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# An Efficient Method for Indefinite $q$-Integrals

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**Abstract:** A method has been introduced recently for deriving indefinite integrals of special functions that satisfy homogeneous (nonhomogeneous) second-order linear differential equations. This paper extends this method to include indefinite Jackson $q$-integrals of special functions satisfying homogeneous (nonhomogeneous) second-order linear $q$-difference equations. Many $q$-integrals, both previously known and completely new, are derived using the method. We introduce samples of indefinite and definite $q$-integrals for Jackson's $q$-Bessel functions, $q$-hypergeometric functions, and some orthogonal polynomials.

**Keywords:** Indefinite $q$-integrals, $q$-special functions, Jackson $q$- Bessel functions, $q$-hypergeometric functions.

## 1 Introduction and preliminaries

Conway in [1] introduced a simple method of deriving indefinite integrals. The method applies to any special function satisfying an ordinary differential equation. The main result derived in [1] is the indefinite integral

$$\int f(x)\left(h''(x)+p(x)h'(x)+q(x)h(x)\right)y(x)=f(x)\left(h'(x)y(x)-h(x)y'(x)\right),$$

where $f(x)$ satisfies the first-order differential equation

$$f(x)=\exp(\int p(x)dx).$$

Here $p(x)$ and $q(x)$ are arbitrary complex-valued differentiable functions of $x$ in $\mathbb{R}$, with $h(x)$ being at least twice differentiable. In a series of papers, see [1,2,3,4,5,6,7,8], Conway developed this method to obtain more indefinite integrals. This paper extends Conway's results to include special functions satisfying second-order q-difference equations.

Throughout this paper, $q$ is a positive number less than 1, $\mathbb{N}$ is the set of positive integers, and $\mathbb{N}_0$ is the set of non-negative integers. We follow Gasper and Rahman [9] for the definition of the $q$-shifted factorial, $q$-gamma, $q$-beta function, and $q$-hypergeometric series. A $q$-natural number $[n]_q$ is defined by $[n]_q=\frac{1-q^n}{1-q}$, $n\in\mathbb{N}_0$. The $q$-derivative $D_qf(x)$ of a function $f$ is defined by [10,11]

$$(D_qf)(x)=\frac{f(x)-f(qx)}{(1-q)x},\ \text{if}\ x\neq 0,$$

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and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. Jackson's $q$-integral of a function $f$ is defined by [12]

$$\int_0^a f(t)\, d_q t := (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n),\; a \in \mathbb{R}, \tag{1}$$

provided that the corresponding series in (1) converges. A function $F(x)$ is called anti-$q$-derivative of a function $f(x)$ if $D_q F(x) = f(x)$. if $F(x)$ is any anti-$q$-derivative of $f(x)$, then the most general anti-$q$-derivative of $f(x)$ is called an indefinite $q$-integral and denoted,

$$\int f(x)\, d_q x = F(x) + C(x),$$

where $C(x)$ is a $q$-periodic function, i.e. $C(x) = C(qx)$, for all $x$. From the $q$-product rules

$$D_q(uv)(x) = D_q u(x)v(x) + u(qx)D_q v(x), \tag{2}$$
$$D_{q^{-1}}(uv)(x) = D_{q^{-1}}u(x)v(x) + u(x/q)D_{q^{-1}}v(x), \tag{3}$$

and the fundamental theorem of calculus, see for example [13], we obtain the indefinite $q$-integration by parts rule:

$$\begin{aligned}\int u(qx)D_q v(x)\, d_q x &= (uv)(x) - \int D_q u(x)v(x)\, d_q x,\\ \int u(x/q)D_{q^{-1}}v(x)\, d_q x &= q(uv)(x/q) - \int D_{q^{-1}}u(x)v(x)\, d_q x.\end{aligned} \tag{4}$$

Moreover, if $f$ and $g$ are defined on an interval $[a,b]$, $a < 0 < b$, and they are continuous at zero, then

$$\frac{1}{q}\int_a^b D_{q^{-1}}u(x)v(x)\, d_q x = u(x/q)v(x/q)\Big|_a^b - \frac{1}{q}\int_a^b u(x/q)D_{q^{-1}}v(x)\, d_q x. \tag{5}$$

We shall use the following proposition from [14].

**Proposition 1.** *Let the functions $f$ and $g$ be defined and continuous on $[0,\infty[$. Assume that the improper Riemann integrals of the functions $f(x)g(x)$ and $f(x/q)g(x)$ exist on $[0,\infty[$. Then*

$$\begin{aligned}\int_0^{\infty} f(x)D_q g(x)dx &= \frac{f(0)g(0)}{1-q}\ln q - \frac{1}{q}\int_0^{\infty} g(x)D_{q^{-1}}f(x)dx\\ &= \frac{f(0)g(0)}{1-q}\ln q - \int_0^{\infty} g(qx)D_q f(x)dx.\end{aligned} \tag{6}$$

The remaining sections of this paper are organized as follows. In Section 2, we derive a $q$-analog of the Euler-Lagrange method. The method will generate indefinite $q$-integrals for functions satisfying homogeneous and non-homogeneous second-order $q$-difference equations. Section 3 contains applications of the derived method to Jackson's $q$-Bessel and Struve functions. Section 4 includes applications of the derived method to other special functions. Finally, we added an appendix for the definitions and main properties of the functions we need in this paper.

## 2 The $q$-type Lagrangian method for second-order $q$-difference equations

In this section, we extend the Lagrangian method introduced in [2] to functions satisfying homogeneous second-order $q$-difference equations of the form (7) or (14) below. We also extend the method introduced in [1] to functions satisfying non-homogeneous second-order $q$-difference equation of the form (21) or (23) below.

