta I & B_w \end{bmatrix},$$

and the disturbance $\tilde{w}(t)$ is

$$\tilde{w}(t) = \begin{bmatrix} \frac{1}{\delta} F(x(t),t) H_{1_i} x(t) \\ 0 \\ \frac{1}{\delta} F(x(t),t) H_{2_i} K_j x(t) \\ w(t) \end{bmatrix}. \tag{19}$$

Let consider a Lyapunov function

$$V(x(t)) = \gamma x^T(t) Q x(t)$$

where $Q = P^{-1}$. Differentiate $V(x(t))$ along the closed-loop system (18) yields

$$\begin{aligned} \dot{V}(x(t)) &= \gamma \dot{x}^T(t) Q x(t) + \gamma x^T(t) Q \dot{x}(t) \\ &= \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j \Big(\gamma x^T(t) (A_i + B_i K_j)^T Q x(t) \\ &\quad + \gamma x^T(t) Q (A_i + B_i K_j) x(t) \\ &\quad + \gamma \tilde{w}^T(t) \tilde{B}_w^T Q x(t) + \gamma x^T(t) Q \tilde{B}_w \tilde{w}(t) \Big). \end{aligned} \tag{20}$$

Adding and subtracting $-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2 \sum_{i=1}^r \sum_{j=1}^r \sum_{m=1}^r \sum_{n=1}^r \mu_i \mu_j \mu_m \mu_n [\tilde{w}^T(t) \tilde{w}(t)]$ to and from (20), we get

$$\begin{aligned} \dot{V}(x(t)) &= \gamma \sum_{i=1}^r \sum_{j=1}^r \sum_{m=1}^r \sum_{n=1}^r \mu_i \mu_j \mu_m \mu_n \Big(\begin{bmatrix} x^T(t) & \tilde{w}^T(t) \end{bmatrix} \times \\ &\quad \left(\begin{matrix} \begin{pmatrix} (A_i + B_i K_j)^T Q \\ + Q(A_i + B_i K_j) \\ + \frac{(\tilde{C}_i + \tilde{D}_i K_j)^T (\tilde{C}_m + \tilde{D}_m K_n)}{\gamma} \end{pmatrix} & (*)^T \\ \tilde{B}_w^T Q & -\gamma I \end{matrix} \right) \\ &\quad \times \begin{bmatrix} x(t) \\ \tilde{w}(t) \end{bmatrix} \Big) - \tilde{z}^T(t) \tilde{z}(t) \\ &\quad + \gamma^2 \sum_{i=1}^r \sum_{j=1}^r \sum_{m=1}^r \sum_{n=1}^r \mu_i \mu_j \mu_m \mu_n [\tilde{w}^T(t) \tilde{w}(t)] \end{aligned} \tag{21}$$

where

$$\tilde{z}(t) = \sum_{i=1}^r \sum_{j=1}^r \mu_i \mu_j [\tilde{C}_i + \tilde{D}_i K_j] x(t) \tag{22}$$

with

$$\tilde{C}_i = \begin{bmatrix} \frac{\gamma\rho}{\delta} H_{1_i}^T & 0 & \sqrt{\alpha}\rho H_{3_i}^T & \sqrt{\alpha} C_i^T \end{bmatrix}^T$$

and $\tilde{D}_i = \begin{bmatrix} 0 & \frac{\gamma\rho}{\delta} H_{2_i}^T & \sqrt{\alpha}\rho H_{4_i}^T & \sqrt{\alpha} D_i^T \end{bmatrix}^T$.

Note that (15) can be rewritten as

$$\begin{pmatrix} \begin{pmatrix} (A_i P + B_i Y_j)^T \\ + (A_i P + B_i Y_j) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_w^T & -\gamma I & (*)^T \\ \tilde{C}_i P + \tilde{D}_i Y_j & 0 & -\gamma I \end{pmatrix} < 0. \tag{23}$$

Thus, pre and post multiply (13)-(14) by $\begin{pmatrix} Q & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$ yields

$$\begin{pmatrix} \begin{pmatrix} (A_i + B_i K_i)^T Q \\ + Q(A_i + B_i K_i) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_w^T Q & -\gamma I & (*)^T \\ \tilde{C}_i + \tilde{D}_i K_i & 0 & -\gamma I \end{pmatrix} < 0, \tag{24}$$

