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Weighted Amarendra Distribution: Properties and Applications to Model Real-Life Data

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Abstract: In this study, a new generalization of Amarendra distribution has been proposed. The new distribution is called the weighted Amarendra distribution. Some statistical properties of the distribution are obtained. These include survival function, hazard function, reverse hazard rate, moments, moment generating function, order statistics, entropies, Bonferroni and Lorenz curves. The maximum likelihood technique is used to estimate the parameters of the model. Finally, the usefulness and application of the distribution has been demonstrated by the two set of real-life data.

Keywords: Weighted technique, Moments, Reliability Properties, Order Statistics, Entropies, Parameter Estimation.

1 Introduction

Distribution fitting has been of great importance in different areas of science. It is used to select a statistical distribution which best describes the fits of a data set generated by some random process. The advantage of distribution fitting to data is that it allows for the development of valid models of random process and by so doing gives possibilities for proper predictions of future occurrences which may help in making better decisions. In most practical situations, some of the well-known statistical distributions do not fit some real life data adequately and one is led to seek modifications of existing statistical distributions that will fit different real-life data more appropriately. Over the years, researchers have been developing new statistical distributions capable of providing more flexible models for many real-life phenomena. One of the trending methods for generating new statistical distribution is the weighted method.

The concept of weighted distributions came into limelight when Fisher [1] studied how methods of ascertainment can affect the form of distributions of recorded observations. Weighted distributions take into account the method of ascertainment by adjusting the probabilities of the actual occurrences of the events to arrive at a specification of those events as observed and

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recorded. Weighted distributions occur frequently in studies related to reliability, survival analysis, biomedicine, ecology and other many areas Stene [2] and Oluyede and George [3]. Rao [4] introduced a unified concept of weighted distribution and identified various sampling situations that can be modeled by weighted distributions. Deusen [5] discussed a Size-biased distribution theory independently and fit it to the distributions of diameter of breast height (DBH). Lappi and Bailey [6], used weighted distributions to analyse the HPS diameter increment data.

In fisheries, there are various good sources which provide the detailed description of weighted distributions. Different authors have reviewed and studied the various weighted probability models and illustrated their applications in diverse fields. Zelen [7] introduced the concept of length-biased sampling. Patil and Ord [8] presented and examined size biased sampling and related invariant weighted distributions. Maryam et al. [9] proposed a weighted Devya distribution along with properties and application that will help to model data sets in practice. Rather and Subramanian [10] discussed on weighted Sushila distribution with properties and applications which shows more flexibility then the subject Distribution. Weighted Lomax distribution with properties and application to data were presented and studied by Kilany [11]. Again Rather and Subramanian [12] studied the weighted version of Akshaya distribution with applications in engineering science. Recently, Rather and Ozel [13] discussed the weighted power lindley distribution with applications on the life time data. Statistical applications of weighted distributions related to human population and ecology was studied by Patil and Rao [14].

Amarendra distribution was introduced by Shanker [15] for modeling different types of data. Its various statistical properties such as mean, variance, coefficient of variation, skewness and kurtosis have been obtained. The parameters of the distribution have been investigated by maximum likelihood technique. The applicability and the goodness of fit of the Amarendra distribution over one parameter Akash, Shanker, Sujatha, Lindley and exponential distributions have been illustrated with two real lifetime data sets from medical science and engineering fields.

2 Weighted Amarendra Distribution: Definition and Properties

The probability density function (pdf) of the Amarendra distribution is given by

$$f(x;\theta)=\frac{\theta^4}{\theta^3+\theta^2+2\theta+6}\left(1+x+x^2+x^3\right)e^{-\theta x};x>0,\theta>0 \quad (1)$$

and the cumulative distribution function (cdf) of the Amarendra distribution is given by

$$F(x;\theta)=1-\left[1+\frac{\theta^3x^3+\theta^2(\theta+2)x^2+\theta(\theta^3+\theta^2+2\theta+6)\,x}{(\theta^3+\theta^2+2\theta+6)}\right]e^{-\theta x}\;;x>0,\ \theta>0$$

Suppose X is a non- negative random variable with probability density function $f(x)$. Let $w(x)$ be the non-negative weight function, then the probability density function of the weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))};\ x\geq 0 \quad (2)$$

Where $w(x)$ be a non-negative weight function.

$$E(w(x)) = \int w(x) f(x) < \infty,\ 0 < E(w(x)) < \infty.$$

For different weighted models, we have different choices of the weight function $w(x)$.when x^c the resulting distribution is termed as weighted distribution. In this paper, we have to find the weighted version of Amarendra distribution, in weights $w(x) = x^c$, in order to get the weighted Amarendra distribution and its probability density function (pdf) is given as

$$f_w(x) = \frac{x^c f(x)}{E(x^c)} \tag{3}$$

where,

$$E(x^c) = \int_0^\infty x^c f_w(x; c, \theta)\, dx \tag{4}$$

$$E(x^c) = \frac{\theta^5 c! + (c+1)!\theta^4 + (c+2)!\theta^3 + (c+3)!\theta^2}{\theta^{c+4}}$$