**Theorem 1.** *Let $p(x)$ and $r(x)$ be continuous at zero functions defined on an interval $(a,b)$, $-\infty \le a < 0 < b \le \infty$. Let $y(x)$ be a solution of the second-order $q$-difference equation*

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + p(x)D_{q^{-1}}y(x) + r(x)y(x) = 0 \;\; (x \in (a,b)). \tag{7}$$

*Then*

$$\int f(x)\Big(\frac{1}{q}D_{q^{-1}}D_q h(x)+p(x)D_{q^{-1}}h(x)+r(x)h(x)\Big)y(x)\,d_qx$$
$$=f(x/q)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big), \tag{8}$$

*where $h(x)$ is an arbitrary function, and $f(x)$ is a solution of the first order $q$-difference equation*

$$\frac{1}{q}D_{q^{-1}}f(x)=p(x)f(x). \tag{9}$$

*Proof.*Applying (4), we obtain

$$\frac{1}{q}\int f(x)y(x)D_{q^{-1}}D_q h(x)\,d_qx=f(x/q)y(x/q)D_{q^{-1}}h(x)$$
$$-\frac{1}{q}\int D_{q^{-1}}h(x)D_{q^{-1}}\big(f(x)y(x)\big)\,d_qx. \tag{10}$$

From (2), we get

$$\frac{1}{q}\int D_{q^{-1}}h(x)D_{q^{-1}}\big(f(x)y(x)\big)\,d_qx=\frac{1}{q}\int f(x/q)D_{q^{-1}}y(x)D_{q^{-1}}h(x)\,d_qx$$
$$+\frac{1}{q}\int D_{q^{-1}}h(x)D_{q^{-1}}\big(f(x)\big)y(x)\,d_qx. \tag{11}$$

Applying (4) on the first $q$-integral on the right hand side of (11) and substituting into (10), we get

$$\frac{1}{q}\int f(x)y(x)D_{q^{-1}}D_q h(x)\,d_qx=f(x/q)\,\Big(y(x/q)D_{q^{-1}}h(x)-h(x/q)D_{q^{-1}}y(x)\Big)$$
$$+\int h(x)f(x)\frac{1}{q}D_{q^{-1}}D_q y(x)\,d_qx+\int h(x)D_{q^{-1}}y(x)\frac{1}{q}D_{q^{-1}}f(x)\,d_qx. \tag{12}$$

Substituting with the value of $\frac{1}{q}D_{q^{-1}}D_q y(x)$ from (7) and using (9), we get

$$\int f(x)\Big(\frac{1}{q}D_{q^{-1}}D_q h(x)+p(x)D_{q^{-1}}h(x)+r(x)h(x)\Big)y(x)\,d_qx$$
$$=f(x/q)\Big(y(x/q)D_{q^{-1}}h(x)-h(x/q)D_{q^{-1}}y(x)\Big). \tag{13}$$

But one can verify that

$$f(x/q)\Big(y(x/q)D_{q^{-1}}h(x)-h(x/q)D_{q^{-1}}y(x)\Big)=f(x/q)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big).$$

This yields (8) and completes the proof.

*Remark.*It is worth noting that the right-hand side of (13) can be represented as $f(\frac{x}{q})W_{q^{-1}}(y,h)(x)$, where $W_{q^{-1}}(y,h)(x)$ is the $q^{-1}$-Wronskian defined by

$$W_{q^{-1}}(y,z)(x)=y(x)D_{q^{-1}}z(x)-z(x)D_{q^{-1}}y(x),$$

see [15,16].

**Theorem 2.***Let $p(x)$ and $r(x)$ be continuous at zero functions defined on an interval $(a,b)$, $-\infty\le a<0<b\le\infty$. Let $y(x)$ be any solution of the second-order $q$-difference equation*

$$\frac{1}{q}D_{q^{-1}}D_q y(x)+p(x)D_q y(x)+r(x)y(x)=0. \tag{14}$$

*Then*

$$\int f(x)\Big(\frac{1}{q}D_{q^{-1}}D_qh(x)+p(x)D_qh(x)+r(x)h(x)\Big)y(x)\,d_qx$$
$$=f(x)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big), \tag{15}$$

*where $h(x)$ is an arbitrary function and $f(x)$ is a solution of the first order $q$-difference equation*

$$D_qf(x)=p(x)f(x). \tag{16}$$

*Proof.*We omit the proof because it is similar to the one of Theorem 1.

**Corollary 1.***In Theorem 1, let $h(x)$ be a solution of the inhomogeneous equation*

$$\frac{1}{q}D_{q^{-1}}D_qh(x)+p(x)D_{q^{-1}}h(x)+r(x)h(x)=\frac{1}{f(x)}, \tag{17}$$

*where $f(x)$ is a solution of (9). Then*

$$\int y(x)\,d_qx=f(x/q)\Big(y(x/q)D_{q^{-1}}h(x)-h(x/q)D_{q^{-1}}y(x)\Big)$$
$$=f(x/q)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big).$$

*Proof.*Let $h(x)$ be a solution of the inhomogeneous Equation (17). Substituting with (17) into (13), we obtain the required result.

**Corollary 2.***In Theorem 2, let $h(x)$ be a solution of the inhomogeneous equation*

$$\frac{1}{q}D_{q^{-1}}D_qh(x)+p(x)D_qh(x)+r(x)h(x)=\frac{1}{f(x)}, \tag{18}$$

*where $f(x)$ is a solution of (16). Then*

$$\int y(x)d_qx=f(x)\Big(y(x)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}y(x)\Big). \tag{19}$$

*Proof.*Let $h(x)$ be a solution of the inhomogeneous Equation (18). Substituting with (18) into (15) yields (19) and completes the proof.