$i = 1,2,\cdots,r$, and

$$\begin{aligned} &\left\{ \begin{pmatrix} \begin{pmatrix} (A_i + B_i K_j)^T Q \\ + Q(A_i + B_i K_j) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_w^T Q & -\gamma I & (*)^T \\ \tilde{C}_i + \tilde{D}_i K_j & 0 & -\gamma I \end{pmatrix} \right. \\ &\left. + \begin{pmatrix} \begin{pmatrix} (A_j + B_j K_i)^T Q \\ + Q(A_j + B_j K_i) \end{pmatrix} & (*)^T & (*)^T \\ \tilde{B}_w^T Q & -\gamma I & (*)^T \\ \tilde{C}_j + \tilde{D}_j K_i & 0 & -\gamma I \end{pmatrix} \right\} < 0, \end{aligned} \tag{25}$$

$i < j \le r$, respectively. Applying the Schur complement on (24)-(25) and rearranging them, then we have

$$\left(\begin{pmatrix} (A_i+B_iK_i)^TQ \\ +Q(A_i+B_iK_i) \\ +\frac{(\tilde{C}_i+\tilde{D}_iK_i)^T(\tilde{C}_i+\tilde{D}_iK_i)}{\gamma} \end{pmatrix} \quad (*)^T \atop \tilde{B}_w^TQ \qquad\qquad -\gamma I\right) < 0, \tag{26}$$

$i = 1,2,\cdots,r$, and

$$\left\{\left(\begin{pmatrix} (A_i+B_iK_j)^TQ \\ +Q(A_i+B_iK_j) \\ +\frac{(\tilde{C}_i+\tilde{D}_iK_j)^T(\tilde{C}_i+\tilde{D}_iK_j)}{\gamma} \end{pmatrix} \quad (*)^T \atop \tilde{B}_w^TQ \qquad\qquad -\gamma I\right) + \left(\begin{pmatrix} (A_j+B_jK_i)^TQ \\ +Q(A_j+B_jK_i) \\ +\frac{(\tilde{C}_j+\tilde{D}_jK_i)^T(\tilde{C}_j+\tilde{D}_jK_i)}{\gamma} \end{pmatrix} \quad (*)^T \atop \tilde{B}_w^TQ \qquad\qquad -\gamma I\right)\right\} < 0, \tag{27}$$

$i < j \le r$, respectively. Using (26)-(27) and the fact that

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n M_{ij}^T N_{mn} \le \frac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j[M_{ij}^TM_{ij}+N_{ij}N_{ij}^T], \tag{28}$$

it is obvious that we have

$$\left(\begin{pmatrix} (A_i+B_iK_j)^TQ \\ +Q(A_i+B_iK_j) \\ +\frac{(\tilde{C}_i+\tilde{D}_iK_j)^T(\tilde{C}_i+\tilde{D}_iK_j)}{\gamma} \end{pmatrix} \quad (*)^T \atop \tilde{B}_w^TQ \qquad\qquad -\gamma I\right) < 0 \tag{29}$$

where $i,j = 1,2,\cdots,r$. Since (29) is less than zero and the fact that $\mu_i \ge 0$ and $\sum_{i=1}^{r}\mu_i = 1$, then (21) becomes

$$\dot{V}(x(t)) \le -\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]. \tag{30}$$

Integrate both sides of (30) yields

$$\int_0^{T_f}\dot{V}(x(t))dt \le \int_0^{T_f}\Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big]dt$$

$$V(x(T_f)) - V(x(0)) \le \int_0^{T_f}\Big[-\tilde{z}^T(t)\tilde{z}(t) + \gamma^2\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]\Big]dt.$$

Using the fact that $x(0) = 0$ and $V(x(T_f)) \ge 0$ for all $T_f \ne 0$, we get

$$\int_0^{T_f}\tilde{z}^T(t)\tilde{z}(t)dt \le \gamma^2\left[\int_0^{T_f}\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{m=1}^{r}\sum_{n=1}^{r}\mu_i\mu_j\mu_m\mu_n[\tilde{w}^T(t)\tilde{w}(t)]dt\right]. \tag{31}$$

