Applied Mathematics & Information Sciences
An International Journal

Robust stability and stabilization of nonlinear uncertain stochastic switched discrete-time systems with interval time-varying delays

G. Rajchakit

Major of Mathematics, Faculty of Science, Maejo University, Chiangmai 50290, Thailand

Received: Dec. 12, 2011; Revised Feb. 13, 2012; Accepted March 6, 2012
Published online: 1 Sep. 2012

Abstract: This paper is concerned with robust stability and stabilization of nonlinear uncertain stochastic switched discrete time-delay systems. The system to be considered is subject to interval time-varying delays, which allows the delay to be a fast time-varying function and the lower bound is not restricted to zero. Based on the discrete Lyapunov functional, a switching rule for the robust stability and stabilization for the nonlinear uncertain stochastic discrete time-delay system is designed via linear matrix inequalities. Numerical examples are included to illustrate the effectiveness of the results.

Keywords: Switching design, nonlinear uncertain stochastic discrete system, robust stability and stabilization, Lyapunov function, linear matrix inequality.

1. Introduction

As an important class of hybrid systems, switched systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models, such as manufacturing, communication networks, automotive engineering control and chemical processes (see, e.g., [1–7] and the references therein). On the other hand, time-delay phenomena are very common in practical systems. A switched system with time-delay individual subsystems is called a switched time-delay system; in particular, when the subsystems are linear, it is then called a switched time-delay linear system.

During the last decades, the stability analysis of switched linear continuous/discrete time-delay systems has attracted a lot of attention [8–11]. The main approach for stability analysis relies on the use of Lyapunov-Krasovskii functionals and linear matrix inequlity (LMI) approach for constructing a common Lyapunov function [12–14]. Although many important results have been obtained for switched linear continuous-time systems, there are few results concerning the stability of switched linear discrete systems with time-varying delays. It was shown in [9, 11, 15, 17–25] that when all subsystems are asymptotically stable, the switching system is asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems has been studied in [14], but the result was limited to constant delays. In [15], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the average dwell time scheme.

This paper studies mean square robust stability and stabilization problem for nonlinear uncertain stochastic switched linear discrete-time delay with interval time-varying delays. Specifically, our goal is to develop a constructive way to design switching rule to mean square robustly stable and stabilize the nonlinear uncertain stochastic discrete-time delay systems. By using improved Lyapunov-Krasovskii functionals combined with LMIs technique, we propose new criteria for the mean square robust stability and stabilization of the nonlinear uncertain stochastic discrete-time delay system. Compared to the existing results, our result has its own advantages. First, the time delay is assumed to be a time-varying function belonging to a given interval, which means that the lower and upper bounds for the time-varying delay are available, the delay function is bounded but not restricted to zero. Second, the approach allows us to design the switching rule for mean square robust stabil-

* Corresponding author: e-mail: griengkrai@yahoo.com

ity and stabilization in terms of LMIs, which can be solvable by utilizing Matlab's LMI Control Toolbox available in the literature to date.

The paper is organized as follows: Section 2 presents definitions and some well-known technical propositions needed for the proof of the main results. Switching rule for the mean square robust stability and stabilization is presented in Section 3. Numerical examples of the result are given in Section 4.

2. Preliminaries

The following notations will be used throughout this paper. R^+ denotes the set of all real non-negative numbers; R^n denotes the n-dimensional space with the scalar product of two vectors $\langle x, y\rangle$ or $x^T y$; $R^{n\times r}$ denotes the space of all matrices of $(n\times r)-$ dimension. N^+ denotes the set of all non-negative integers; A^T denotes the transpose of A; a matrix A is symmetric if $A = A^T$.
Matrix A is semi-positive definite ($A \geq 0$) if $\langle Ax, x\rangle \geq 0$, for all $x \in R^n$; A is positive definite ($A > 0$) if $\langle Ax, x\rangle > 0$ for all $x \neq 0$; $A \geq B$ means $A - B \geq 0$. $\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{Re\lambda : \lambda \in \lambda(A)\}$.

Consider a nonlinear uncertain stochastic discrete systems with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= (A_\gamma + \Delta A_\gamma(k))x(k) \\ &+ (B_\gamma + \Delta B_\gamma(k))x(k-d(k)) \\ &+ f_\gamma(k, x(k-d(k))) \\ &+ \sigma_\gamma(x(k), x(k-d(k)), k)\omega(k), \\ &\quad k \in N^+, \quad x(k) = v_k, \\ &\quad k = -d_2, -d_2+1, ..., 0, \end{aligned} \tag{2.1}$$

where $x(k) \in R^n$ is the state, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i, B_i, i = 1, 2, ..., N$ are given constant matrices.

The nonlinear perturbations $f_i(k, x(k-d(k))), i = 1, 2, ..., N$ satisfies the following condition

$$\begin{aligned} &f_i^T(k, x(k-d(k)))f_i(k, x(k-d(k))) \\ &\leq \beta_i^2 x^T(k-d(k))x(k-d(k)), \quad i = 1, 2, ..., N, \end{aligned}$$

where $\beta_i, i = 1, 2, ..., N$ is positive constants. For simplicity, we denote $f_i(k, x(k-d(k))$ by f_i, respectively.

The time-varying uncertain matrices $\Delta A_i(k)$ and $\Delta B_i(k)$ are defined by:

$$\Delta A_i(k) = E_{ia}F_{ia}(k)H_{ia}, \quad \Delta B_i(k) = E_{ib}F_{ib}(k)H_{ib}, \tag{2.2}$$

where $E_{ia}, E_{ib}, H_{ia}, H_{ib}$ are known constant real matrices with appropriate dimensions.
$F_{ia}(k), F_{ib}(k)$ are unknown uncertain matrices satisfying

$$F_{ia}^T(k)F_{ia}(k) \leq I, \quad F_{ib}^T(k)F_{ib}(k) \leq I, \quad k = 0, 1, 2, ..., \tag{2.3}$$

where I is the identity matrix of appropriate dimention, $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E[\omega(k)] = 0, \quad E[\omega^2(k)] = 1, \quad E[\omega(i)\omega(j)] = 0 (i \neq j), \tag{2.4}$$

and σ_i: $R^n \times R^n \times R \to R^n, i = 1, 2, ..., N$ is the continuous function, and is assumed to satisfy that

$$\begin{aligned} &\sigma_i^T(x(k), x(k-d(k)), k)\sigma_i(x(k), x(k-d(k)), k) \\ &\leq \rho_{i1}x^T(k)x(k) + \rho_{i2}x^T(k-d(k))x(k-d(k), \\ &x(k), x(k-d(k) \in R^n, \end{aligned} \tag{2.5}$$

where $\rho_{i1} > 0$ and $\rho_{i2} > 0, i = 1, 2, ..., N$ are khown constant scalars. The time-varying function $d(k) : N^+ \to N^+$ satisfies the following condition:

$$0 < d_1 \leq d(k) \leq d_2, \quad \forall k \in N^+$$

Remark 2.1. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Definition 2.1. The uncertain stochastic switched system (2.1) is robustly stable if there exists a switching function $\gamma(.)$ such that the zero solution of the uncertain stochastic switched system is robustly stable.