**Theorem 3.***Let $p(x)$ and $r(x)$ be continuous functions at zero. Let $y(x)$ be any solution of the second-order $q$-difference Equation (7). Then*

$$\int_0^\infty f(x)\Big[\frac{1}{q}D_{q^{-1}}D_qh(x)+p(x)D_{q^{-1}}h(x)+r(x)h(x)\Big]y(x)dx$$
$$=\frac{f(0)\ln q}{1-q}\left(D_qh(0)y(0)-h(0)D_qy(0)\right), \tag{20}$$

*where $h(x)$ is an arbitrary function and $f(x)$ is a solution of (9).*

*Proof.*Applying Proposition 1, we get

$$\int_0^\infty f(x)y(x)D_qD_{q^{-1}}h(x)dx=\frac{D_qh(0)f(0)y(0)}{1-q}\ln q-\frac{1}{q}\int_0^\infty D_{q^{-1}}h(x)D_{q^{-1}}(f(x)y(x))\,dx.$$

From the $q$-product rule, we obtain

$$\int_0^\infty f(x)y(x)D_qD_{q^{-1}}h(x)dx=\frac{D_qh(0)f(0)y(0)}{1-q}\ln q$$
$$-\frac{1}{q}\int_0^\infty D_{q^{-1}}h(x)f(x/q)D_{q^{-1}}y(x)\,dx-\frac{1}{q}\int_0^\infty D_{q^{-1}}h(x)y(x)D_{q^{-1}}f(x)\,dx.$$

Applying the Proposition 1 again, we obtain

$$-\int_0^\infty D_q h(x/q) f(x/q) D_{q^{-1}} y(x)\, dx = -\frac{h(0)f(0)D_q y(0)}{1-q} \ln q$$
$$+\int_0^\infty h(x) f(x) D_q D_{q^{-1}} y(x)\, dx + \int_0^\infty h(x) D_{q^{-1}} y(x) D_q f(x/q) dx.$$

Using Equations (7) and (9) yields

$$\frac{1}{q}\int_0^\infty f(x)y(x)D_{q^{-1}}D_q h(x)dx = \frac{D_q h(0)f(0)y(0)}{1-q}\ln q - \frac{h(0)f(0)D_q y(0)}{1-q}\ln q$$
$$-\int_0^\infty D_{q^{-1}} h(x)y(x)f(x)p(x)dx - \int_0^\infty r(x)h(x)y(x)f(x)dx.$$

Thus, we get the desired result.

**Theorem 4.** *Let $p(x)$, $r(x)$, and $g(x)$ be continuous functions at zero. Let $y(x)$ be any solution of the second-order q-difference equation*

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + p(x) D_{q^{-1}} y(x) + r(x) y(x) = g(x). \tag{21}$$

*Then*

$$\begin{aligned} &\int f(x)\left(\frac{1}{q}D_{q^{-1}}D_q h(x) + p(x)D_{q^{-1}}h(x) + r(x)h(x)\right) y(x)\, d_q x = \\ &f(x/q)\left(y(x)D_{q^{-1}}h(x) - h(x)D_{q^{-1}}y(x)\right) + \int f(x)h(x)g(x)\, d_q x = \\ &f(x/q)\left(y(x/q)D_{q^{-1}}h(x) - h(x/q)D_{q^{-1}}y(x)\right) + \int f(x)h(x)g(x)\, d_q x, \end{aligned} \tag{22}$$

*where $h(x)$ is an arbitrary function and $f(x)$ is a solution of (9).*

*Proof.* The proof is similar to the proof of Theorem 1. In Equation (12), substituting with the value of $\frac{1}{q}D_{q^{-1}}D_q y(x)$ from (7) and using (9), we obtain (22) and completes the proof.

**Theorem 5.** *Let $p(x)$, $r(x)$ and $g(x)$ be continuous functions at zero. Let $y(x)$ be any solution of the second-order q-difference equation*

$$\frac{1}{q} D_{q^{-1}} D_q y(x) + p(x) D_q y(x) + r(x) y(x) = g(x). \tag{23}$$

*Then,*

$$\begin{aligned} &\int f(x)\left(\frac{1}{q}D_{q^{-1}}D_q h(x) + p(x)D_q h(x) + r(x)h(x)\right) y(x)\, d_q x = \\ &f(x)\left(y(x)D_{q^{-1}}h(x) - h(x)D_{q^{-1}}y(x)\right) + \int f(x)h(x)g(x)\, d_q x, \end{aligned} \tag{24}$$

*where $h(x)$ is an arbitrary function and $f(x)$ is a solution of the first order q-difference Equation (16).*

*Proof.* The proof follows similarly as the proof of Theorem 4.

## 3 Applications to $q$-Bessel and Struve functions

There are unlimited indefinite $q$-integrals follow from (24) as $h(x)$ is an arbitrary function. The art of using Equation (13) is to choose $h(x)$ to give interesting $q$-integrals. In this section, we introduce applications to Theorems 1 and 2 to define $q$-integrals of $q$-Bessel and Struve functions.

### 3.1 q-Bessel functions

**Theorem 6.** *Let $m$ and $\nu$ be complex numbers, such that $\Re(\nu) > -1$ and $\Re(m+\nu) > 0$. If $J_\nu^{(3)}(.;q^2)$ is the third Jackson $q$-Bessel function defined in (I.3), then*

$$\int x^{m+1}\left(\frac{1}{(1-q)^2}-\frac{[\nu]_q^2-q^{\nu-m}[m]_q^2}{x^2}\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$=q^{\nu-m}x^{m+1}\left(\frac{[m]_q}{x}J_\nu^{(3)}(x;q^2)-q^{m-1}D_{q^{-1}}J_\nu^{(3)}(x;q^2)\right), \tag{25}$$

*or equivalently,*

$$\int x^{m+1}\left(\frac{1}{(1-q)^2}-\frac{[\nu]_q^2-q^{\nu-m}[m]_q^2}{x^2}\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$=x^{m+1}\left(\frac{(q^{\nu-m}[m]_q-[\nu]_q)}{x}J_\nu^{(3)}(x;q^2)+\frac{J_{\nu+1}^{(3)}(x;q^2)}{1-q}\right). \tag{26}$$

*In particular,*

$$\int x^{\nu+1}J_\nu^{(3)}(x;q^2)\,d_qx=(1-q)x^{\nu+1}J_{\nu+1}^{(3)}(x;q^2), \tag{27}$$

*and*

$$\int x^{1-\nu}J_\nu^{(3)}(x;q^2)\,d_qx=-(1-q)\left(\frac{x}{q}\right)^{1-\nu}J_{\nu-1}^{(3)}(\frac{x}{q};q^2). \tag{28}$$