Putting $\tilde{z}(t)$ and $\tilde{w}(t)$ respectively given in (22) and (19) into (31) and using the fact that $\|F(x(t),t)\| \le \rho$, and (28), we have

$$\int_0^{T_f}\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(\alpha x^T(t)[C_i+D_iK_j]^T[C_i+D_iK_j]x(t) + \alpha\rho^2x^T(t)[H_{3_i}+H_{4_i}K_j]^T[H_{3_i}+H_{4_i}K_j]x(t)\Big)dt \le \gamma^2\left[\int_0^{T_f}w^T(t)w(t)\,dt\right]. \tag{32}$$

Adding and subtracting

$$z^T(t)z(t) = \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \times \Big(x^T(t)\,[C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]^T [C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]\,x(t)\Big)$$

to and from (32), one obtains

$$\int_0^{T_f}\Big\{z^T(t)z(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j \times \Big(\alpha x^T(t)[C_i+D_iK_j]^T[C_i+D_iK_j]x(t) + \alpha\rho^2x^T(t)[H_{3_i}+H_{4_i}K_j]^T[H_{3_i}+H_{4_i}K_j]x(t) - x^T(t)[C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]^T [C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]x(t)\Big)\Big\}dt \le \gamma^2\left[\int_0^{T_f}w^T(t)w(t)\,dt\right]. \tag{33}$$

Using the triangular inequality and the fact that $\|F(x(t),t)\| \le \rho$, we have

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(x^T(t) \times [C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]^T [C_i+F(x(t),t)H_{3_i}+D_iK_j+F(x(t),t)H_{4_i}K_j]\,x(t)\Big)$$
$$\le \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j\Big(\Big\{\alpha x^T(t)\,[C_i+D_iK_j]^T\,[C_i+D_iK_j]\,x(t)\Big\} + \alpha\rho^2x^T(t)\,[H_{3_i}+H_{4_i}K_j]^T\,[H_{3_i}+H_{4_i}K_j]\,x(t)\Big). \tag{34}$$

Using (34) on (33), we obtain

$$\int_0^{T_f}z^T(t)z(t)\,dt \le \gamma^2\int_0^{T_f}w^T(t)w(t)\,dt. \tag{35}$$

Hence, the inequality (9) holds. ■

6 Simulation results

The system from (5) can be described by the following state equations:

$$\begin{aligned}
\dot{x}_1(t) &= -\tfrac{B}{J}x_1(t) + \tfrac{3}{2}\tfrac{P}{J}\lambda_m x_2(t) - \tfrac{P}{J}w_3(t)\\
\dot{x}_2(t) &= -\tfrac{\lambda_m}{L_q}x_1(t) - \tfrac{R}{L_q}x_2(t) - \tfrac{L_d}{L_q}x_1(t)x_3(t) + 0.1w_2(t))\\
\dot{x}_3(t) &= \tfrac{L_q}{L_d}x_1(t)x_2(t) + \tfrac{R}{L_d}x_3(t) + 0.1w_1(t)\\
z(t) &= x_1(t)
\end{aligned} \tag{36}$$

where $x_1(t) = \omega_r(t)$, $x_2(t) = i_q(t)$, $x_3(t) = i_d(t)$, $w_1(t)$ and $w_2(t)$ are the process noise, $w_3(t)$ is the disturbance factor from torque load and $z(t)$ is the controlled output. It is found that currents and the speed in dynamic model of BLDC motor from (5) and (36) are highly nonlinear. Simultaneously, it deals with the load torque change. Thus, the nonlinearity and various uncertainties including external disturbances have to be taken into account [21]. The nonlinear system plant can be approximated by TS fuzzy rules. Let us choose the membership functions of the fuzzy sets as Figure 2. The membership function can be write as

$$M_1(x_1(t)) = \frac{-x_1(t)+N_2}{N_2-N_1} \text{ and } M_2(x_1(t)) = \frac{x_1(t)-N_1}{N_2-N_1}.$$

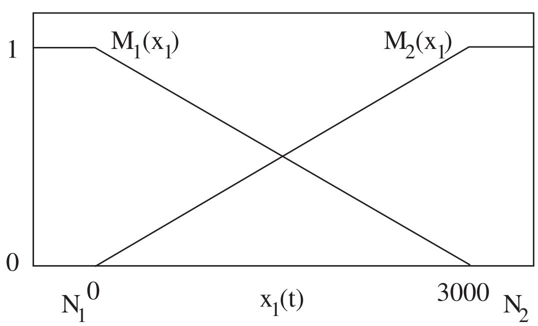

Fig. 2: Membership function for the two fuzzy set..

