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Improving Numerical Solutions for the Generalized Huxley Equation: The New Iterative Method (NIM)

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Abstract: The focus of our research was to address the generalized Huxley equation using the recently developed iterative method called the new iterative method (NIM). Our study entailed a comprehensive investigation of the convergence characteristics of the NIM. Additionally, we compared the outcomes obtained from the NIM with other established iterative techniques, including the variational iteration method (VIM) and Adomian decomposition method (ADM), as well as the exact solution.

Keywords: The new iterative method (NIM),Generalized Huxley equation, Variational iteration method (VIM), Adomian decomposition method (ADM).

1 Introduction

Nonlinear systems are ubiquitous in various scientific and engineering fields, from physics and chemistry to biology and economics. They arise when a system's output is not proportional to its input, leading to complex and often chaotic dynamics. The behavior of these systems can be modeled mathematically using nonlinear models, which can capture their intricate behavior more accurately than linear models. However, solving real-life nonlinear models is often challenging both theoretically and numerically, as they may not have explicit solutions or may require complex algorithms to solve.

To make nonlinear models tractable, researchers often resort to making simplifying assumptions, which can reduce the complexity of the problem and make it easier to analyze. These assumptions may include approximations, neglecting small terms, reducing the number of variables, or assuming symmetry or periodicity. However, these simplifications can lead to inaccurate results or may obscure important features of the system's behavior. In some cases, the assumptions made may render the model irrelevant to real-world applications.

Despite the challenges of modeling and analyzing nonlinear systems, significant progress has been made in recent years. Various techniques, such as perturbation methods, numerical simulations, bifurcation analysis, and chaos theory, have been developed to analyze and solve nonlinear models. Moreover, recent advances in machine learning and artificial intelligence have shown promising results in solving complex systems while maintaining accuracy. These new methods offer exciting opportunities for solving challenging problems in various fields [1,2,3,4,5,6,7,8,9,10].

The introduction of a new mathematical approach called the new iterative method (NIM) by Daftardar-Gejji and Jafari [11] has enabled the solution of both linear and nonlinear functional equations. The NIM technique has demonstrated its effectiveness in solving various types of nonlinear equations, including algebraic, integral, ordinary and partial differential equations of both fractional and integer order. Compared to other established methods such as ADM [13], HPM [14], and VIM [15], NIM is simple to understand and implement using computer software. Research has indicated that NIM delivers superior results [12].

The generalized Huxley equation,

$$u_t - u_{xx} = \beta u(1-u^\delta)(u^\delta - \gamma), \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (1)$$

where the initial condition is defined as follows:

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{1/\delta}, \quad (2)$$

illustrate the propagation of nerve impulses in nerve fibres and the movement of liquid crystals. Wang et al. derived

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the exact solution to the equation using nonlinear transformations, as outlined in their paper [16]:

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left\{\sigma\gamma\left(x + \left\{\frac{\alpha\rho}{2(1+\delta)}\right\}t\right)\right\}\right]^{1/\delta} \quad (3)$$

where $\sigma = \delta\rho/4(1+\delta)$, $\alpha = 1+\delta-\gamma$ and $\rho = \sqrt{4\beta(1+\delta)}$.

Several methods have been utilized for obtaining approximate solutions of the generalized Huxley equation. For example, Hashim et al. [17] implemented the Adomian decomposition method, Hashemi et al. [18] implemented both the HPM and the ADM, Batiha et al. [19] discussed the use of the variational iteration method, and Hemida and Mohamed [20] developed a scheme based on the homotopy analysis method to approximate solutions to the equation.

The new iterative method (NIM) was utilized to derive an analytical solution for the generalized Huxley equation, as outlined in this article. The accuracy of NIM was assessed by comparing its results with those of other iterative methods, including VIM and ADM, as well as the exact solution. The outcomes obtained via NIM were discovered to be consistent and closely aligned with those obtained using ADM, VIM, and the exact solution.