*Proof.* The third Jackson $q$-Bessel function $J_\nu^{(3)}(x;q^2)$ is defined in (I.3) and satisfies the second-order $q$-difference equation (I.15). By comparing (I.15) with (7), we obtain

$$p(x)=\frac{1}{qx},\qquad r(x)=\frac{q^{-\nu}}{(1-q)^2}\left(1-\frac{(1-q^\nu)^2}{x^2}\right). \tag{29}$$

Thus $f(x)=x$ is a solution of (9). Also, Equation (8) associated with the third Jackson $q$-Bessel function is

$$\int x(L_{q,\nu}^{(3)}h)(x)J_\nu^{(3)}(x;q^2)\,d_qx=\frac{x}{q}\left(J_\nu^{(3)}(x;q^2)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}J_\nu^{(3)}(x;q^2)\right), \tag{30}$$

where

$$(L_{q,\nu}^{(3)}h)(x)=\frac{1}{q}D_{q^{-1}}D_qh(x)+\frac{1}{qx}D_{q^{-1}}h(x)+\frac{q^{-\nu}}{(1-q)^2}\left(1-\frac{(1-q^\nu)^2}{x^2}\right)h(x).$$

Substituting with $h(x)=x^m$ into (30), and noting that

$$(L_{q,\nu}^{(3)})x^m=x^m\left(\frac{q^{-\nu}}{(1-q)^2}-\frac{q^{-m}[m]_q^2-q^{-\nu}[\nu]_q^2}{x^2}\right),$$

we obtain

$$\int x^{m+1}\left(\frac{1}{(1-q)^2}-\frac{[\nu]_q^2-q^{\nu-m}[m]_q^2}{x^2}\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$=q^{\nu-m}x^{m+1}\left(\frac{[m]_qJ_\nu^{(3)}(x;q^2)}{x}-q^{m-1}D_{q^{-1}}J_\nu^{(3)}(x;q^2)\right),$$

using the $q$-difference equation [17, Eq.(3.5)]

$$D_{q^{-1}}J_\nu^{(3)}(x;q^2) = \frac{q^{1-\nu}[\nu]_q}{x}J_\nu^{(3)}(x;q^2) - \frac{q^{1-\nu}}{1-q}J_{\nu+1}^{(3)}(x;q^2), \tag{31}$$

we obtain

$$\begin{aligned}&\int x^{m+1}\left(\frac{1}{(1-q)^2} - \frac{[\nu]_q^2 - q^{\nu-m}[m]_q^2}{x^2}\right)J_\nu^{(3)}(x;q^2)\,d_qx \\ &= x^{m+1}\left(\frac{(q^{\nu-m}[m]_q - [\nu]_q)}{x}J_\nu^{(3)}(x;q^2) + \frac{J_{\nu+1}^{(3)}(x;q^2)}{1-q}\right).\end{aligned} \tag{32}$$

Substituting with $m=\nu$ into (26) gives (27). Substituting with $m=-\nu$ into (25) yields

$$\begin{aligned}&\int x^{1-\nu}J_\nu^{(3)}(x;q^2)\,d_qx = \\ &q^{2\nu}(1-q)^2x^{1-\nu}\left(\frac{-q^{-\nu}[\nu]_q}{x}J_\nu^{(3)}(x;q^2) - q^{-\nu-1}D_{q^{-1}}J_\nu^{(3)}(x;q^2)\right).\end{aligned} \tag{33}$$

Applying [17, Eq.(2.10)] (with $x$ is replaced by $\frac{x}{q}$ )

$$\frac{q^{1-\nu}[\nu]_q}{x}J_\nu^{(3)}(\frac{x}{q};q^2) + D_{q^{-1}}J_\nu^{(3)}(x;q^2) = \frac{q^{-\nu}}{1-q}J_{\nu-1}^{(3)}(\frac{x}{q};q^2),$$

using $J_\nu^{(3)}(\frac{x}{q};q^2) = x(q^{-1}-1)D_{q^{-1}}J_\nu^{(3)}(x;q^2) + J_\nu^{(3)}(x;q^2)$, we obtain

$$\frac{q^{1-\nu}[\nu]_q}{x}J_\nu^{(3)}(x;q^2) + q^{-\nu}D_{q^{-1}}J_\nu^{(3)}(x;q^2) = \frac{q^{-\nu}}{1-q}J_{\nu-1}^{(3)}(\frac{x}{q};q^2).$$

Substituting into (33), we get (28) and completes the proof of the theorem.

*Remark.*Equation (27) is equivalent to [17, Eq.(2.10)] (with $\nu$ is replaced by $\nu+1$)

$$D_q\left(x^{\nu+1}J_{\nu+1}^{(3)}(x;q^2)\right) = \frac{x^{\nu+1}}{1-q}J_\nu^{(3)}(x;q^2).$$

**Proposition 2.***Let $\mu$ and $\nu$ be a complex numbers. Assume that $\Re(\mu) > -1$ and $\Re(\nu) > -1$. Then*

$$\begin{aligned}&\int x\left(\frac{q^{-\nu}[\nu-\mu]_q}{1-q} + \frac{q^{-\mu}[\mu]_q^2 - q^{-\nu}[\nu]_q^2}{x^2}\right)J_\nu^{(3)}(x;q^2)J_\mu^{(3)}(x;q^2)\,d_qx \\ &= ([\mu]_q - [\nu]_q)J_\nu^{(3)}(x;q^2)J_\mu^{(3)}(x;q^2) \\ &+ \frac{x}{1-q}\left(J_{\nu+1}^{(3)}(x;q^2)J_\mu^{(3)}(x;q^2) - J_\nu^{(3)}(x;q^2)J_{\mu+1}^{(3)}(x;q^2)\right).\end{aligned}$$

*Proof.*The proof follows by substituting with $h(x) = J_\mu^{(3)}(x;q^2)$ into Equation (30).