Knowing that $x_1(t) \in [N_1\ N_2]$, the nonlinear system (36) can be approximated by the following two rules TS model:

Plant Rule 1: IF $x_1(t)$ is $M_1(x_1(t))$ THEN

$$\dot{x}(t) = [A_1 + \Delta A_1]x(t) + B_w w(t) + B_1 u(t),\ x(0) = 0,$$
$$z(t) = C_1 x(t).$$

Plant Rule 2: IF $x_1(t)$ is $M_2(x_1(t))$ THEN

$$\dot{x}(t) = [A_2 + \Delta A_2]x(t) + B_w w(t) + B_2 u(t),\ x(0) = 0,$$
$$z(t) = C_2 x(t),$$

where

$$A_1 = \begin{bmatrix} -\frac{B}{J} & \frac{3}{2}\frac{P}{J}\lambda_m & 0\\ -\frac{\lambda_m}{L_q} & -\frac{R}{L_q} & -\frac{L_d}{L_q}N_1\\ 0 & \frac{L_q}{L_d}N_1 & -\frac{R}{L_d}\end{bmatrix},$$

$$A_2 = \begin{bmatrix} -\frac{B}{J} & \frac{3}{2}\frac{P}{J}\lambda_m & 0\\ -\frac{\lambda_m}{L_q} & -\frac{R}{L_q} & -\frac{L_d}{L_q}N_2\\ 0 & \frac{L_q}{L_d}N_2 & -\frac{R}{L_d}\end{bmatrix},$$

$$B_1 = B_2 = \begin{bmatrix} 0 & 0\\ \frac{1}{L_q} & 0\\ 0 & \frac{1}{L_d}\end{bmatrix},\quad B_w = \begin{bmatrix} 0 & 0 & -\frac{P}{J}\\ 0 & 0.1 & 0\\ 0.1 & 0 & 0\end{bmatrix},$$

$$C_1 = C_2 \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$\Delta A_1 = F(x(t),t)H_{1_1},\ \Delta A_2 = F(x(t),t)H_{1_2},$$

$$x(t) = [x_1^T(t)\ x_2^T(t)\ x_3^T(t)]^T$$

$$\text{and } w(t) = [w_1^T(t)\ w_2^T(t)\ w_3^T(t)]^T.$$

Let us choose the value of $[N_1\ \ N_2]$ in the membership function as $[0\ \ 3,000]$. Now, by assuming that in (8), $\|F(x(t),t)\| \le \rho = 1$ and the values of R are uncertain but bounded within 10% of their nominal values given in (36), we have

$$H_{1_1} = H_{1_2} = \begin{bmatrix} 0 & 0 & 0\\ 0 & -\frac{0.1R}{L_q} & 0\\ 0 & 0 & -\frac{0.1R}{L_d}\end{bmatrix}.$$

From the parameters in Table 1,

Table 1: Motor Parameters [21].

Symbol	Quantity	Units
P	No. of pole pair	2
$L = L_q = L_d$	Stator inductance	0.0018 H
R	Stator resistance	0.5 Ω
J	Motor inertia	0.003 kg$\cdot m^2$
B	Friction coefficient	0.0008 N$\cdot$m/rad/sec
λ_m	Magnetic flux	0.311 volts/rad/sec

the LMI optimization algorithm and Theorem 1 with assuming $\gamma = 1$, $\delta = 1$ and $\alpha = 2$, we obtain the results as follows:

$$P = \begin{bmatrix} 0.2987 & 0.0041 & 0\\ 0.0041 & 1.4933 & 0\\ 0 & 0 & 1.4933\end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.2019 & 0.0029 & -0.0157\\ 0 & 0.0190 & 0.0054\end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.0180 & 12.2413 & 0.0053 \\ 0.2019 & 0.0029 & 0.8601 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.9577 & 0.0001 & -0.0149 \\ -0.0002 & 0.0180 & 0.0054 \end{bmatrix},$$

$$K_2 = \begin{bmatrix} -0.0722 & 11.6125 & 0.0050 \\ 0.9577 & 0.0001 & 0.8159 \end{bmatrix}.$$