Definition 2.2. The discrete-time system (2.1) is robustly stable in the mean square if there exists a positive definite scalar function $V(k, x(k) : R^n \times R^n \to R$ such that
$E[\Delta V(k, x(k))]$
$= E[V(k+1, x(k+1)) - V(k, x(k))] < 0,$
along any trajectory of solution of the system for all uncertainties which satisfy (2.1), (2.2) and (2.3).

Proposition 2.1. (Cauchy inequality) *For any symmetric positive definite marix $N \in M^{n\times n}$ and $a, b \in R^n$ we have*

$$\underline{+}a^T b \leq a^T N a + b^T N^{-1} b.$$

Proposition 2.2. [16] *Let E, H and F be any constant matrices of appropriate dimensions and $F^T F \leq I$. For any $\epsilon > 0$, we have*

$$EFH + H^T F^T E^T \leq \epsilon EE^T + \epsilon^{-1} H^T H.$$

3. Main results

A. Stability.

Let us set

$$W_i(S_1,S_2,P,Q)=\begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14}\\ * & W_{i22} & W_{i23} & W_{i24}\\ * & * & W_{i33} & W_{i34}\\ * & * & * & W_{i44}\end{bmatrix},$$

$$\begin{aligned}
&W_{i11}=(d_2-d_1+1)Q-P-S_1A_i-A_i^TS_1^T\\
&+2S_1E_{ia}E_{ia}^TS_1^T+S_1E_{ib}E_{ib}^TS_1^T+S_2E_{ia}E_{ia}^TS_2^T\\
&+H_{ia}^TH_{ia}+2\rho_{i1}I,\\
&W_{i12}=S_1-S_1A_i,\\
&W_{i13}=-S_1B_i,\\
&W_{i14}=-S_1-S_2A_i,\\
&W_{i22}=P+S_1+S_1^T+S_1E_{ib}E_{ib}^TS_1^T+H_{ia}^TH_{ia},\\
&W_{i23}=-S_1B_i,\\
&W_{i24}=S_2-S_1,\\
&W_{i33}=-Q+S_2E_{ib}E_{ib}^TS_2^T+2H_{ib}^TH_{ib}+2\rho_{i2}I,\\
&W_{i34}=-S_2B_i,\\
&W_{i44}=-S_2-S_2^T+H_{ia}^TH_{ia}+H_{ib}^TH_{ib}.
\end{aligned}$$

The main result of this paper is summarized in the following theorem.

Theorem 3.1. *The uncertain stochastic switched system (2.1) is robustly stable in the mean square if there exist symmetric positive definite matrices $P>0,Q>0$ and matrices S_1,S_2 satisfying the following conditions*

$$W_i(S_1,S_2,P,Q)<0,\quad i=1,2,...,N. \tag{3.1}$$

The switching rule is chosen as $\gamma(x(k))=i$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (2.1)

$$V(k)=V_1(k)+V_2(k)+V_3(k),$$

where

$$V_1(k)=x^T(k)Px(k),\quad V_2(k)=\sum_{i=k-d(k)}^{k-1}x^T(i)Qx(i),$$

$$V_3(k)=\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1}x^T(l)Qx(l),$$

We can verify that

$$\lambda_1\|x(k)\|^2\le V(k). \tag{3.2}$$

Let us set $\xi(k)=[x(k)\,x(k+1)\,x(k-d(k))\,f_i(k,x(k-d(k)))\,\omega(k)]^T$ and

$$H=\begin{pmatrix}0&0&0&0&0\\0&P&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix},\quad G=\begin{pmatrix}P&0&0&0&0\\I&I&0&0&0\\0&0&I&0&0\\0&0&0&I&0\\0&0&0&0&I\end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system (2.1) and taking the mathematical expectation, we obtained

$$\begin{aligned}E[\Delta V_1(k)]&=E[x^T(k+1)Px(k+1)-x^T(k)Px(k)]\\&=E[\xi^T(k)H\xi(k)-2\xi^T(k)G^T\begin{pmatrix}0.5x(k)\\0\\0\\0\\0\end{pmatrix}].\end{aligned} \tag{3.3}$$

because of

$$\xi^T(k)H\xi(k)=x(k+1)Px(k+1),$$

$$2\xi^T(k)G^T\begin{pmatrix}0.5x(k)\\0\\0\\0\\0\end{pmatrix}=x^T(k)Px(k).$$

Using the expression of system (2.1)

$$\begin{aligned}
0&=-S_1x(k+1)+S_1(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
&\quad+S_1(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+S_1f_i+S_1\sigma_i\omega(k),\\
0&=-S_2x(k+1)+S_2(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
&\quad+S_2(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+S_2f_i+S_2\sigma_i\omega(k),\\
0&=-\sigma_i^Tx(k+1)+\sigma_i^T(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
&\quad+\sigma_i^T(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+\sigma_i^Tf_i+\sigma_i^T\sigma_i\omega(k),
\end{aligned}$$

we have
$E[-2\xi^T(k)G^T$

$$\begin{pmatrix}
0.5x(k)\\
[-S_1x(k+1)+S_1(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
+S_1(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+S_1f_i\\
+S_1\sigma_i\omega(k)]\\
\\
0\\
\\
[-S_2x(k+1)+S_2(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
+S_2(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+S_2f_i\\
+S_2\sigma_i\omega(k)]\\
\\
[-\sigma_i^Tx(k+1)+\sigma_i^T(A_i+E_{ia}F_{ia}(k)H_{ia})x(k)\\
+\sigma_i^T(B_i+E_{ib}F_{ib}(k)H_{ib})x(k-d(k))+\sigma_i^Tf_i\\
+\sigma_i^T\sigma_i\omega(k)]
\end{pmatrix}]$$

Therefore, from (3.3) it follows that

$$
\begin{aligned}
&E[\Delta V_1(k)]\\
&= E[x^T(k)[-P - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}\\
&- A_i^T S_1^T - H_{ia}^T F_{ia}^T(k)E_{ia}S_1^T]x(k)\\
&+ 2x^T(k)[S_1 - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}]\\
&x(k+1)\\
&+ 2x^T(k)[-S_1B_i - S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k))\\
&+ 2x^T(k)[-S_1 - S_2A_i - S_2E_{ia}F_{ia}(k)H_{ia}]\\
&f_i(k, x(k-d(k)))\\
&+ 2x^T(k)[-S_1\sigma_i - \sigma_i^T A_i - \sigma_i^T E_{ia}F_{ia}(k)H_{ia}]\omega(k)\\
&+ x(k+1)[P + S_1 + S_1^T]x(k+1)\\
&+ 2x(k+1)[-S_1B_i - S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k))\\
&+ 2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k)))\\
&+ 2x(k+1)[\sigma_i^T - S_1\sigma_i]\omega(k)\\
&+ 2x^T(k-d(k))[-S_2B_i - S_2E_{ib}F_{ib}(k)H_{ib}]\\
&f_i(k, x(k-d(k)))\\
&+ 2x^T(k-d(k))[-\sigma_i^T B_i - \sigma_i^T E_{ib}F_{ib}(k)H_{ib}]\omega(k)\\
&+ f_i(k, x(k-d(k)))^T[-S_2 - S_2^T]f_i(k, x(k-d(k)))\\
&+ 2f_i(k, x(k-d(k)))^T(k)[-S_2\sigma_i - \sigma_i^T]\omega(k)\\
&+ \omega^T(k)[-2\sigma_i^T\sigma_i]\omega(k)].
\end{aligned}
$$