For the convenience of the reader, we set the following notations in the remaining of this section.

$$A_{n,\nu} := q^{-n}[n-\nu]_q[n+\nu]_q, \tag{34}$$

$$C_{n,\nu} := q^{\frac{1}{2}}[n+\nu]_q + q^{-\frac{1}{2}}[n-\nu+1]_q, \tag{35}$$

$$K_{n,\nu} := q^{-n-\frac{3}{2}}\left(q^{-\nu}[n+\nu]_q + q^{\nu}[n-\nu+1]_q\right), \tag{36}$$

$$C_\nu(q) := \frac{(1+q)^{-\nu+1}}{\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\nu+\frac{1}{2})}. \tag{37}$$

**Theorem 7.** *Let $m$ and $\nu$ be complex numbers, such that $\Re(\nu) > -1$. Let $\sin(\cdot;q)$ and $\cos(\cdot;q)$ be the functions defined in (I.10) and (I.11), respectively. Set*

$$\hat{N}_{n,\nu} := q^{\frac{3n+\nu+1}{2}}(1-q)A_{n,\nu}$$

$$D_{n,\nu}(x) := \frac{q^{-n+\nu}[n-\nu]_q}{x}\sin(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q) + \frac{q^{\frac{-1}{2}(n+\nu+2)}}{1-q}\cos(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q),$$

$$\tilde{D}_{n,\nu}(x) := \frac{q^{-n+\nu}[n-\nu]_q}{x}\cos(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q) - \frac{q^{\frac{-1}{2}(n+\nu+1)}}{1-q}\sin(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q).$$

*Then*

$$\int\left([2n+1]_q x^n\cos(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q) + q^{\frac{1}{2}}x^{n-1}\hat{N}_{n,\nu}\sin(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q)\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$= q^{\frac{3n+\nu+2}{2}}(1-q)x^{n+1}D_{n,\nu}(x)J_\nu^{(3)}(\frac{x}{q};q^2) + q^{\frac{n+\nu+2}{2}}x^{n+1}\sin(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q)J_{\nu+1}^{(3)}(x;q^2), \tag{38}$$

*and*

$$\int\left(x^{n-1}\hat{N}_{n,\nu}\cos(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q) - [2n+1]_q x^n\sin(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q)\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$= q^{\frac{3n+\nu+1}{2}}(1-q)x^{n+1}\tilde{D}_{n,\nu}(x)J_\nu^{(3)}(\frac{x}{q};q^2) + q^{\frac{n+\nu+1}{2}}x^{n+1}\cos(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q)J_{\nu+1}^{(3)}(x;q^2). \tag{39}$$

*Proof.* From Theorem 6, we get $f(x) = x$ is a solution of (9). Equation (13) associated with the third Jackson $q$-Bessel function is

$$\int x(L_{q,\nu}^{(3)}h)(x)J_\nu^{(3)}(x;q^2)\,d_qx = \frac{x}{q}\Big(J_\nu^{(3)}(\frac{x}{q};q^2)D_{q^{-1}}h(x) - h(x/q)D_{q^{-1}}J_\nu^{(3)}(x;q^2)\Big), \tag{40}$$

where

$$(L_{q,\nu}^{(3)}h)(x) = \frac{1}{q}D_{q^{-1}}D_qh(x) + \frac{1}{qx}D_{q^{-1}}h(x) + \frac{q^{-\nu}}{(1-q)^2}\left(1 - \frac{(1-q^\nu)^2}{x^2}\right)h(x).$$

Substituting with $h(x) = x^n\sin(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q)$ into (40), we get

$$(L_{q,\nu}^{(3)})h(x) = \frac{q^{\frac{-1}{2}(3n+\nu+2)}[2n+1]_q}{1-q}x^{n-1}\cos(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q) + A_{n,\nu}x^{n-2}\sin(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q).$$

Hence, substituting into (40) yields

$$\int\left([2n+1]_q x^n\cos(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q) + q^{\frac{1}{2}}x^{n-1}\hat{N}_{n,\nu}\sin(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q)\right)J_\nu^{(3)}(x;q^2)\,d_qx$$
$$= \left(\frac{q^{\frac{n+\nu+2}{2}}(1-q^n)}{x}\sin(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q) + q^n\cos(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q)\right)x^{n+1}J_\nu^{(3)}(\frac{x}{q};q^2)$$
$$-q^{\frac{n+\nu}{2}}(1-q)x^{n+1}\sin(\frac{q^{\frac{-1}{2}(n+\nu+2)}x}{1-q};q)D_{q^{-1}}J_\nu^{(3)}(x;q^2).$$

Then, from

$$D_{q^{-1}}J_\nu^{(3)}(x;q^2) = \frac{q[\nu]_q}{x}J_\nu^{(3)}(\frac{x}{q};q^2) - \frac{q}{(1-q)}J_{\nu+1}^{(3)}(x;q^2),$$

see [17, Eq.(3.5)], we obtain (38). Similarly, (39) follows by substituting with $h(x) = x^n\cos(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q)$ into (40) and using

$$(L_{q,\nu}^{(3)})h(x) = A_{n,\nu}x^{n-2}\cos(\frac{q^{\frac{-1}{2}(n+\nu)}x}{1-q};q) - \frac{q^{\frac{-1}{2}(3n+\nu+1)}[2n+1]_q}{1-q}x^{n-1}\sin(\frac{q^{\frac{-1}{2}(n+\nu+1)}x}{1-q};q).$$

**Corollary 3.** *For $\Re(\nu) > -1$,*

$$\int x^{\nu}\cos(\frac{q^{-(\nu+\frac{1}{2})}x}{1-q};q)J_{\nu}^{(3)}(x;q^2)\,d_qx$$
$$=\frac{x^{\nu+1}}{[2\nu+1]_q}\left(q^{\nu}\cos(\frac{q^{-(\nu+\frac{1}{2})}x}{1-q};q)J_{\nu}^{(3)}(\frac{x}{q};q^2)+q^{\nu+1}\sin(\frac{q^{-(\nu+1)}x}{1-q};q)J_{\nu+1}^{(3)}(x;q^2)\right), \tag{41}$$

*and*

$$\int x^{\nu}\sin(\frac{q^{-(\nu+\frac{1}{2})}x}{1-q};q)J_{\nu}^{(3)}(x;q^2)\,d_qx$$
$$=\frac{x^{\nu+1}}{[2\nu+1]_q}\left(q^{\nu}\sin(\frac{q^{-(\nu+\frac{1}{2})}x}{1-q};q)J_{\nu}^{(3)}(\frac{x}{q};q^2)-q^{\nu+\frac{1}{2}}\cos(\frac{q^{-(\nu+1)}x}{1-q};q)J_{\nu+1}^{(3)}(x;q^2)\right). \tag{42}$$

*Proof.* The proof follows by setting $n=\nu$ in (38) and (39), respectively.