The resulting fuzzy controller is

$$u(t) = \Sigma_{j=1}^{2} \mu_j K_j x(t)$$

where

$$\mu_1 = M_1(x_1(t)) \text{ and } \mu_2 = M_2(x_1(t)).$$

Remark 1: To verify the validity of the proposed controller, the $\mathscr{H}_\infty$ fuzzy controller based on LMI is compared with the conventional PID controller. Figure 3 shows the result of the rotate speed using $\mathscr{H}_\infty$ fuzzy controller with no load while Figure 4 and Figure 5 show the speed curve when the motor is with load by setting the target speed at 2400 rpm. In Figure 4, the speed curve of the proposed $\mathscr{H}_\infty$ fuzzy controller is given. Figure 5 is the result of the PID controller using Ziegler Nichols method [46], the tuning parameters are determined as $K_p = 0.43$, $K_i = 2.30$ and $K_d = 0.28$. The disturbance input signals $w_1(t)$, $w_2(t)$ and $w_3(t)$ which were used during the simulation are given in Figure 6. After 0.2 second, the ratio of the regulated output energy to the disturbance noise energy to a constant value which is about 0.096 as shown in Figure 7. Thus $\gamma = \sqrt{0.0825} = 0.2864$ is less than the possible value. The proposed $\mathscr{H}_\infty$ fuzzy controller provided the faster response speed and completely eliminated the overshoot. Finally, Table 2 shows the performance result between the proposed $\mathscr{H}_\infty$ fuzzy controller and PID controller. However, in the simulated example, the fuzzy model is an approximated model by the defined membership functions as shown in Figure 2, thus the rotate speed may not be in agreement with the reference value (2400rpm). However, it can be solved this situations by examining the new membership function as so to exactly represent the nonlinear dynamic BLDC motor system. □

Table 2: Performance Analysis of Fuzzy and PID Controller.

Controller	Setting time (sec)	% Overshoot
$\mathscr{H}_\infty$ Fuzzy	0.04	-
PID	0.22	4.16

7 Conclusions

The performance analysis of a BLDC motor drive system with two types of speed controllers namely PID and $\mathscr{H}_\infty$ fuzzy controller based on LMI approach is presented. The BLDC motor is subjected to uncertain nonlinearities, transient and steady-state behaviour of the system. By comparison with both controllers, it is observed that $\mathscr{H}_\infty$ fuzzy controller based on LMI approach gives much better dynamic response for the system. It is found that the system responds faster and no overshoot, including a stability criterion in terms of Lyapunov method can guarantee the stability of the nonlinear fuzzy system. As a result, a better dynamic performance of the system is obtained. Simulation results are also shown that the

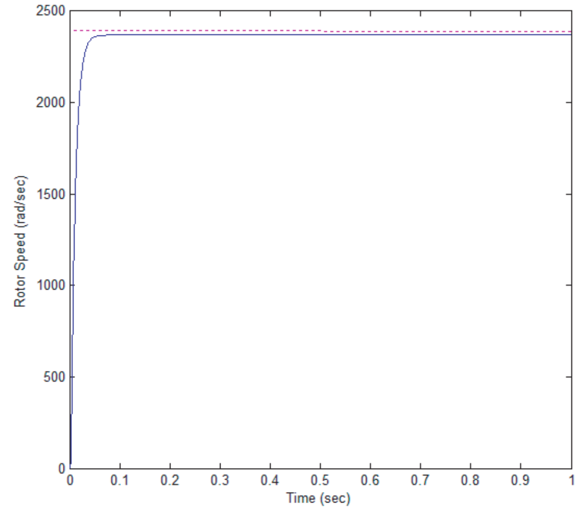

Fig. 3: Rotate speed simulation using $\mathscr{H}_\infty$ fuzzy controller based on LMI with no load.