2 The new iterative method (NIM)

In this section, the NIM numerical method will be outlined as follows [21,22,23,24]:

$$u = f + L(u) + N(u), \quad (4)$$

In the equation above, f is a known function, and L and N are linear and nonlinear operators, respectively.
The NIM solution for Eq. (4) has the form

$$u = \sum_{i=0}^{\infty} u_i. \quad (5)$$

Since L is linear then

$$L\left(\sum_{i=0}^{\infty} u_i\right) = \sum_{i=0}^{\infty} L(u_i). \quad (6)$$

The nonlinear operator N in Eq. (4) is decomposed as below

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}.$$
$$= \sum_{i=0}^{\infty} A_i, \quad (7)$$

where

$$A_0 = N(u_0)$$
$$A_1 = N(u_0+u_1) - N(u_0)$$
$$A_2 = N(u_0+u_1+u_2) - N(u_0+u_1)$$
$$\vdots$$
$$A_i = \left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\}, \quad i \geq 1.$$

Using Eqs.(5), (6) and (7) in Eq. (4), we get

$$\sum_{i=0}^{\infty} u_i = f + \sum_{i=0}^{\infty} L(u_i) + \sum_{i=0}^{\infty} A_i. \quad (8)$$

The solution of Eq. (4) can be expressed as

$$u = \sum_{i=0}^{\infty} u_i = u_0 + u_1 + u_2 + \ldots + u_n + \ldots, \quad (9)$$

where

$$u_0 = f$$
$$u_1 = L(u_0) + A_0$$
$$u_2 = L(u_1) + A_1$$
$$\vdots$$
$$u_n = L(u_{n-1}) + A_{n-1}$$
$$\vdots \quad (10)$$

Algorithm

$$\begin{aligned}
INPUT\ &: Read\ M(Number\ of\ iterations);\\
&Read\ L(u);\ N(u);\ f\\
&Step-1\ : u_{-1} = 0,\ u_0 = f\\
&Step-2:\ For(n=0,\ n \leq M, n++)\\
&\{\\
&Step-3: A_n = f(u_n) - f(u_{n-1});\\
&Step-4: u_{n+1} = f + L(u_n) + A_n;\\
&Step-5: u = u_{n+1}\\
&\}\ end\\
OUTPUT &: u
\end{aligned} \quad (11)$$

3 The convergence of the NIM

Theorem 1: For any n and for some real $L > 0$ and $||u_i|| \leq M < \frac{1}{e}, i = 1,2,\ldots,$ if N is $C^{(\infty)}$ in the neighborhood of u_0 and $||N^{(n)}(u_0)|| \leq L$, then $\sum_{n=0}^{\infty} H_n$ is convergent absolutely and $||H_n|| \leq LM^n e^{n-1}(e-1),\ n = 1,2,\ldots.$

Proof:

$$
\begin{aligned}
||H_n|| &\leq LM^n \sum_{i_n=1}^{\infty} \sum_{i_{n-1}=0}^{\infty} \cdots \sum_{i_1=0}^{\infty} \left(\prod_{j=1}^{n} \frac{1}{i_j!} \right) \\
&= LM^n e^{n-1}(e-1). \qquad (12)
\end{aligned}
$$

Thus the series $\sum_{n=1}^{\infty} ||H_n||$ is dominated by the convergent series $LM(e-1)\sum_{n=1}^{\infty}(Me)^{n-1}$, where $M < 1/e$. Hence, $\sum_{n=0}^{\infty} H_n$ is absolutely convergent, due to the comparison test.

As it is difficult to show boundedness of u_i, for all i, a more useful result is provedin the following theorem, where conditions on $N^{(k)}(u_0)$ are given which are sufficient to guarantee convergence of the series.

Theorem 2: The series $\sum_{n=0}^{\infty} H_n$ is convergent absolutely if N is $C^{(\infty)}$ and $||N^{(n)}(u_0)|| \leq M \leq e^{-1}, \forall n$.