**Theorem 8.** *Let $\Re(\nu) > -1$, $\lambda\in\mathbb{C}$ and $\Re(m+\nu)>0$ . If $J_{\nu}^{(1)}(\lambda x|q^2)$ is the first Jackson q-Bessel function defined in* (I.1)*, then*

$$\int x^{m+1}\left(q^m\lambda^2+\frac{q^{-m}[m]_q^2-q^{-\nu}[\nu]_q^2}{x^2}\right)J_{\nu}^{(1)}(\lambda x|q^2)\,d_qx=$$
$$x^{m+1}\left(\frac{q^{-m}[m]_q-q^{-\nu}[\nu]_q}{x}J_{\nu}^{(1)}(\lambda x|q^2)+q^{\nu}\lambda J_{\nu+1}^{(1)}(\lambda x|q^2)\right). \tag{43}$$

*In particular,*

$$\int x^{\nu+1}J_{\nu}^{(1)}(\lambda x|q^2)\,d_qx=\frac{x^{\nu+1}}{\lambda}J_{\nu+1}^{(1)}(\lambda x|q^2), \tag{44}$$

*and*

$$\int x^{1-\nu}J_{\nu}^{(1)}(\lambda x|q^2)\,d_qx=\frac{-x^{1-\nu}}{\lambda}J_{\nu-1}^{(1)}(\lambda x|q^2). \tag{45}$$

*Proof.* The second Jackson $q$-Bessel function $J_{\nu}^{(2)}(\lambda x|q^2)$ is defined in (I.2) and satisfies the second-order $q$-difference equation (I.16). By comparing (I.16) with (14), we obtain

$$p(x)=\frac{1-q\lambda^2x^2(1-q)}{x},\quad r(x)=\frac{q\lambda^2x^2-q^{1-\nu}[\nu]_q^2}{x^2}.$$

Thus $f(x)=\dfrac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}$ is a solution of Equation (16). Also, Equation (15) associated with the second Jackson $q$-Bessel function will be

$$\int\frac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}(L_{\nu,q}^{(2)}h)(x)J_{\nu}^{(2)}(\lambda x|q^2)d_qx=$$
$$\frac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(J_{\nu}^{(2)}(\lambda x|q^2)D_{q^{-1}}h(x)-h(x)D_{q^{-1}}J_{\nu}^{(2)}(\lambda x|q^2)\right), \tag{46}$$

where

$$(L_{\nu,q}^{(2)}h)(x)=\frac{1}{q}D_{q^{-1}}D_qh(x)+\frac{1-q\lambda^2x^2(1-q)}{x}D_qh(x)+\frac{q\lambda^2x^2-q^{1-\nu}[\nu]_q^2}{x^2}h(x). \tag{47}$$

Substituting with $h(x)=x^m$, we get

$$(L^{(2)}_{\nu,q}h)(x)=qx^m\left(q^m\lambda^2+\frac{q^{-m}[m]_q^2-q^{-\nu}[\nu]_q^2}{x^2}\right).$$

Hence, substituting into (46) yields

$$\int\frac{x^{m+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(q^m\lambda^2+A_{m,\nu}x^{-2}\right)J^{(2)}_\nu(\lambda x|q^2)\,d_qx=\frac{x^{m+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(\frac{q^{-m}[m]_qJ^{(2)}_\nu(\lambda x|q^2)}{x}-\frac{1}{q}D_{q^{-1}}J^{(2)}_\nu(\lambda x|q^2)\right),\tag{48}$$

using [18, Eq.(2.1)],

$$D_{q^{-1}}J^{(2)}_\nu(\lambda x|q^2)=\frac{q^{1-\nu}[\nu]_q}{x}J^{(2)}_\nu(\lambda x|q^2)-\lambda q^{1+\nu}J^{(2)}_{\nu+1}(\lambda x|q^2),\tag{49}$$

and by substituting into (48), we obtain

$$\int\frac{x^{m+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(q^m\lambda^2+A_{m,\nu}x^{-2}\right)J^{(2)}_\nu(\lambda x|q^2)\,d_qx=\frac{x^{m+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(\frac{q^{-m}[m]_q-q^{-\nu}[\nu]_q}{x}J^{(2)}_\nu(\lambda x|q^2)+q^\nu\lambda J^{(2)}_{\nu+1}(\lambda x|q^2)\right).\tag{50}$$

By using (I.4), we get (43). Substituting with $\nu=m$ into (50) yields

$$\int\frac{x^{\nu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}J^{(2)}_\nu(\lambda x|q^2)\,d_qx=\frac{x^{\nu+1}}{\lambda(-x^2\lambda^2(1-q)^2;q^2)_\infty}J^{(2)}_{\nu+1}(\lambda x|q^2),\tag{51}$$

using (I.4), we get (44). Substituting with $m=-\nu$, into (50) and using [9, Eq.(1.25)] (with $x$ is replaced by $2x\lambda(1-q)$ and $q$ by $q^2$ )

$$q^{2\nu}J^{(2)}_{\nu+1}(\lambda x|q^2)=\frac{[2\nu]_q}{\lambda x}J^{(2)}_\nu(\lambda x|q^2)-J^{(2)}_{\nu-1}(\lambda x|q^2),\tag{52}$$

we get

$$\int\frac{x^{1-\nu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}J^{(2)}_\nu(\lambda x|q^2)\,d_qx=\frac{-x^{1-\nu}}{\lambda(-x^2\lambda^2(1-q)^2;q^2)_\infty}J^{(2)}_{\nu-1}(\lambda x|q^2),\tag{53}$$

using (I.4), we get (45).