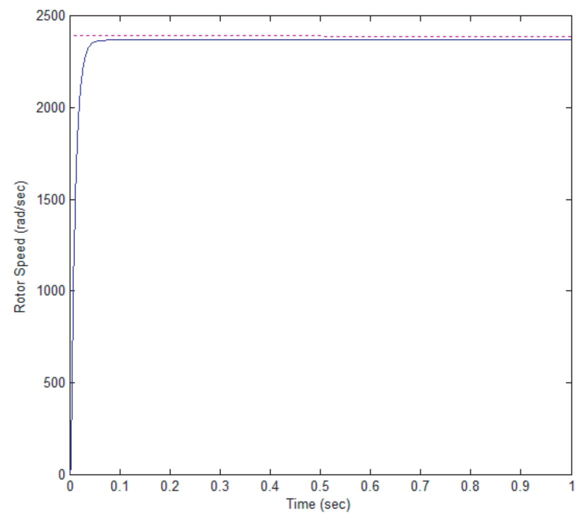

Fig. 4: Rotate speed simulation using $\mathscr{H}_\infty$ fuzzy controller based on LMI with load change.

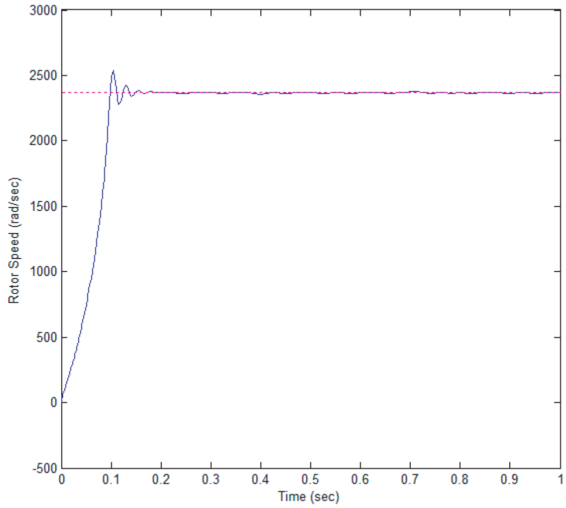

Fig. 5: Rotate speed simulation using PID controller with load change.

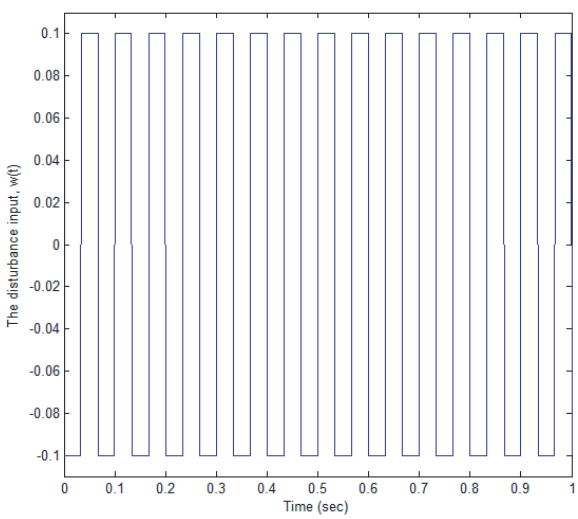

Fig. 6: The disturbance input signals, $w_1(t)$, $w_2(t)$ and $w_3(t)$.

proposed methodology is provided a high performance robust control system for the BLDC motor.

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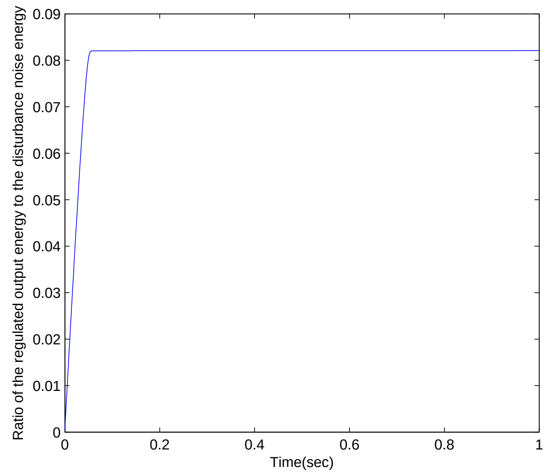

Fig. 7: The ratio of the regulated output energy to the disturbance noise energy: $\left(\frac{\int_0^{T_f} z^T(t)z(t)dt}{\int_0^{T_f} w^T(t)w(t)dt}\right)$

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