Substitute, the values of equation (1) and (4) in equation (3), we will get the probability density function (pdf) of weighted Amarendra distribution.

$$f_w(x; c, \theta) = \frac{\theta^{c+4}}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} x^c (1 + x + x^2 + x^3) e^{-\theta x} \quad ; x > 0,\ \theta > 0,\ c > 0 \tag{5}$$

Now, the cumulative distribution function (cdf) of the weighted Amarendra distribution is obtained as

$$F_w(x; c, \theta) = \int_0^x f_w(x; c, \theta) dx$$

$$F_w(x; c, \theta) = \int_0^x \frac{\theta^{c+4}}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} x^c (1 + x + x^2 + x^3) e^{-\theta x}\, dx$$

After simplification, we get the cumulative distribution function (cdf) of weighted Amarendra distribution as

$$F_w(x; c, \theta) = \frac{\theta^3 \gamma(c+1; \theta x) + \left(\frac{1}{\theta}\right) \gamma(c+2; \theta x) + \left(\frac{1}{\theta^2}\right) \gamma(c+3; \theta x) + \left(\frac{1}{\theta^3}\right) \gamma(c+4; \theta x)}{\theta^3 c! + \theta^2 (c+1)! + (c+2)!\theta + (c+3)!} \tag{6}$$

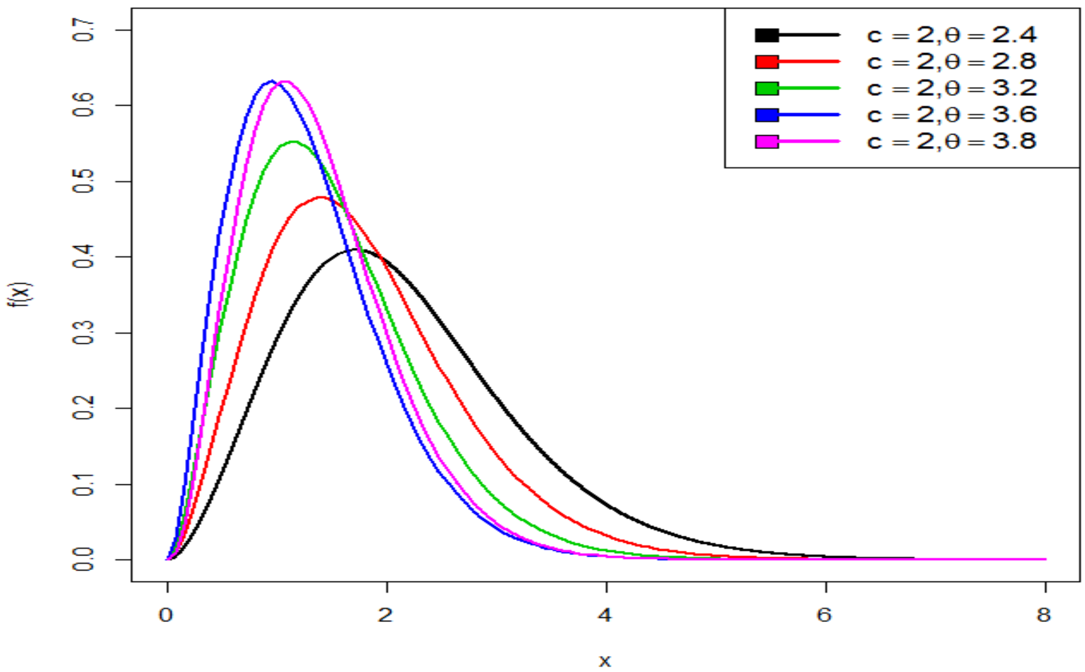

Fig. 1:pdf plot of Weighted Amarendra Distribution.