*Example 1.*For $\nu=0$ and $m=1$, Equation (43) will be

$$\int\left(1+\lambda^2q^2x^2\right)J^{(1)}_0(\lambda x|q^2)\,d_qx=xJ^{(1)}_0(\lambda x|q^2)+q\lambda x^2J^{(1)}_1(\lambda x|q^2).$$

Moreover, if $\lambda$ is a zero of $J^{(1)}_0(x|q^2)$, then

$$\int_0^1\left(1+\lambda^2q^2x^2\right)J^{(1)}_0(\lambda x|q^2)\,d_qx=q\lambda J^{(1)}_1(\lambda|q^2).$$

**Theorem 9.***Let $\nu$ and $n$ be a complex numbers with $\Re(\nu)>-1$. Let $Sin_q(\cdot)$ and $Cos_q(\cdot)$ be the functions defined in* (I.8) *and* (I.9)*, respectively. Set*

$$I_{n,\nu}(x):=\frac{q^{-n}[n-\nu]_q}{x}Sin_q(q^{n+\frac12}x)+q^{-\frac12}Cos_q(q^{n+\frac12}x),$$
$$\tilde{I}_{n,\nu}(x):=\frac{q^{-n}[n-\nu]_q}{x}Cos_q(q^{n+\frac12}x)-q^{-\frac12}Sin_q(q^{n+\frac12}x).$$

*Then*

$$\int \Big(x^{n-1}A_{n,\nu}Sin_q(q^{n+\frac{3}{2}}x)+x^nC_{n,\nu}(x)Cos_q(q^{n+\frac{3}{2}}x)\Big)J_\nu^{(1)}(x|q^2)\,d_qx$$
$$= x^{n+1}\Big(I_{n,\nu}(x)J_\nu^{(1)}(x|q^2)+q^\nu Sin_q(q^{n+\frac{1}{2}}x)J_{\nu+1}^{(1)}(x|q^2)\Big), \tag{54}$$

*and*

$$\int \Big(x^{n-1}A_{n,\nu}Cos_q(q^{n+\frac{3}{2}}x)-x^nC_{n,\nu}(x)Sin_q(q^{n+\frac{3}{2}}x)\Big)J_\nu^{(1)}(x|q^2)\,d_qx$$
$$= x^{n+1}\Big(\tilde{I}_{n,\nu}(x)J_\nu^{(1)}(x|q^2)+q^\nu Cos_q(q^{n+\frac{1}{2}}x)J_{\nu+1}^{(1)}(x|q^2)\Big), \tag{55}$$

*where $A_{n,\nu}$ is defined in (34).*

*Proof.*From Theorem 8, we get $f(x)=\dfrac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}$ is a solution of (16). Substitute with $h(x)=x^n\mathrm{Sin}_q(q^{n+\frac{1}{2}}x)$ and $\lambda=1$ into (47), we get

$$(L_{\nu,q}^{(2)}h)(x)=qx^{n-1}\Big(x^{-1}A_{n,\nu}\mathrm{Sin}_q(q^{n+\frac{3}{2}}x)+C_{n,\nu}(x)\mathrm{Cos}_q(q^{n+\frac{3}{2}}x)\Big).$$

Hence, substituting into Equation (46), using (49) and the identity

$$\mathrm{Sin}_q(q^{n+\frac{1}{2}}x)=q^{n+\frac{1}{2}}(1-q)x\mathrm{Cos}_q(q^{n+\frac{3}{2}}x)+\mathrm{Sin}_q(q^{n+\frac{3}{2}}x),$$

yields

$$\int \frac{1}{(-x^2(1-q)^2;q^2)_\infty}\Big(x^{n-1}A_{n,\nu}\mathrm{Sin}_q(q^{n+\frac{3}{2}}x)+x^nC_{n,\nu}(x)\mathrm{Cos}_q(q^{n+\frac{3}{2}}x)\Big)J_\nu^{(2)}(x|q^2)\,d_qx$$
$$= \frac{x^{n+1}}{(-x^2(1-q)^2;q^2)_\infty}\Big(I_{n,\nu}(x)J_\nu^{(2)}(x|q^2)+q^\nu\mathrm{Sin}_q(q^{n+\frac{1}{2}}x)J_{\nu+1}^{(2)}(x|q^2)\Big), \tag{56}$$

using ($I.4$), we get (54). Similarly, we get the $q$-integral (55) by substituting with $h(x)=x^n\mathrm{Cos}_q(q^{n+\frac{1}{2}}x)$ into (46) and using ($I.4$) and

$$(L_{\nu,q}^{(2)}h)(x)=qx^{n-1}\Big(x^{-1}A_{n,\nu}\mathrm{Cos}_q(q^{n+\frac{3}{2}}x)-C_{n,\nu}(x)\mathrm{Sin}_q(q^{n+\frac{3}{2}}x)\Big).$$

**Corollary 4.***For $\Re(\nu)>-1$,*

$$\int x^\nu Cos_q(q^{\nu+\frac{3}{2}}x)J_\nu^{(1)}(x|q^2)\,d_qx$$
$$= \frac{x^{\nu+1}}{[2\nu+1]_q}\Big(Cos_q(q^{\nu+\frac{1}{2}}x)J_\nu^{(1)}(x|q^2)+q^{\frac{1}{2}+\nu}Sin_q(q^{\nu+\frac{1}{2}}x)J_{\nu+1}^{(1)}(x|q^2)\Big), \tag{57}$$

*and*

$$\int x^\nu Sin_q(q^{\nu+\frac{3}{2}}x)J_\nu^{(1)}(x|q^2)\,d_qx$$
$$= \frac{x^{\nu+1}}{[2\nu+1]_q}\Big(Sin_q(q^{\nu+\frac{1}{2}}x)J_\nu^{(1)}(x|q^2)-q^{\frac{1}{2}+\nu}Cos_q(q^{\nu+\frac{1}{2}}x)J_{\nu+1}^{(1)}(x|q^2)\Big). \tag{58}$$

*Proof.*The proof follows by setting $n=\nu$ in (54) and (55), respectively.