By asumption (2.4), we have

$$
\begin{aligned}
&E[\Delta V_1(k)]\\
&= E[x^T(k)[-P - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}\\
&- A_i^T S_1^T - H_{ia}^T F_{ia}^T(k)E_{ia}S_1^T]x(k)\\
&+ 2x^T(k)[S_1 - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1)\\
&+ 2x^T(k)[-S_1B_i - S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k))\\
&+ 2x^T(k)[-S_1 - S_2A_i - S_3E_{ia}F_{ia}(k)H_{ia}]\\
&f_i(k, x(k-d(k)))\\
&+ x(k+1)[P + S_1 + S_1^T]x(k+1)\\
&+ 2x(k+1)[-S_1B_i - S_1E_{ib}F_{ib}(k)H_{ib}]x(k-d(k))\\
&+ 2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k)))\\
&+ 2x^T(k-d(k))[-S_2B_i - S_2E_{ib}F_{ib}(k)H_{ib}]\\
&f_i(k, x(k-d(k)))\\
&+ f_i(k, x(k-d(k)))^T[-S_2 - S_2^T]f_i(k, x(k-d(k)))\\
&+ \omega^T(k)[-2\sigma_i^T\sigma_i]\omega(k)].
\end{aligned}
$$

Applying Propositon 2.2, Propositon 2.3, condition (2.3) and asumption (2.5), the following estimations hold

$$
\begin{aligned}
&-S_1E_{ia}F_{ia}(k)H_{ia} - H_{ia}^T F_{ia}^T(k)E_{ia}^T S_1^T\\
&\le S_1E_{ia}E_{ia}^T S_1^T + H_{ia}^T H_{ia},
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k)S_1E_{ia}F_{ia}(k)H_{ia}x(k+1)\\
&\le x^T(k)S_1E_{ia}E_{ia}^T S_1^T x(k) + x(k+1)^T H_{ia}^T H_{ia}x(k+1),
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k))\\
&\le x^T(k)S_1E_{ib}E_{ib}^T S_1^T x(k) + x(k-d(k))^T H_{ib}^T H_{ib}\\
&x(k-d(k)),
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k)S_2E_{ia}F_{ia}(k)H_{ia}f_i\\
&\le x^T(k)S_2E_{ia}E_{ia}^T S_2^T x(k) + f_i^T H_{ia}^T H_{ia}f_i,
\end{aligned}
$$

$$
\begin{aligned}
&-2x(k-d(k))^T(k)S_2E_{ib}F_{ib}(k)H_{ib}f_i\\
&\le x(k-d(k))^T(k)S_2E_{ib}E_{ib}^T S_2^T x(k-d(k)) + f_i^T H_{ib}^T H_{ib}f_i,
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k+1)S_1E_{ib}F_{ib}(k)H_{ib}x(k-d(k))\\
&\le x^T(k+1)S_1E_{ib}E_{ib}^T S_1^T x(k+1) + x(k-d(k))^T H_{ib}^T H_{ib}\\
&x(k-d(k)),
\end{aligned}
$$

$$
\begin{aligned}
&-\sigma_i^T(x(k), x(k-d(k)), k)\sigma_i(x(k), x(k-d(k)), k)\\
&\le \rho_{i1}x^T(k)x(k) + \rho_{i2}x^T(k-d(k))x(k-d(k)).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&E[\Delta V_1(k)]\\
&\le E[x^T(k)[-P - S_1A_i - A_i^T S_1^T + 2S_1E_{ia}E_{ia}^T S_1^T\\
&+ S_1E_{ib}E_{ib}^T S_1^T + S_2E_{ia}E_{ia}^T S_2^T + H_{ia}^T H_{ia} + 2\rho_{i1}I]x(k)\\
&+ 2x^T(k)[S_1 - S_1A_i]x(k+1)\\
&+ 2x^T(k)[-S_1B_i]x(k-d(k))\\
&+ 2x^T(k)[-S_1 - S_2A_i]f_i(k, x(k-d(k)))\\
&+ x(k+1)[P + S_1 + S_1^T + S_1E_{ib}E_{ib}^T S_1^T\\
&+ H_{ia}^T H_{ia}]x(k+1) + 2x(k+1)[-S_1B_i]x(k-d(k))\\
&+ 2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k)))\\
&+ x^T(k-d(k))[S_2E_{ib}E_{ib}^T S_2^T + 2H_{ib}^T H_{ib}\\
&+ 2\rho_{i2}I]x(k-d(k))\\
&+ 2x^T(k-d(k))[-S_2B_i]f_i(k, x(k-d(k)))\\
&+ f_i(k, x(k-d(k)))^T[-S_2 - S_2^T + H_{ia}^T H_{ia}\\
&+ H_{ib}^T H_{ib}]f_i(k, x(k-d(k)))].
\end{aligned}
\tag{3.4}
$$

The difference of $V_2(k)$ is given by

$$\begin{aligned}
&E[\Delta V_2(k)]\\
&= E[\sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i) - \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)]\\
&= E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k)\\
&- x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i)\\
&- \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i)].
\end{aligned} \tag{3.5}$$

Since $d(k) \geq d_1$ we have

$$\sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i) \leq 0,$$

and hence from (3.5) we have

$$\begin{aligned}
E[\Delta V_2(k)] \leq E[&\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i)\\
&+ x^T(k)Qx(k)\\
&- x^T(k-d(k))Qx(k-d(k))].
\end{aligned} \tag{3.6}$$

The difference of $V_3(k)$ is given by

$$\begin{aligned}
E[\Delta V_3(k)] &= E[\sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j}^{k} x^T(l)Qx(l)\\
&- \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Qx(l)]\\
&= E[\sum_{j=-d_2+2}^{-d_1+1} [\sum_{l=k+j}^{k-1} x^T(l)Qx(l)\\
&+ x^T(k)Q(\xi)x(k) - \sum_{l=k+j}^{k-1} x^T(l)Qx(l)\\
&- x^T(k+j-1)Qx(k+j-1)]]\\
&= E[\sum_{j=-d_2+2}^{-d_1+1} [x^T(k)Qx(k)\\
&- x^T(k+j-1)Qx(k+j-1)]]\\
&= E[(d_2-d_1)x^T(k)Qx(k)\\
&- \sum_{j=k+1-d_2}^{k-d_1} x^T(j)Qx(j)].
\end{aligned} \tag{3.7}$$