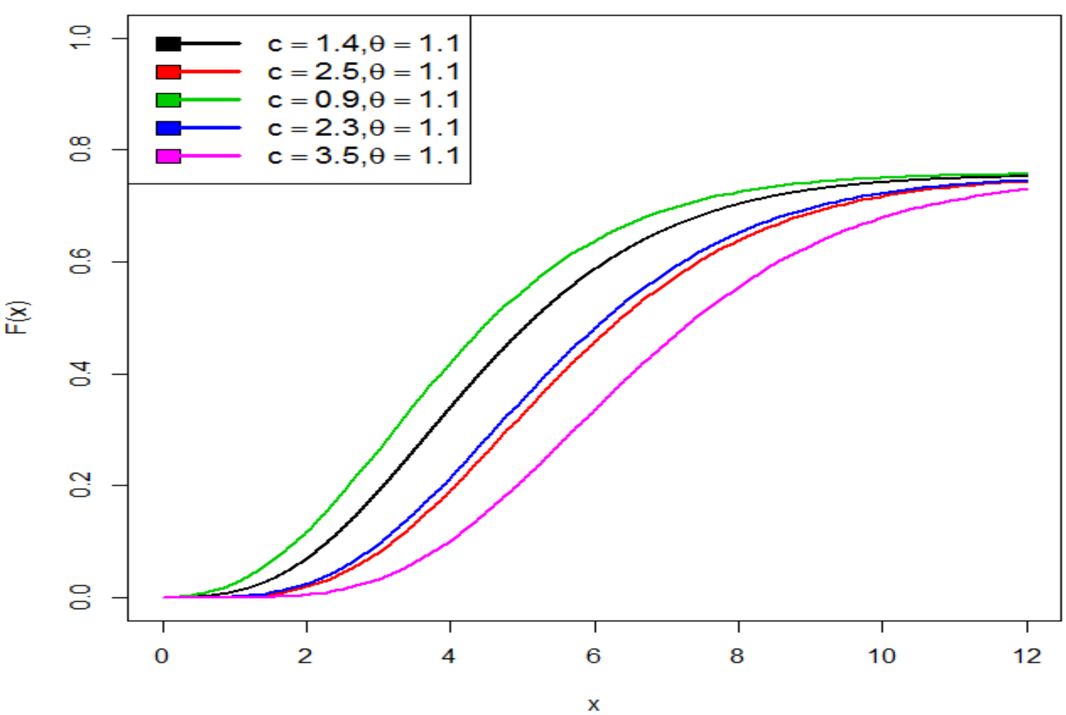

Fig. 2: cdf plot of Weighted Amarendra Distribution.

3 Reliability Properties

In this section, we have obtained the reliability function, failure rate, reverse hazard rate of the weighted Amarendra distribution. The survival function or the reliability function of the weighted Amarendra distribution is obtained as

$$R(x;c,\theta)=1-F_w(x;c,\theta)$$

$$R_w(x;c,\theta)=1-\frac{\theta^3\gamma\left(c+1;\theta x\right)+\left(\frac{1}{\theta}\right)\gamma\left(c+2;\theta x\right)+\left(\frac{1}{\theta^2}\right)\gamma\left(c+3;\theta x\right)+\left(\frac{1}{\theta^3}\right)\gamma\left(c+4;\theta x\right)}{\theta^3 c!+\theta^2(c+1)!+(c+2)!\theta+(c+3)!}\quad ;\ x>0,\ \theta>0, c>0$$

The corresponding hazard function or failure rate can beobtained as

$$h(x;c,\theta)=\frac{f_w(x;c,\theta)}{R(x;c,\theta)}$$

$$h(x;c,\theta)=\frac{f_w(x;c,\theta)}{1-F_w(x;c,\theta)}$$

$$h(x;c,\theta)=\frac{\theta^{c+4}x^c\left(1+x+x^2+x^3\right)e^{-\theta x}}{\left(\theta^3 c!+\theta^2(c+1)!+(c+2)!\theta+(c+3)!\right)-\theta^3\left[\gamma\left(c+1;\theta x\right)+\left(\frac{1}{\theta}\right)\gamma\left(c+2;\theta x\right)+\left(\frac{1}{\theta^2}\right)\gamma\left(c+3;\theta x\right)+\left(\frac{1}{\theta^3}\right)\gamma\left(c+4;\theta x\right)\right]}$$

The Reverse hazard rate of WAD defined in (5) can be calculated as

$$h_r(x;c,\theta)=\frac{f_w(x;c,\theta)}{F_w(x;c,\theta)}$$

$$h_r(x;c,\theta)=\frac{\theta^{c+4}x^c(1+x+x^2+x^3)e^{-\theta x}}{\left[\gamma\left(c+1;\theta x\right)+\left(\frac{1}{\theta}\right)\gamma\left(c+2;\theta x\right)+\left(\frac{1}{\theta^2}\right)\gamma\left(c+3;\theta x\right)+\left(\frac{1}{\theta^3}\right)\gamma\left(c+4;\theta x\right)\right]}$$

The Mills ratio of the weighted Amarendra distribution is defined in (5) can be calculated as

$$Mills\ Ratio=\frac{1}{h_r(x;c,\theta)}=\frac{\left[\gamma\left(c+1;\theta x\right)+\left(\frac{1}{\theta}\right)\gamma\left(c+2;\theta x\right)+\left(\frac{1}{\theta^2}\right)\gamma\left(c+3;\theta x\right)+\left(\frac{1}{\theta^3}\right)\gamma\left(c+4;\theta x\right)\right]}{\theta^{c+4}x^c(1+x+x^2+x^3)e^{-\theta x}}$$