Proof: Consider the recurrence relation

$$\varepsilon_n = \varepsilon_0 exp(\varepsilon_{n-1}), \;\; n = 1, 2, 3, ..., \qquad (13)$$

where $\varepsilon_0 = M$. Define $\eta_n = \varepsilon_n - \varepsilon_{n-1}, n = 1, 2, 3, \cdots$. We observe that

$$||H_n|| \leq \eta_n, \;\; n = 1, 2, 3, \cdots. \qquad (14)$$

Let

$$\sigma_n = \sum_{i=1}^{n} \eta_i = \varepsilon_n - \varepsilon_0. \qquad (15)$$

Not that $\varepsilon_0 = e^{-1} > 0$, $\varepsilon_1 = \varepsilon_0 exp(\varepsilon_0) > \varepsilon_0$ and $\varepsilon_2 = \varepsilon_0 exp(\varepsilon_1) > \varepsilon_0 exp(\varepsilon_0) = \varepsilon_1$. In general, $\varepsilon_n > \varepsilon_{n-1} > 0$. Hence $\sum \eta_n$ is a series of positive real numbers. Note that

$$
\begin{gathered}
0 < \varepsilon_0 = M = e^{-1} < 1, \\
0 < \varepsilon_1 = \varepsilon_0 exp(\varepsilon_0) < \varepsilon_0 e^1 = e^{-1} e^1 = 1, \qquad (16) \\
0 < \varepsilon_2 = \varepsilon_0 exp(\varepsilon_1) < \varepsilon_0 e^1 = 1.
\end{gathered}
$$

In general $0 < \varepsilon_n < 1$. Hence, $\sigma = \varepsilon_n - \varepsilon_0 < 1$. This implies that $\{\sigma_n\}_{n=1}^{\infty}$ is bounded above by 1, and hence convergent. Therefore, $\sum H_n$ is absolutely convergent by comparison test.

4 Discussion of Numerical Results

This section applies NIM to obtain the solution of the generalized Huxley equation. We will perform integration of equation (1) and utilize equation (2) to derive the solution of the generalized Huxley equation (1) based on the specified initial condition.

$$
\begin{aligned}
u = &\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{1/\delta} \\
&+ \int_0^t \left[u_{xx} + (\beta u)\left(1 - u^{\delta}\right)\left(u^{\delta} - \gamma\right)\right] dt \qquad (17)
\end{aligned}
$$

By using algorithm (11) we obtain:

$$u_0 = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{1/\delta} \qquad (18)$$

$$
\begin{aligned}
u_1 = &\beta 2^{-\frac{1}{\delta}-2} t(\gamma(\tanh(\frac{M}{2N}) + 1))^{1/\delta} \\
&(4(\gamma+1)((\frac{\gamma}{e^{-\frac{M}{N}}+1})^{1/\delta})^{\delta} \\
&- 4((\frac{\gamma}{e^{-\frac{M}{N}}+1})^{1/\delta})^{2\delta} \\
&+ \frac{\gamma(2\gamma - 4(\delta+1) - \gamma(\delta + \sinh(\frac{M}{N}) + 1)\text{sech}^2(\frac{M}{2N}))}{\delta+1}), \qquad (19)
\end{aligned}
$$

where $M = \beta\gamma\delta x$, and $N = \sqrt{\beta(\delta+1)}$.

So,

$$\sum_{i=0}^{1} u_i = u_0 + u_1. \qquad (20)$$

By utilizing computer algebra software such as Mathematica, it is easy to obtain the remaining components required for the repetition formula.

To compare the precision and effectiveness of the new iterative method (NIM) with the ADM [17] and the VIM for solving Eq. (1), we will utilize the same parameter values for the generalized Huxley equation (1) as those used in [17]. By doing so, we aim to demonstrate the superiority of NIM over ADM and VIM for the same equation.

Numerical comparisons of the results obtained using NIM, VIM, ADM, and exact solutions for the case where $\beta = 1$, $\gamma = 0.001$, and $\delta = 1, 2$, and 3 are presented in Tables 1–3. The results demonstrate that NIM outperforms VIM and ADM in terms of efficiency.This is because NIM obviates the requirement to calculate Adomian polynomials, which can pose difficulties in certain circumstances.