Since $d(k) \leq d_2$, and

$$\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i) \leq 0,$$

we obtain from (3.6) and (3.7) that

$$\begin{aligned}
&E[\Delta V_2(k) + \Delta V_3(k)]\\
&\leq E[(d_2-d_1+1)x^T(k)Qx(k)\\
&- x^T(k-d(k))Qx(k-d(k))].
\end{aligned} \tag{3.8}$$

Therefore, combining the inequalities (3.4), (3.8) gives

$$\begin{aligned}
&E[\Delta V(k)]\\
&\leq \psi^T(k)W_i(S_1,S_2,P,Q)\psi(k)],\\
&\forall i = 1,2,....,N, k = 0,1,2,...,
\end{aligned} \tag{3.9}$$

where

$$\psi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f_i(k,x(k-d(k)))]^T,$$

$$W_i(S_1,S_2,P,Q) = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14}\\ * & W_{i22} & W_{i23} & W_{i24}\\ * & * & W_{i33} & W_{i34}\\ * & * & * & W_{i44} \end{bmatrix},$$

$$\begin{aligned}
W_{i11} &= (d_2-d_1+1)Q - P - S_1A_i - A_i^TS_1^T\\
&+ 2S_1E_{ia}E_{ia}^TS_1^T + S_1E_{ib}E_{ib}^TS_1^T\\
&+ S_3E_{ia}E_{ia}^TS_3^T + H_{ia}^TH_{ia} + 2\rho_{i1}I,\\
W_{i12} &= S_1 - S_1A_i,\\
W_{i13} &= -S_1B_i - S_2A_i,\\
W_{i14} &= -S_1 - S_3A_i,\\
W_{i14} &= -S_1 - S_3A_i,\\
W_{i22} &= P + S_1 + S_1^T + S_1E_{ib}E_{ib}^TS_1^T + H_{ia}^TH_{ia},\\
W_{i23} &= -S_1B_i,\\
W_{i24} &= S_3 - S_1,\\
W_{i33} &= -Q + S_3E_{ib}E_{ib}^TS_3^T + 2H_{ib}^TH_{ib} + 2\rho_{i2}I,\\
W_{i34} &= -S_3B_i,\\
W_{i44} &= -S_3 - S_3^T + H_{ia}^TH_{ia} + H_{ib}^TH_{ib}.
\end{aligned}$$

Therefore, we finally obtain from (3.9) and the condition (3.1) that

$$E[\Delta V(k)] < 0,$$

by choosing switching rule as $\gamma(x(k)) = i$, which, combining the condition (3.2), and Definition 2.2., concludes the proof of the theorem in the mean square.

Remark 3.1. Note that theresult sproposed in [8–10] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [13] was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

B. Stabilization.

Consider a nonlinear uncertain stochastic switched control discrete-time systems with interval time-varying delay of the form

$$\begin{aligned} x(k+1) &= (A_\gamma + \Delta A_\gamma(k))x(k) + (B_\gamma + \Delta B_\gamma(k))u(k) \\ &\quad + f_\gamma(k, x(k-d(k))) \\ &\quad + \sigma_\gamma(x(k), x(k-d(k)), k)\omega(k), \quad k \in N^+, \\ x(k) &= v_k, \quad k = -d_2, -d_2+1, ..., 0, \end{aligned} \tag{3.10}$$

where $x(k) \in R^n$ is the state, $u(k) \in R^m, m \le n$, is the control input, $\gamma(.) : R^n \to \mathcal{N} := \{1, 2, \ldots, N\}$ is the switching rule, which is a function depending on the state at each time and will be designed. A switching function is a rule which determines a switching sequence for a given switching system. Moreover, $\gamma(x(k)) = i$ implies that the system realization is chosen as the i^{th} system, $i = 1, 2, ..., N$. It is seen that the system (2.1) can be viewed as an autonomous switched system in which the effective subsystem changes when the state $x(k)$ hits predefined boundaries. $A_i, B_i, i = 1, 2, ..., N$ are given constant matrices.

The nonlinear perturbations $f_i(k, x(k-d(k))), i = 1, 2, ..., N$ satisfies the following condition

$$\begin{aligned} & f_i^T(k, x(k-d(k)))f_i(k, x(k-d(k))) \\ & \le \beta_i^2 x^T(k-d(k))x(k-d(k)), i = 1, 2, ..., N, \end{aligned} \tag{3.11}$$

where $\beta_i, i = 1, 2, ..., N$ is positive constants. For simplicity, we denote $f_i(k, x(k-d(k))$ by f_i, respectively.

We consider a delayed feedback control law

$$u(k) = (C_i + \Delta C_i(k))x(k-d(k)), \quad k = -h_2, ..., 0, \tag{3.12}$$

and $C_i + \Delta C_i(k), i = 1, 2, ..., N$ is the controller gain to be determined.

The time-varying uncertain matrices $\Delta A_i(k)$, $\Delta B_i(k)$, and $\Delta C_i(k)$ are defined by:

$$\Delta A_i(k) = E_{ia}F_{ia}(k)H_{ia}, \Delta B_i(k) = E_{ib}F_{ib}(k)H_{ib},$$
$$\Delta C_i(k) = E_{ic}F_{ic}(k)H_{ic},$$

where $E_{ia}, E_{ib}, E_{ic}, H_{ia}, H_{ib}, H_{ic}$ are known constant real matrices with appropriate dimensions. $F_{ia}(k), F_{ib}(k), F_{ic}(k)$ are unknown uncertain matrices satisfying

$$F_{ia}^T(k)F_{ia}(k) \le I, \quad F_{ib}^T(k)F_{ib}(k) \le I,$$
$$F_{ic}^T(k)F_{ic}(k) \le I, \quad k = 0, 1, 2, ...$$

where I is the identity matrix of appropriate dimention, $\omega(k)$ is a scalar Wiener process (Brownian Motion) on $(\Omega, \mathcal{F}, \mathcal{P})$ with

$$E[\omega(k)] = 0, \quad E[\omega^2(k)] = 1, \quad E[\omega(i)\omega(j)] = 0 (i \ne j), \tag{3.13}$$

and σ_i: $R^n \times R^n \times R \to R^n, i = 1, 2, ..., N$ is the continuous function, and is assumed to satisfy that

$$\begin{aligned} & \sigma_i^T(x(k), x(k-d(k)), k)\sigma_i(x(k), x(k-d(k)), k) \\ & \le \rho_{i1}x^T(k)x(k) + \rho_{i2}x^T(k-d(k))x(k-d(k), \\ & x(k), x(k-d(k) \in R^n, \end{aligned} \tag{3.14}$$

where $\rho_{i1} > 0$ and $\rho_{i2} > 0, i = 1, 2, ..., N$ are khown constant scalars. The time-varying function $d(k) : N^+ \to N^+$ satisfies the following condition:

$$0 < d_1 \le d(k) \le d_2, \quad \forall k = 0, 1, 2, ...$$

Remark 3.2. It is worth noting that the time delay is a time-varying function belonging to a given interval, in which the lower bound of delay is not restricted to zero.