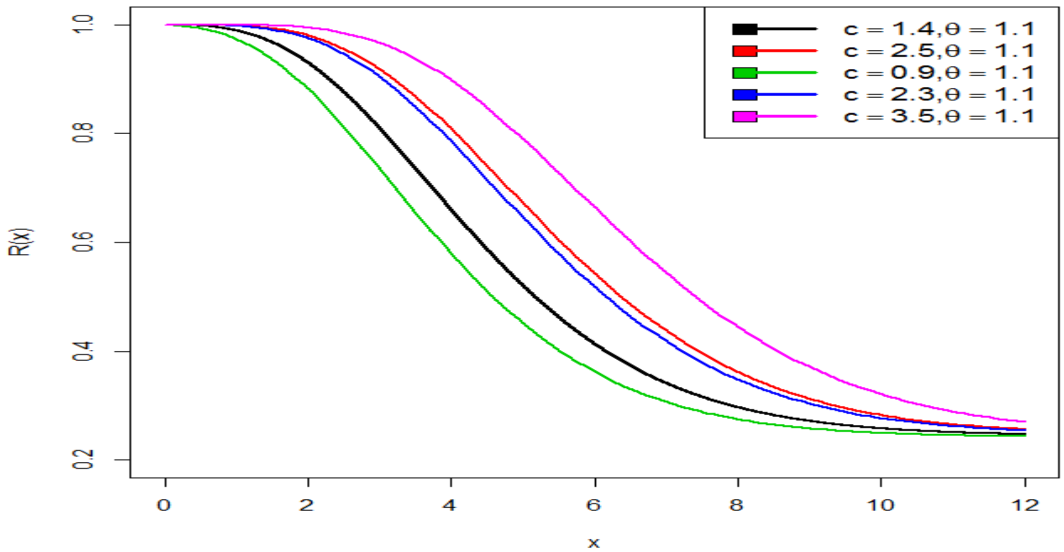

Fig.3: Survival plot of Weighted Amarendra Distribution.

4 Statistical Properties

In this section, some properties of the proposed distribution are discussed.

4.1 Moments

Moments are very important and play an essential role in statistical analysis. If a random variable X has the pdf given by the equation (5) then the r^{th} moment is given by

$$E(X^r) = \mu_r' = \int_0^\infty x^r f_w(x;c,\theta)\,dx$$

$$E(X^r) = \mu_r' = \int_0^\infty x^r \frac{\theta^{c+4}}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} x^c (1+x+x^2+x^3) e^{-\theta x} dx$$

$$E(X^r) = \left(\frac{\theta^3 (c+r)! + (c+r+1)!\theta^2 + (c+r+2)!\theta + (c+r+3)!}{\theta^r \left(\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!\right)} \right) \qquad (7)$$

Putting $r = 1$ in equation (7), we get the mean of weighted Amarendra distribution which is given by

$$\mu_1' = \left(\frac{\theta^3 (c+1)! + (c+2)!\theta^2 + (c+3)!\theta + (c+4)!}{\theta \left(\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!\right)} \right)$$

and putting r = 2 in equation (7), we get second moment as

$$\mu_2' = \left(\frac{\theta^3(c+2)!+(c+3)!\theta^2+(c+4)!\theta+(c+5)!}{\theta^2\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right)$$

Variance = $\mu_2' - (\mu_1')^2$

$$\mu_2 = \left(\frac{\theta^3(c+2)!+(c+3)!\theta^2+(c+4)!\theta+(c+5)!}{\theta^2\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right) - \left(\frac{\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!}{\theta\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right)^2$$

$$\mu_2 = \left(\frac{\left(\theta^3(c+2)!+(c+3)!\theta^2+(c+4)!\theta+(c+5)!\right)\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right) - \left(\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!\right)^2}{\theta^2\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)^2}\right)$$

$$S.D(\sigma) = \sqrt{\left(\frac{\left(\theta^3(c+2)!+(c+3)!\theta^2+(c+4)!\theta+(c+5)!\right)\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right) - \left(\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!\right)^2}{\theta^2\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)^2}\right)}$$

4.2 Harmonic Mean

The Harmonic mean of the Weighted Amarendra distribution can be obtained as

$$H.M = E\left(\frac{1}{x}\right)$$

$$H.M = \int_0^\infty \frac{1}{x} f_w(x;c,\theta)\,dx$$

$$H.M = \int_0^\infty \frac{1}{x}\frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}x^c(1+x+x^2+x^3)e^{-\theta x}\,dx$$

$$H.M = \frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\int_0^\infty \left(1+x+x^2+x^3\right)x^{c-1}e^{-\theta x}\,dx$$

$$H.M=\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}\int_{0}^{\infty}x^{c-1}e^{-\theta x}dx+\int_{0}^{\infty}x^{c}e^{-\theta x}dx+\int_{0}^{\infty}x^{c+1}e^{-\theta x}dx+\int_{0}^{\infty}x^{c+2}e^{-\theta x}dx$$

On simplification we get,

$$H.M=\frac{\theta^{4}(\theta^{3}\Gamma c+\theta^{2}\Gamma c+1+\theta\Gamma c+2+\Gamma c+3}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}$$