5 Conclusion

The successful application of the new iterative method (NIM) to solve the generalized Huxley equation is presented in this paper. The solution obtained through NIM is presented as a series, and the accuracy of this solution is compared with the ADM, VIM, and exact solutions. The results of the comparison demonstrate that NIM is both efficient and reliable in solving partial differential equations. In fact, the use of NIM offers a straightforward method for finding highly accurate solutions to such equations. The implications of these findings are significant, as they provide researchers and practitioners with a promising new tool for tackling challenging mathematical problems in a variety of fields, including physics, engineering, and finance, among

Table 1: Numerical and exact solutions at $\gamma = 0.001$, $\beta = 1$, and $\delta = 1$.

x	t	Exact	NIM	ADM [17]	VIM [19]
0.1	0.05	0.000500030171	0.000500005184	0.000500005184	0.000500005184
	0.1	0.000500042665	0.000499992695	0.000499992690	0.000499992690
	1	0.000500267553	0.000499767825	0.000499767803	0.000499767803
0.5	0.05	0.000500100882	0.000500075894	0.000500075895	0.000500075895
	0.1	0.000500113376	0.000500063402	0.000500063401	0.000500063401
	1	0.000500338263	0.000499838513	0.000499838513	0.000499838513
0.9	0.05	0.000500171593	0.000500146606	0.000500146605	0.000500146605
	0.1	0.000500184087	0.000500134112	0.000500134111	0.000500134111
	1	0.000500408974	0.000499909228	0.000499909224	0.000499909224

Table 2: Numerical exact solutions at $\gamma = 0.001$, $\beta = 1$, and $\delta = 2$.

x	t	Exact	NIM	ADM [17]	VIM [19]
0.1	0.05	0.0223618841	0.0223607666	0.0223607664	0.0223607664
	0.1	0.0223624429	0.0223602077	0.0223602076	0.0223602076
	1	0.0223724988	0.0223501490	0.0223501462	0.0223501490
0.5	0.05	0.0223644658	0.0223633485	0.0223633483	0.0223633483
	0.1	0.0223650245	0.0223627896	0.0223627895	0.0223627895
	1	0.0223750792	0.0223527299	0.0223527292	0.0223527320
0.9	0.05	0.0223670472	0.0223659299	0.0223659298	0.0223659298
	0.1	0.0223676058	0.0223653713	0.0223653711	0.0223653711
	1	0.0223776594	0.0223553125	0.0223553120	0.0223553148

Table 3: Numerical and exact solutions at $\gamma = 0.001$ $\beta = 1$, and $\delta = 3$.

x	t	Exact	NIM	ADM [17]	VIM [19]
0.1	0.05	0.0793740204	0.0793700536	0.0793700531	0.0793700531
	0.1	0.0793760039	0.0793680697	0.0793680693	0.0793680695
	1	0.0794116901	0.0793323441	0.0793323439	0.0793323637
0.5	0.05	0.0793819558	0.0793779900	0.0793779893	0.0793779894
	0.1	0.0793839389	0.0793760061	0.0793760059	0.0793760061
	1	0.0794196179	0.0793402882	0.0793402876	0.0793403074
0.9	0.05	0.0793898897	0.0793859242	0.0793859239	0.0793859240
	0.1	0.0793918724	0.0793839413	0.0793839409	0.0793839411
	1	0.0794275442	0.0793482315	0.0793482298	0.0793482496

others. With its demonstrated effectiveness and simplicity, NIM is sure to become a valuable asset in the arsenal of numerical methods available for solving partial differential equations.

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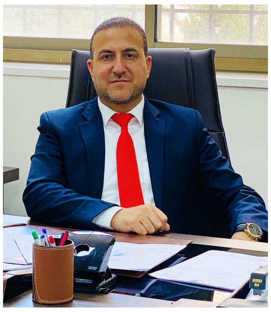

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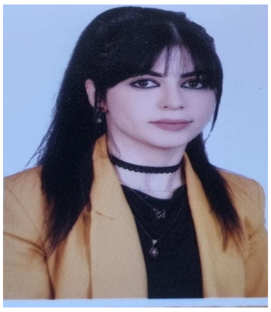

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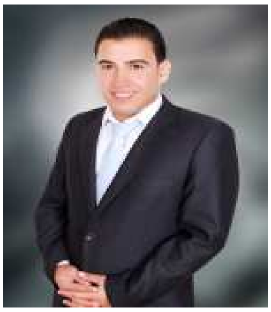

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