Applying the feedback controller (3.11) to the system (3.10), the closed-loop discrete time-delay system is

$$\begin{aligned} x(k+1) &= (A_i + \Delta A_i(k))x(k) \\ &\quad + (B_i + \Delta B_i(k))(C_i + \Delta C_i(k))x(k-d(k)) \\ &\quad + f_i(k, x(k-d(k))) \\ &\quad + \sigma_i(x(k), x(k-d(k)), k)\omega(k), \\ &\quad k = 0, 1, 2, ... \end{aligned} \tag{3.15}$$

Definition 3.1. The nonlinear uncertain stochastic switched control system (3.10) is stablilizable if there is a delayed feedback control (3.12) such that the nonlinear uncertain stochastic switched system (3.14) is robustly stable.
Let us set

$$W_i(S_1, S_2, P, Q) = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14} \\ * & W_{i22} & W_{i23} & W_{i24} \\ * & * & W_{i33} & W_{i34} \\ * & * & * & W_{i44} \end{bmatrix},$$

$$\begin{aligned} W_{i11} &= (d_2 - d_1 + 1)Q - P - S_1A_i - A_i^TS_1^T \\ &\quad + 2S_1E_{ia}E_{ia}^TS_1^T + S_2E_{ia}E_{ia}^TS_2^T + H_{ia}^TH_{ia} + 2\rho_{i1}I, , \\ W_{i12} &= S_1 - S_1A_i, \\ W_{i13} &= -S_1, \\ W_{i14} &= -S_1 - S_2A_i, \\ W_{i22} &= P + S_1 + S_1^T + H_{ia}^TH_{ia}, \\ W_{i23} &= -S_1, \\ W_{i24} &= S_2 - S_1, \\ W_{i33} &= -Q + H_{ia}^TH_{ia} + 2\rho_{i2}I, \\ W_{i34} &= -S_2, \\ W_{i44} &= -S_2 - S_2^T + H_{ia}^TH_{ia}. \end{aligned}$$

Theorem 3.2. *The nonlinear uncertain stochastic switched control system (3.10) is stabilizable in the mean square by the delayed feedback control (3.12), where*
$(C_i + \Delta C_i(k))$
$= (B_i + \Delta B_i(k))^T[(B_i + \Delta B_i(k))(B_i + \Delta B_i(k))^T]^{-1}$,
$i = 1, 2, ..., N$,
if there exist symmetric matrices $P > 0, Q > 0$ and matrices S_1, S_2 satisfying the following conditions

$$W_i(S_1, S_2, P, Q) < 0, \quad i = 1, 2, ..., N. \tag{3.16}$$

The switching rule is chosen as $\gamma(x(k)) = i$.

Proof. Consider the following Lyapunov-Krasovskii functional for any ith system (3.10)

$$V(k) = V_1(k) + V_2(k) + V_3(k),$$

where

$$V_1(k) = x^T(k)Px(k), \quad V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i),$$

$$V_3(k) = \sum_{j=-d_2+2}^{-d_1+1} \sum_{l=k+j+1}^{k-1} x^T(l)Qx(l),$$

We can verify that

$$\lambda_1\|x(k)\|^2 \le V(k). \tag{3.17}$$

Let us set $\xi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f_i(k, x(k-d(k)))\, \omega(k)]^T$ and

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} P & 0 & 0 & 0 & 0 \\ I & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

Then, the difference of $V_1(k)$ along the solution of the system (3.10) and taking the mathematical expectation, we obtained

$$E[\Delta V_1(k)] = E[x^T(k+1)Px(k+1) - x^T(k)Px(k)]$$
$$= E[\xi^T(k)H\xi(k) - 2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}]. \tag{3.18}$$

because of

$$\xi^T(k)H\xi(k) = x(k+1)Px(k+1),$$

$$2\xi^T(k)G^T \begin{pmatrix} 0.5x(k) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = x^T(k)Px(k).$$

Using the expression of system (3.10)

$$\begin{aligned} 0 &= -S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ &\quad + S_1x(k-d(k)) + S_1f_i + S_1\sigma_i\omega(k), \\ 0 &= -S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ &\quad + S_2x(k-d(k)) + S_2f_i + S_2\sigma_i\omega(k), \\ 0 &= -\sigma_i^Tx(k+1) + \sigma_i^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ &\quad + \sigma_i^Tx(k-d(k)) + \sigma_i^Tf_i + \sigma_i^T\sigma_i\omega(k), \end{aligned}$$

we have
$E[-2\xi^T(k)G^T$

$$\begin{pmatrix} 0.5x(k) \\ [-S_1x(k+1) + S_1(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ +S_1x(k-d(k)) + S_1f_i + S_1\sigma_i\omega(k)] \\ 0 \\ [-S_2x(k+1) + S_2(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ +S_2x(k-d(k)) + S_3f_i + S_2\sigma_i\omega(k)] \\ [-\sigma_i^Tx(k+1) + \sigma_i^T(A_i + E_{ia}F_{ia}(k)H_{ia})x(k) \\ +\sigma_i^Tx(k-d(k)) + \sigma_i^Tf_i + \sigma_i^T\sigma_i\omega(k)] \end{pmatrix}]$$

Therefore, from (3.19) it follows that

$$\begin{aligned} &E[\Delta V_1(k)] \\ &= E[x^T(k)[-P - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia} \\ &- A_i^TS_1^T - H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \\ &+ 2x^T(k)[S_1 - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1) \\ &+ 2x^T(k)[-S_1]x(k-d(k)) \\ &+ 2x^T(k)[-S_1 - S_2A_i - S_2E_{ia}F_{ia}(k)H_{ia}] \\ &f_i(k, x(k-d(k))) \\ &+ 2x^T(k)[-S_1\sigma_i - \sigma_i^TA_i - \sigma_i^TE_{ia}F_{ia}(k)H_{ia}]\omega(k) \\ &+ x(k+1)[P + S_1 + S_1^T]x(k+1) \\ &+ 2x(k+1)[S_2 - S_1]x(k-d(k)) \\ &+ 2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k))) \\ &+ 2x(k+1)[\sigma_i^T - S_1\sigma_i]\omega(k) \\ &+ 2x^T(k-d(k))[-S_2]f_i(k, x(k-d(k))) \\ &+ 2x^T(k-d(k))[-\sigma_i^T]\omega(k) \\ &+ f_i(k, x(k-d(k)))^T[-S_2 - S_2^T]f_i(k, x(k-d(k))) \\ &+ 2f_i(k, x(k-d(k)))^T(k)[-S_2\sigma_i - \sigma_i^T]\omega(k) \\ &+ \omega^T(k)[-2\sigma_i^T\sigma_i]\omega(k)]. \end{aligned}$$