4.3 Moment Generating Function and Characteristic Function

Let X have a Weighted Amarendra distribution, then the MGF of $X$is given by

$$M_{X}(t)=E(e^{tx})=\int_{0}^{\infty}e^{tx}f_{w}(x;c,\theta)dx$$

$$M_{X}(t)=\int_{0}^{\infty}e^{tx}\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}x^{c}(1+x+x^{2}+x^{3})e^{-\theta x}dx$$

$$M_{X}(t)=\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}\int_{0}^{\infty}e^{-(\theta-t)x}x^{c}(1+x+x^{2}+x^{3})dx$$

$$M_{X}(t)=\left\{\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}\left[\int_{0}^{\infty}x^{c}e^{-(\theta-t)x}xdx+\int_{0}^{\infty}e^{-(\theta-t)x}x^{c+1}dx+\int_{0}^{\infty}e^{-(\theta-t)x}x^{c+2}dx+\int_{0}^{\infty}e^{-(\theta-t)x}x^{c+3}dx\right]\right\}$$

$$M_{X}(t)=\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}\left[\frac{\Gamma c+1}{(\theta-t)^{c+1}}+\frac{\Gamma c+2}{(\theta-t)^{c+2}}+\frac{\Gamma c+3}{(\theta-t)^{c+3}}+\frac{\Gamma c+4}{(\theta-t)^{c+4}}\right]$$

Using binomial expansion: $\frac{1}{(1-x)^{s}}=\sum_{k=0}^{\infty}\binom{s+k-1}{k}x^{k}$; $\binom{n}{k}=\frac{n!}{(n-k)!k!}$

$$M_{X}(t)=\frac{\theta^{c+4}}{\theta^{3}c!+(c+1)!\theta^{2}+(c+2)!\theta+(c+3)!}$$

$$\left(\frac{c!}{\theta^{c+1}}\sum_{k=0}^{\infty}\binom{c+k}{k}+\frac{(c+1)!}{\theta^{c+2}}\sum_{k=0}^{\infty}\binom{c+k+1}{k}+\frac{(c+2)!}{\theta^{c+3}}\sum_{k=0}^{\infty}\binom{c+k+2}{k}+\frac{(c+3)!}{\theta^{c+4}}\sum_{k=0}^{\infty}\binom{c+k+3}{k}\right)\left(\frac{t}{\theta}\right)^{k}$$

On simplification we get

$$M_X(t)=\sum_{k=0}^{\infty}\frac{(c+k)!}{\theta^k}\left(\frac{\theta^3+\theta^2(c+k+1)+\theta(c+k+1)(c+k+2)+(c+k+1)(c+k+2)(c+k+3)}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)\frac{t^k}{k!}$$

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\,\mu_r'$$

$$\mu_r'=\frac{(c+r)!}{\theta^r}\left(\frac{\theta^3+\theta^2(c+r+1)+\theta(c+r+1)(c+r+2)+(c+r+1)(c+r+2)(c+r+3)}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)$$

$$\mu_1'=\frac{(c+1)!}{\theta}\left(\frac{\theta^3+\theta^2(c+2)+\theta(c+2)(c+3)+(c+2)(c+3)(c+4)}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)$$

$$\mu_2'=\frac{(c+2)!}{\theta^2}\left(\frac{\theta^3+\theta^2(c+3)+\theta(c+3)(c+4)+(c+3)(c+4)(c+5)}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)$$

Similarly, we obtain the characteristic function of the weighted Amarendra Distribution and is defined as

$$\varphi_X(t)=M_X(it)$$

$$\varphi_X(t)=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\,\mu_r'$$

$$M_X(it)=\sum_{r=0}^{\infty}\frac{(c+r)!}{\theta^r}\left(\frac{\theta^3+\theta^2(c+r+1)+\theta(c+r+1)(c+r+2)+(c+r+1)(c+r+2)(c+r+3)}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)\frac{(it)^r}{r!}$$

5 Order Statistics

Order statistics are among the essential tools in inferential and non-parametric statistics. The applicability of the order statistics is in the field of reliability and life testing. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a random sample $X_1, X_2, \ldots, X_n$ drawn from the continuous population with probability density function $f_X(x)$ and cumulative density function with $F_X(x)$, then the pdf of r^{th} order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-r)!}f_X(x)\left[F_X(x)\right]^{r-1}\left[1-F_X(x)\right]^{n-r}\,(8)$$

Using the equations (5) and (6) in equation (8), the probability density function of r^{th}order statistics $X_{(r)}$of Weighted Amarendra distribution is given by

$$f_{X(r)}(x)=\frac{n!}{(r-1)!(n-1)!}\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}x^c(1+x+x^2+x^3)e^{-\theta x}\right)$$

$$\left(\frac{\theta^3}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\left(\gamma(c+1;\theta x)+\left(\frac{1}{\theta}\right)\gamma(c+2;\theta x)+\left(\frac{1}{\theta^2}\right)\gamma(c+3;\theta x)+\left(\frac{1}{\theta^3}\right)\gamma(c+4;\theta x)\right)\right)^{r-1}$$

$$\left(1-\left(\frac{\theta^3\left[\gamma(c+1;\theta x)+\left(\frac{1}{\theta}\right)\gamma(c+2;\theta x)+\left(\frac{1}{\theta^2}\right)\gamma(c+3;\theta x)+\left(\frac{1}{\theta^3}\right)\gamma(c+4;\theta x)\right]}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)\right)^{n-r}$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ can be obtained as