By asumption (3.14), we have

$$\begin{aligned} &E[\Delta V_1(k)] \\ &= E[x^T(k)[-P - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia} \\ &- A_i^TS_1^T - H_{ia}^TF_{ia}^T(k)E_{ia}S_1^T]x(k) \end{aligned}$$

$$
\begin{aligned}
&+2x^T(k)[S_1 - S_1A_i - S_1E_{ia}F_{ia}(k)H_{ia}]x(k+1)\\
&+2x^T(k)[-S_1]x(k-d(k))\\
&+2x^T(k)[-S_1 - S_2A_i - S_2E_{ia}F_{ia}(k)H_{ia}]\\
&f_i(k, x(k-d(k)))\\
&+x(k+1)[P + S_1 + S_1^T]x(k+1)\\
&+2x(k+1)[-S_1]x(k-d(k))\\
&+2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k)))\\
&+2x^T(k-d(k))[-S_2]f_i(k, x(k-d(k)))\\
&+f_i(k, x(k-d(k)))^T[-S_2 - S_2^T]f_i(k, x(k-d(k)))\\
&+\omega^T(k)[-2\sigma_i^T\sigma_i]\omega(k)].
\end{aligned}
$$

Applying Propositon 2.2, Propositon 2.3, condition (3.13) and asumption (3.15), the following estimations hold

$$
\begin{aligned}
&-S_1E_{ia}F_{ia}(k)H_{ia} - H_{ia}^TF_{ia}^T(k)E_{ia}^TS_1^T\\
&\leq S_1E_{ia}E_{ia}^TS_1^T + H_{ia}^TH_{ia},
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k)S_1E_{ia}F_{ia}(k)H_{ia}x(k+1)\\
&\$leq x^T(k)S_1E_{ia}E_{ia}^TS_1^Tx(k) + x(k+1)^TH_{ia}^TH_{ia}x(k+1),
\end{aligned}
$$

$$
\begin{aligned}
&-2x^T(k)S_2E_{ia}F_{ia}(k)H_{ia}f_i\\
&\leq x^T(k)S_2E_{ia}E_{ia}^TS_2^Tx(k) + f_i^TH_{ia}^TH_{ia}f_i,
\end{aligned}
$$

$$
\begin{aligned}
&-\sigma_i^T(x(k), x(k-d(k)), k)\sigma_i(x(k), x(k-d(k)), k)\\
&\leq \rho_{i1}x^T(k)x(k) + \rho_{i2}x^T(k-d(k))x(k-d(k)).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&E[\Delta V_1(k)]\\
&\leq E[x^T(k)[-P - S_1A_i - A_i^TS_1^T\\
&+2S_1E_{ia}E_{ia}^TS_1^T + S_2E_{ia}E_{ia}^TS_2^T\\
&+H_{ia}^TH_{ia} + 2\rho_{i1}I]x(k)\\
&+2x^T(k)[S_1 - S_1A_i]x(k+1)\\
&+2x^T(k)[-S_1]x(k-d(k))\\
&+2x^T(k)[-S_1 - S_2A_i]f_i(k, x(k-d(k)))\\
&+x(k+1)[P + S_1 + S_1^T + H_{ia}^TH_{ia}]x(k+1)\\
&+2x(k+1)[-S_1]x(k-d(k))\\
&+2x(k+1)[S_2 - S_1]f_i(k, x(k-d(k)))\\
&+x^T(k-d(k))[2\rho_{i2}I]x(k-d(k))\\
&+2x^T(k-d(k))[-S_2]f_i(k, x(k-d(k)))\\
&+f_i(k, x(k-d(k)))^T[-S_2 - S_2^T + H_{ia}^TH_{ia}]\\
&f_i(k, x(k-d(k)))].
\end{aligned} \tag{3.19}
$$

The difference of $V_2(k)$ is given by

$$
\begin{aligned}
E[\Delta V_2(k)] = E[&\sum_{i=k+1-d(k+1)}^{k} x^T(i)Qx(i)\\
&-\sum_{i=k-d(k)}^{k-1} x^T(i)Qx(i)]
\end{aligned}
$$

$$
\begin{aligned}
&= E[\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i) + x^T(k)Qx(k)\\
&-x^T(k-d(k))Qx(k-d(k)) + \sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i)\\
&-\sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i)].
\end{aligned} \tag{3.20}
$$

Since $d(k) \geq d_1$ we have

$$
\sum_{i=k+1-d_1}^{k-1} x^T(i)Qx(i) - \sum_{i=k+1-d(k)}^{k-1} x^T(i)Qx(i) \leq 0,
$$

and hence from (3.21) we have

$$
\begin{aligned}
E[\Delta V_2(k)] \leq E[&\sum_{i=k+1-d(k+1)}^{k-d_1} x^T(i)Qx(i)\\
&+x^T(k)Qx(k)\\
&-x^T(k-d(k))Qx(k-d(k))].
\end{aligned} \tag{3.21}
$$

The difference of $V_3(k)$ is given by

$$
\begin{aligned}
E[\Delta V_3(k)] &= E[\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j}^{k} x^T(l)Qx(l)\\
&-\sum_{j=-d_2+2}^{-d_1+1}\sum_{l=k+j+1}^{k-1} x^T(l)Qx(l)]\\
&= E[\sum_{j=-d_2+2}^{-d_1+1}[\sum_{l=k+j}^{k-1} x^T(l)Qx(l)\\
&+x^T(k)Q(\xi)x(k)\\
&-\sum_{l=k+j}^{k-1} x^T(l)Qx(l)\\
&-x^T(k+j-1)Qx(k+j-1)]]\\
&= E[\sum_{j=-d_2+2}^{-d_1+1}[x^T(k)Qx(k)\\
&-x^T(k+j-1)Qx(k+j-1)]]\\
&= E[(d_2-d_1)x^T(k)Qx(k)\\
&-\sum_{j=k+1-d_2}^{k-d_1} x^T(j)Qx(j)].
\end{aligned} \tag{3.22}
$$

Since $d(k) \leq d_2$, and

$$
\begin{aligned}
&\sum_{i=k=1-d(k+1)}^{k-d_1} x^T(i)Qx(i) - \sum_{i=k+1-d_2}^{k-d_1} x^T(i)Qx(i)\\
&\leq 0,
\end{aligned}
$$

we obtain from (3.22) and (3.23) that

$$\begin{aligned} &E[\Delta V_2(k) + \Delta V_3(k)] \\ &\leq E[(d_2 - d_1 + 1)x^T(k)Qx(k) \\ &- x^T(k - d(k))Qx(k - d(k))]. \end{aligned} \tag{3.23}$$

Therefore, combining the inequalities (3.20), (3.24) gives

$$\begin{aligned} &E[\Delta V(k)] \leq E[\psi^T(k)W_i(S_1, S_2, P, Q)\psi(k)], \\ &\forall i = 1, 2, ..., N, k = 0, 1, 2, ..., \end{aligned} \tag{3.24}$$

where

$$\psi(k) = [x(k)\, x(k+1)\, x(k-d(k))\, f_i(k, x(k-d(k)))]^T,$$

$$W_i(S_1, S_2, P, Q) = \begin{bmatrix} W_{i11} & W_{i12} & W_{i13} & W_{i14} \\ * & W_{i22} & W_{i23} & W_{i24} \\ * & * & W_{i33} & W_{i34} \\ * & * & * & W_{i44} \end{bmatrix},$$

$$\begin{aligned} &W_{i11} = (d_2 - d_1 + 1)Q - P - S_1A_i - A_i^T S_1^T \\ &+ 2S_1E_{ia}E_{ia}^T S_1^T + S_2E_{ia}E_{ia}^T S_2^T + H_{ia}^T H_{ia} + 2\rho_{i1}I, \\ &W_{i12} = S_1 - S_1A_i, \\ &W_{i13} = -S_1, \\ &W_{i14} = -S_1 - S_2A_i, \\ &W_{i22} = P + S_1 + S_1^T + H_{ia}^T H_{ia}, \\ &W_{i23} = -S_1, \\ &W_{i24} = S_2 - S_1, \\ &W_{i33} = -Q + H_{ia}^T H_{ia} + 2\rho_{i2}I, \\ &W_{i34} = -S_2, \\ &W_{i44} = -S_2 - S_2^T + H_{ia}^T H_{ia}. \end{aligned}$$

Therefore, we finally obtain from (3.25) and the condition (3.17) that

$$E[\Delta V(k)] < 0,$$

by choosing switching rule as $\gamma(x(k)) = i$, which, combining the condition (3.18), and Definition 2.2 and 3.1., concludes the proof of the theorem in the mean square.

Remark 3.3. Note that theresult sproposed in [8–10] for switching systems to be asymptotically stable under an arbitrary switching rule. The asymptotic stability for switching linear discrete time-delay systems studied in [13] was limited to constant delays. In [14], a class of switching signals has been identified for the considered switched discrete-time delay systems to be stable under the averaged well time scheme.

4. Numerical examples

Example 4.1. (Stability) Consider the nonlinear uncertain stochastic switched discrete time-delay system (2.1), where the delay function $d(k)$ is given by

$$d(k) = 1 + 6sin^2\frac{k\pi}{2}, \quad k = 0, 1, 2, \ldots.$$

and

$$\begin{aligned} (A_1, B_1) &= \left(\begin{bmatrix} -1 & 0.1 \\ 0.2 & -0.2 \end{bmatrix}, \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \right), \\ (A_2, B_2) &= \left(\begin{bmatrix} -2 & 0.3 \\ 0.5 & -0.3 \end{bmatrix}, \begin{bmatrix} 0.3 & 0.1 \\ 0.24 & 0.18 \end{bmatrix} \right), \\ (H_{1a}, H_{1b}) &= \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right), \\ (H_{2a}, H_{2b}) &= \left(\begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \right), \\ (E_{1a}, E_{1b}) &= \left(\begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} \right), \\ (E_{2a}, E_{2b}) &= \left(\begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right), \\ (F_{1a}, F_{1b}) &= \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right), \\ (F_{2a}, F_{2b}) &= \left(\begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \right), \\ f_1(k, x(k - d(k)) &= \begin{bmatrix} 0.1699cos(k)x_1(k - d(k)) \\ 0.1699sin(k)x_2(k - d(k)) \end{bmatrix}, \\ f_2(k, x(k - d(k)) &= \begin{bmatrix} 0.2699sin(k)x_1(k - d(k)) \\ 0.2699cos(k)x_2(k - d(k)) \end{bmatrix}. \end{aligned}$$

By LMI toolbox of Matlab, we find that the conditions (3.1) of Theorem 3.1 are satisfied with $\beta_1 = 0.1699, \beta_2 = 0.2699, d_1 = 1, d_2 = 7, \rho_{11} = 0.5, \rho_{12} = 0.2, \rho_{21} = 0.3, \rho_{22} = 0.2$, and

$$P = \begin{bmatrix} 114.4629 & 4.5328 \\ 4.5328 & 132.1362 \end{bmatrix}, Q = \begin{bmatrix} 16.4921 & 0.3656 \\ 0.3656 & 18.5234 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} -1.5293 & 1.6966 \\ 0.7474 & 2.1570 \end{bmatrix}, S_2 = \begin{bmatrix} 6.8785 & 1.1641 \\ 0.4721 & 8.4612 \end{bmatrix}.$$

By Theorem 3.1 the nonlinear uncertain stochastic switched discrete time-delay system is robustly stable and the switching rule is chosen as $\gamma(x(k)) = i$.

Example 4.2. (Stabilization) Consider the nonlinear uncertain stochastic switched discrete time-delay control system (3.10), where the delay function $d(k)$ is given by

$$d(k) = 1 + 8sin^2\frac{k\pi}{2}, \quad k = 0, 1, 2, \ldots.$$

and

$$(A_1, B_1) = \left(\begin{bmatrix} -1 & 0.1 \\ 0.2 & -0.2 \end{bmatrix}, \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.3 \end{bmatrix} \right),$$
$$(A_2, B_2) = \left(\begin{bmatrix} -2 & 0.3 \\ 0.5 & -0.3 \end{bmatrix}, \begin{bmatrix} 0.3 & 0.1 \\ 0.24 & 0.18 \end{bmatrix} \right),$$
$$(H_{1a}, H_{1b}) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right),$$
$$(H_{2a}, H_{2b}) = \left(\begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \right),$$
$$(E_{1a}, E_{1b}) = \left(\begin{bmatrix} 0.3 & 0 \\ 0 & 0.4 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix} \right),$$
$$(E_{2a}, E_{2b}) = \left(\begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right),$$
$$(F_{1a}, F_{1b}) = \left(\begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}, \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix} \right),$$
$$(F_{2a}, F_{2b}) = \left(\begin{bmatrix} 0.2 & 0 \\ 0 & 0.5 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix} \right),$$
$$f_1(k, x(k-d(k)) = \begin{bmatrix} 0.8931cos(k)x_1(k-d(k)) \\ 0.8931sin(k)x_2(k-d(k)) \end{bmatrix},$$
$$f_2(k, x(k-d(k)) = \begin{bmatrix} 0.7314sin(k)x_1(k-d(k)) \\ 0.7314cos(k)x_2(k-d(k)) \end{bmatrix}.$$