$$f_{X(n)}(x)=n\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}x^c(1+x+x^2+x^3)e^{-\theta x}\right)$$

$$\left(\frac{\theta^3\left[\gamma(c+1;\theta x)+\left(\frac{1}{\theta}\right)\gamma(c+2;\theta x)+\left(\frac{1}{\theta^2}\right)\gamma(c+3;\theta x)+\left(\frac{1}{\theta^3}\right)\gamma(c+4;\theta x)\right]}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^{n-1}$$

and the pdf of *1st* order statistic $X_{(1)}$ can be obtained as

$$f_{X(1)}(x)=\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}x^c(1+x+x^2+x^3)e^{-\theta x}\right)\times$$

$$\left(1-\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\left[\gamma(c+1;\theta x)+\left(\frac{1}{\theta}\right)\gamma(c+2;\theta x)+\left(\frac{1}{\theta^2}\right)\gamma(c+3;\theta x)+\left(\frac{1}{\theta^3}\right)\gamma(c+4;\theta x)\right]\right)\right)^{n-1}$$

6 Entropy Measures

Entropy has been used in many situations in science and technology. Entropy of a random variable X is a measure of variation of the uncertainty. There exist many entropy definitions and they are not equally good in applications. While the most famous Shannon entropy which quantifies the encoding length is extremely useful in information theory.

6.1Renyi Entropy

The Renyi entropy [16] is important in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\int f^{\gamma}(x;c,\theta)dx\right)$$

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\int_0^{\infty}\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}x^c(1+x+x^2+x^3)e^{-\theta x}\right)^{\gamma}dx\right)$$

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^{\gamma}\int_0^{\infty}\left(\left(x^c(1+x+x^2+x^3)\right)^{\gamma}e^{-\gamma\theta x}\right)dx\right)$$

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^{\gamma}\int_0^{\infty}\left(x^{\gamma c}(1+x+x^2+x^3)^{\gamma}e^{-\gamma\theta x}\right)dx\right)$$

Using binomial expansion to above equation we get,

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\gamma}{i}\binom{i}{j}\binom{j}{k}\int_0^{\infty}x^{(\gamma c+i+j+k+1)-1}e^{-\beta\theta x}dx$$

$$e(\gamma)=\frac{1}{1-\gamma}\log\left(\frac{\theta^{c+4}}{\theta^3c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^{\gamma}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\gamma}{i}\binom{i}{j}\binom{j}{k}\left(\frac{\Gamma(\gamma c+i+j+k+1)}{(\theta\gamma)^{(\gamma c+i+j+k+1)}}\right)$$

6.2 Tsallis Entropy

Tsallis entropy is the generalization of Boltzmann-Gibbs (B-G) entropy to describe the properties of physical system with long range of forces and complex dynamics in equilibrium system. The statistical mechanics initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy [17] for a continuous random variable X is defined as follows

$$S_{\lambda}=\frac{1}{1-\lambda}\left(1-\left(\int f^{\lambda}(x;c,\theta)\,dx\right)\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1-\left(\left(\frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^\lambda \int_0^\infty \left(\left(x^c(1+x+x^2+x^3)\right)^\lambda e^{-\lambda\theta x}\right)dx\right)\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1-\left(\left(\frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^\lambda \int_0^\infty \left(x^{\lambda c}(1+x+x^2+x^3)^\lambda e^{-\lambda\theta x}\right)dx\right)\right)$$

Using binomial expansion:

$$S_\lambda = \frac{1}{1-\lambda}\left(1-\left(\frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^\lambda \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\lambda}{i}\binom{i}{j}\binom{j}{k}\int_0^\infty x^{(\lambda c+i+j+k+1)-1} e^{-\lambda\theta x}dx\right)$$

$$S_\lambda = \frac{1}{1-\lambda}\left(1-\left(\frac{\theta^{c+4}}{\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!}\right)^\lambda \sum_{i=0}^\infty\sum_{j=0}^\infty\sum_{k=0}^\infty \binom{\lambda}{i}\binom{i}{j}\binom{j}{k}\left(\frac{\Gamma(\lambda c+i+j+k+1)}{(\theta\lambda)^{(\lambda c+i+j+k+1)}}\right)\right)$$

7 Bonferroni And Lorenz Curves

7.1 Bonferroni Curve

Bonferroni [18] and Lorenz curves are widely used tool for analyzing and visualizing income inequality. The Lorenz curve can be regarded as the proportion of total income volume accumulated by those units with income lower than or equal to the volume x. The Bonferroni and Lorenz curves for a non-negative random variable X with density function $f(x)$ and distribution function $F(x)$ are respectively.

$$B(p) = \frac{1}{p\mu_1'}\int_0^q x\, f(x;c,\theta)dx$$

and

$$L(p) = \frac{1}{\mu_1'}\int_0^q x\, f(x;c,\theta)dx$$

Where,

$$\mu_1' = \frac{\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!}{\theta\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)} \quad \text{and} \quad q = F^{-1}(p)$$

$$B(p)=\frac{1}{p\mu_1'}\int_0^q \frac{\theta^{c+4}}{\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)} x^{c+1}(1+x+x^2+x^3)e^{-x\theta}dx$$

$$B(p)=\left(\frac{\theta^{c+4}}{p\mu_1'\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right)\int_0^q x^{c+1}(1+x+x^2+x^3)e^{-x\theta}dx$$

$$B(p)=\left(\frac{\theta^{c+4}}{p\mu_1'\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right)\int_0^q x^{c+1}e^{-\theta x}dx+\int_0^q x^{c+2}e^{-\theta x}dx+\int_0^q x^{c+3}e^{-\theta x}dx+\int_0^q x^{c+4}e^{-\theta x}dx$$

put $\theta x = t$; $\theta dx = dt\,; dx = \frac{dt}{\theta} x = \frac{t}{\theta}$ as $x \to 0, t \to 0$; $x \to q, t \to \theta q$

$$B(p)=\left(\frac{\theta^{c+4}}{p\mu_1'\left(\theta^3 c!+(c+1)!\theta^2+(c+2)!\theta+(c+3)!\right)}\right)\int_0^{\theta q}\left(\frac{t}{\theta}\right)^{c+1}e^{-t}dx+\int_0^{\theta q}\left(\frac{t}{\theta}\right)^{c+2}e^{-t}dx+$$

$$\int_0^{\theta q}\left(\frac{t}{\theta}\right)^{c+3}e^{-t}dx+\int_0^{\theta q}\left(\frac{t}{\theta}\right)^{c+4}e^{-t}dx$$

On simplification, we get

$$B(p)=\left(\frac{\theta^3}{p(\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!)}\right)\left(\gamma((c+2),\theta q)+\frac{1}{\theta}\gamma((c+3),\theta q)+\frac{1}{\theta^2}\gamma((c+4),\theta q)+\frac{1}{\theta^3}\gamma((c+5),\theta q)\right)$$

7.2 Lorenz Curve

The Lorenz curve is a graphical representation of the degree of inequality of income in a country. He was an American economist Max O. Lorenz [19] who proposed Lorenz curve as a method for making the comparison between in income distribution of the population at different time points.

$$L(p)=pB(p)$$

$$L(p)=\left(\frac{\theta^3}{(\theta^3(c+1)!+(c+2)!\theta^2+(c+3)!\theta+(c+4)!)}\right)\left(\gamma((c+2),\theta q)+\frac{1}{\theta}\gamma((c+3),\theta q)+\frac{1}{\theta^2}\gamma((c+4),\theta q)+\frac{1}{\theta^3}\gamma((c+5),\theta q)\right)$$

8 Likelihood Ratio Test

Let$X_1, X_2, \ldots, X_n$ be a random sample taken from the weighted Amarendra distribution.

To test the hypothesis

$H_o : f(x) = f(x;\theta)$ against $H_1 : f(x) = f_w(x;c,\theta)$

In order to test whether the random sample of size n comes from the Amarendra distribution or weighted Amarendra distribution, the following test statistic is used

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n} \frac{f_w(x;c,\theta)}{f(x;\theta)}$$

$$\Delta = \prod_{i=1}^{n} \left(\frac{\theta^c \left(\theta^3 + \theta^2 + 2\theta + 6\right)}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} \right) x_i^{\,c}$$

$$\Delta = \left(\frac{\theta^c \left(\theta^3 + \theta^2 + 2\theta + 6\right)}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} \right)^n \prod_{i=1}^{n} x_i^{\,c}$$

We reject the null hypothesis if

$$\Delta = \left(\frac{\theta^c \left(\theta^3 + \theta^2 + 2\theta + 6\right)}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} \right)^n \prod_{i=1}^{n} x_i^{\,c} > k$$

$$\text{or, } \Delta^* = \prod_{i=1}^{n} x^c{}_i > k \left(\frac{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!}{\theta^c \left(\theta^3 + \theta^2 + 2\theta + 6\right)} \right)^n$$

$$\Delta^* = \prod_{i=1}^{n} x_i^{\,c} > k^*, \ \text{ where } k^* = k \left(\frac{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!}{\theta^c \left(\theta^3 + \theta^2 + 2\theta + 6\right)} \right)^n$$

For large sample size n, $2\log\Delta$ is distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus we reject the null hypothesis, when the probability value is given by

$p(\Delta^* > \alpha^*)$ Where, $\alpha^* = \prod_{i=1}^{n} x^c{}_i$ is less than a specified level of significance and $\prod_{i=1}^{n} x_i^{\,c}$ is the observed value of the statistic Δ^*.