By LMI toolbox of Matlab, we find that the conditions (3.17) of Theorem 3.2 are satisfied with $\beta_1 = 0.8931, \beta_2 = 0.7314, d_1 = 1, d_2 = 9, \rho_{11} = 0.5, \rho_{12} = 0.2, \rho_{21} = 0.3, \rho_{22} = 0.4$, and

$$P = \begin{bmatrix} 141.2605 & 2.0171 \\ 2.0171 & 147.8758 \end{bmatrix}, Q = \begin{bmatrix} 14.3329 & 0.0839 \\ 0.0839 & 14.1303 \end{bmatrix},$$
$$S_1 = \begin{bmatrix} -1.9723 & 0.9741 \\ 0.5738 & 1.9825 \end{bmatrix}, S_2 = \begin{bmatrix} 6.2643 & 0.9507 \\ 0.9380 & 9.6081 \end{bmatrix}.$$

By Theorem 3.2, the nonlinear uncertain stochastic switched discrete time-delay control system is stabilizable and the switching rule is $\gamma(x(k)) = i$, the delayed feedback control is:

$$u_1(k) = \begin{bmatrix} 19.9884x_1^1(k-d(k)) - 11.5875x_1^2(k-d(k)) \\ -5.7937x_1^1(k-d(k)) + 6.2572x_1^2(k-d(k)) \end{bmatrix},$$
$$u_2(k) = \begin{bmatrix} 5.6497x_2^1(k-d(k)) - 2.9426x_2^2(k-d(k)) \\ -7.0621x_2^1(k-d(k)) + 8.8865x_2^2(k-d(k)) \end{bmatrix}.$$

5. Conclusion

This paper has proposed a switching design for the robust stability and stabilization of nonlinear uncertain stochastic switched discrete time-delay systems with interval time-varying delays. Based on the discrete Lyapunov functional, a switching rule for the robust stability and stabilization for the nonlinear uncertain stochastic switched discrete time-delay system is designed via linear matrix inequalities.

Acknowledgement

This work was supported by the Thai Research Fund Grant, the Higher Education Commission and Faculty of Science, Maejo University, Thailand. The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] D. Liberzon, A.S. Morse, Basic problems in stability and design of switched systems, IEEE Control Syst. Mag., **19**(1999), 57-70.
[2] A.V. Savkin and R.J. Evans, Hybrid Dynamical Systems: Controller and Sensor Switching Problems, Springer, New York, 2001.
[3] Z. Sun and S.S. Ge, Switched Linear Systems: Control and Design, Springer, London, 2005.
[4] K. Ratchagit, Asymptotic stability of nonlinear delay-difference system via matrix inequalities and application, *International Journal of Computational Methods*, pp. 389-397, 2009.
[5] VN. Phat, Y. Kongtham, and K. Ratchagit, LMI approach to exponential stability of linear systems with interval time-varying delays, *Linear Algebra Appl.*, Vol. 436, pp. 243-251, 2012.
[6] VN. Phat, K. Ratchagit, Stability and stabilization of switched linear discrete-time systems with interval time-varying delay, *Nonlinear Analysis: Hybrid Systems*, Vol. 5, pp. 605-612, 2011.
[7] K. Ratchagit, VN. Phat, Stability criterion for discrete-time systems, *Journal of Inequalities and Applications*, Vol. 10, pp. 1-6, 2010.
[8] F. Gao, S. Zhong and X. Gao, Delay-dependent stability of a type of linear switching systems with discrete and distributed time delays, Appl. Math. Computation, **196**(2008), 24-39.
[9] C.H. Lien, K.W. Yu, Y.J. Chung, Y.F. Lin, L.Y. Chung and J.D. Chen, Exponential stability analysis for uncertain switched neutral systems with interval-time-varying state delay, Nonlinear Analysis: Hybrid systems, **3**(2009),334–342.
[10] G. Xie, L. Wang, Quadratic stability and stabilization of discrete-time switched systems with state delay, In: Proc. of the IEEE Conference on Decision and Control, Atlantics, December 2004, 3235-3240.
[11] S. Boyd, L.E. Ghaoui, E. Feron and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM, Philadelphia, 1994.
[12] D.H. Ji, J.H. Park, W.J. Yoo and S.C. Won, Robust memory state feedback model predictive control for discrete-time uncertain state delayed systems, Appl. Math. Computation, **215**(2009), 2035-2044.
[13] G.S. Zhai, B. Hu, K. Yasuda, and A. Michel, Qualitative analysis of discrete- time switched systems. In: Proc. of the American Control Conference, 2002, 1880-1885.
[14] W.A. Zhang, Li Yu, Stability analysis for discrete-time switched time-delay systems, Automatica, **45**(2009), 2265-2271.

[15] F. Uhlig, A recurring theorem about pairs of quadratic forms and extensions, Linear Algebra Appl., **25**(1979), 219-237.
[16] R.P. Agarwal, Difference Equations and Inequalities, Second Edition, Marcel Dekker, New York, 2000.
[17] M. S. Mahmoud, Switched Time-Delay Systems: Stability and Control, Boston, MA : Springer Science+Business Media, LLC, 2010.
[18] M. S. Mahmoud, M. N. Nounou and H. N. Nounou, Analysis and Synthesis of Uncertain Switched Discrete-Time Systems, IMA J. Math. Contr. Inf., vol. 24, 2007, pp. 245257.
[19] M. S. Mahmoud, Delay-Dependent H_2 Filtering of a Class of Switched Discrete-Time State-Delay Systems, J. Signal Process., vol. 88, no. 11, 2008, pp. 27092719.
[20] M. S. Mahmoud, Switched discrete-time systems with time-varying delays: A generalized H_2 approach , J. Computers and Mathematics with Applications., vol. 57, no. 1, January 2009, pp. 7995.
[21] M. S. Mahmoud, Generalized H_2 control of switched discrete-time systems with unknown delays, Applied Mathematics and Computation, vol. 311, no. 1, May 2009, pp. 3344.
[22] M. S. Mahmoud, A. W. Saif and P. Shi, Stabilization of Linear Switched Delay Systems: H_2 and H_∞ Methods", J. Optimization Theory and Applications, vol. 142, no. 3, September 2009, pp. 583-601.
[23] M. S. Mahmoud, F. M. AL-Sunni and Y. Shi, Switched Discrete-Time Delay Systems: Delay-Dependent Analysis and Synthesis, J. Circuits, Systems and Signal Processing, vol. 28, no. 5, September 2009, pp. 735-761.
[24] X. Li and H. Gao, A new model transformation of discrete-time systems with time-varying delay and its application to stability analysis. IEEE Trans. Autom. Control, 2011, 56, pp. 2172-2178.
[25] H. Gao and X. Li, H_∞ filtering for discrete-time state-delayed systems with finite frequency specifications. IEEE Trans. Autom. Control, 2011, 56, pp. 2935-2941.

G. Rajchakit received the Ph.D. degrees in Applied Mathematics at the King's Mungkut University of Thonburi, Bangkok, Thailand in 2008. Currently, he is a lecturer at the Department of Mathematics, Maejo University, Chiang Mai, Thailand. His research interests include stability and stabilization of dynamical systems, qualitative theory of differential/discrete-time systems.