9 Maximum Likelihood Estimation

In this section, we estimate the parameters of the weighted Amarendra distribution by using the method of maximum likelihood technique. Consider $x_1, x_2, x_3, \ldots, x_n$ be a random sample of size n from weighted Amarendra distribution then the likelihood function is given by

$$L(x;c,\theta) = \left(\frac{\theta^{c+4}}{\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!} \right)^n \prod_{i=1}^{n} x_i^{\,c} \left(\left(1 + x_i + x_i^{\,2} + x_i^{\,3}\right)\right) e^{-\theta x_i} \quad (9)$$

The log-likelihood function is

$$\log L = n(c+4)\log\theta - n\log\left(\theta^3 c! + (c+1)!\theta^2 + (c+2)!\theta + (c+3)!\right) + \sum_{i=1}^{n}\log x_i^{\,c}\left(1+x_i+x_i^{\,2}+x_i^{\,3}\right) - \theta\sum_{i=1}^{n}x_i \quad (10)$$

The maximum likelihood estimates of θ ,c can be obtained by differentiating log-likelihood equation (10) with respect to θ , c and setting the results to zero, we have

$$\frac{\partial \log L}{\partial \theta} = \frac{n(c+4)}{\theta} - \frac{6n}{\theta} - \sum_{i=1}^{n}x_i = 0 \quad (11)$$

$$\frac{\partial \log L}{\partial c} = n\log\theta - n\left(\psi(c-1)+\psi(c+2)+\psi(c+3)+\psi(c+4)\right) + \sum_{i=1}^{n}\log x_i = 0 \quad (12)$$

From equation (11) we get,

$$\hat{\theta} = \frac{(c-2)}{\overline{X}}$$

The maximum likelihood estimator's are obtained by setting (12) to zero and then solving the nonlinear equations.

10 Applications

In this section, we consider a two real life data to demonstrate the applicability of the weighted Amarendra distribution whether our new model provides better fit to the data as compared to the baseline distribution. The goodness of fit measures such as Akaike information criteria (*AIC),*Bayesian information criteria (*BIC*) and Akaike information criteria Corrected (*AICc*) are considered to compare proposed model to the baseline distribution.

10.1 Data Set I

The first data set consists of lifetimes (in days) of 40 patients suffering from blood cancer (leukemia) reported from one of ministry of Health Hospitals In Saudi Arabia [20].

0.315, 0.496, 0.616, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.37, 2.532, 2.693, 2.805, 2.91, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.767, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074, 5.382.

10.2 Data Set 2

we consider a data set reported by Badar and Priest [21], which represents the strength measured in GPa for single carbon fibers and impregnated at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 100 fibers were tested at gauge lengths of 20, 50, 150 and 300 mm. Here, we consider the data set of single fibers of 20 mm in the gauge with a sample of size 63.

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

In order to compare the weighted Amarendra distribution with Amarendra distribution, we consider the criteria like Bayesian information criterion *(BIC),* Akaike Information Criterion *(AIC),* Corrected Akaike Information Criterion *(AICc)* and *-2 log L*. The better distribution is which corresponds to lesser values of *AIC, BIC, AICc* and *– 2 log L*. wecan analyze the distribution by using the formulas as follows:

$$AIC = 2k - 2logL, \qquad BIC = klogn - 2logL, \qquad AICc = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where k is the number of parameters, n is the sample size and L is the maximized value of log likelihood function.

Table 1: Maximum likelihood estimates and goodness of fit criteria AIC, BIC AICc

Data Set	Distribution	MLE	S.E	-2 log L	AIC	BIC	AICc
1	Weighted Amarendra Distribution	$\hat{\theta}$ =1.404507 $\hat{c}$ =1.038081	$\hat{\theta}$ = 0.242576 $\hat{c}$ = 0.653178	140.3644	144.3644	147.6915	145.0501
	Amarendra Distribution	$\hat{\theta}$ =1.036868	$\hat{\theta}$ =0.079906	144.2464	146.2464	147.910	146.9321
2	Weighted Amarendra Distribution	$\hat{\theta}$ = 8.55937 $\hat{c}$ =22.649167	$\hat{\theta}$ = 1.494386 $\hat{c}$ =4.522672	113.8273	117.8274	122.1136	118.2342
	Amarendra Distribution	$\hat{\theta}$ =1.059065	$\hat{\theta}$ =0.064188	208.6055	210.6055	212.7486	211.0123

Table 1 provide the MLEs of the model parameters. The model with minimum AIC, BIC, AICc values is chosen as the best model to fit the data. We observe that weighted Amarendra distribution has the lowest values for the AIC, BIC, AICc criterion as compared to Amarendra distribution. So, we conclude that the weighted Amarendra distribution leads to a better fit than the Amarendra distribution. This distribution can be used to improve the known mathematical modeling which can be used to help the community respond to the COVID-19 pandemic by informing decisions about pandemic distribution and propagations, and implementation of social distancing measures and other interventions [22,30]. Also, more applications can be extended to quasi probability distribution in different models [31,35].

11 Conclusion

In this study, Weighted Amarendra distribution has been proposed. The proposed distribution is generated by using the weighted technique to the baseline distribution. Some statistical properties along with reliability measures has been discussed. The parameters of the new distribution has been obtained by using maximum likelihood technique. Goodness of fit criteria illustrating the usefulness of the flexible new distribution totwo real-life data sets was performed. The AIC, BIC, and AICc criteria suggested that proposed distribution provided better fit to the data as compared Amarendra distribution.

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