**Theorem 10.***Let $\mu$ and $\nu$ be complex numbers. Assume that $\Re(\nu)>-1$ and $\Re(m+\nu)>0$. If $J_\nu^{(2)}(\lambda x|q^2)$ is the second Jackson $q$-Bessel function defined in (I.2), then*

$$\int x^{m+1}\left(\frac{\lambda^2}{q^2}+\frac{q^{-m}[m]_q^2-q^{-\nu}[\nu]_q^2}{q^{-m}x^2}\right)J_\nu^{(2)}(\lambda x|q^2)\,d_qx=$$
$$x^{m+1}\left(\frac{[m]_q-q^{m-\nu}[\nu]_q}{x}J_\nu^{(2)}(\lambda x|q^2)+q^{m-1}\lambda J_{\nu+1}^{(2)}(\frac{\lambda x}{q}|q^2)\right). \tag{59}$$

*In particular,*

$$\int x^{\nu+1} J_\nu^{(2)}(\lambda x|q^2)\, d_q x = \frac{(qx)^{\nu+1}}{\lambda} J_{\nu+1}^{(2)}(\frac{\lambda x}{q}|q^2),$$

*and*

$$\int x^{1-\nu} J_\nu^{(2)}(\lambda x|q^2)\, d_q x = -\frac{(qx)^{1-\nu}}{\lambda} J_{\nu-1}^{(2)}(\frac{\lambda x}{q}|q^2).$$

**Theorem 11.** *Let $\nu$ and $n$ be a complex numbers with $\Re(\nu) > -1$. Let $\sin_q(\cdot)$ and $\cos_q(\cdot)$ be the functions defined in (I.6) and (I.7), respectively. Set*

$$P_{n,\nu}(x) := \frac{q^{-n+\nu}[n-\nu]_q}{x} \sin_q(q^{-n-\frac{3}{2}}x) + q^{-n-\frac{3}{2}} \cos_q(q^{-n-\frac{3}{2}}x),$$
$$\tilde{P}_{n,\nu}(x) := \frac{q^{-n+\nu}[n-\nu]_q}{x} \cos_q(q^{-n-\frac{3}{2}}x) - q^{-n-\frac{3}{2}} \sin_q(q^{-n-\frac{3}{2}}x).$$

*Then*

$$\int \Big(A_{n,\nu} x^{n-1} \sin_q(q^{-n-\frac{3}{2}}x) + K_{n,\nu} x^n \cos_q(q^{-n-\frac{3}{2}}x)\Big) J_\nu^{(2)}(x|q^2)\, d_q x$$
$$= x^{n+1}\Big(P_{n,\nu}(x) J_\nu^{(2)}(\frac{x}{q}|q^2) + q^{\nu-n-1} \sin_q(q^{-n-\frac{3}{2}}x) J_{\nu+1}^{(2)}(\frac{x}{q}|q^2)\Big), \tag{60}$$

*and*

$$\int \Big(A_{n,\nu} x^{n-1} \cos_q(q^{-n-\frac{3}{2}}x) - K_{n,\nu} x^n \sin_q(q^{-n-\frac{3}{2}}x)\Big) J_\nu^{(2)}(x|q^2)\, d_q x$$
$$= x^{n+1}\Big(\tilde{P}_{n,\nu}(x) J_\nu^{(2)}(\frac{x}{q}|q^2) + q^{\nu-n-1} \cos_q(q^{-n-\frac{3}{2}}x) J_{\nu+1}^{(2)}(\frac{x}{q}|q^2)\Big), \tag{61}$$

*where $A_{n,\nu}$ is defined in (34).*

**Corollary 5.** *For $\Re(\nu) > -1$,*

$$\int x^\nu \cos_q(q^{-\nu-\frac{3}{2}}x) J_\nu^{(2)}(x|q^2)\, d_q x$$
$$= \frac{x^{\nu+1}}{q^{-\nu}[2\nu+1]_q}\Big(\cos_q(q^{-\nu-\frac{3}{2}}x) J_\nu^{(2)}(\frac{x}{q}|q^2) + q^{\nu+\frac{1}{2}} \sin_q(q^{-\nu-\frac{3}{2}}x) J_{\nu+1}^{(2)}(\frac{x}{q}|q^2)\Big),$$

*and*

$$\int x^\nu \sin_q(q^{-\nu-\frac{3}{2}}x) J_\nu^{(2)}(x|q^2)\, d_q x$$
$$= \frac{x^{\nu+1}}{q^{-\nu}[2\nu+1]_q}\Big(\sin_q(q^{-\nu-\frac{3}{2}}x) J_\nu^{(2)}(\frac{x}{q}|q^2) - q^{\nu+\frac{1}{2}} \cos_q(q^{-\nu-\frac{3}{2}}x) J_{\nu+1}^{(2)}(\frac{x}{q}|q^2)\Big).$$

### *3.2 q-Struve functions*

**Theorem 12.** *Let $\Re(\nu) > \frac{-1}{2}$, $\lambda \in \mathbb{C}$, and $\Re(\mu+\nu) > -1$. If $H_\nu^{(3)}(\cdot;q^2)$ is the q-Struve function defined in (I.19), then*

$$\int x^{\mu+1}\left(A_{\mu,\nu} x^{-2} + q^{-\nu-1}\lambda^2\right) H_\nu^{(3)}(\lambda x;q^2)\, d_q x = \frac{\lambda^{1+\nu} C_\nu(q) x^{\nu+\mu+1}}{q^{\nu+1}[\nu+\mu+1]_q}$$
$$+ x^{\mu+1}\left(\frac{q^{-\mu}[\mu]_q}{x} H_\nu^{(3)}(\lambda x;q^2) - q^{-1} D_{q^{-1}} H_\nu^{(3)}(\lambda x;q^2)\right), \tag{62}$$

*or equivalently,*

$$\int x^{\mu+1}\left(q^{\nu}A_{\mu,\nu}x^{-2}+\frac{\lambda^2}{q}\right)H_\nu^{(3)}(\lambda x;q^2)\,d_qx=\frac{\lambda^{1+\nu}C_\nu(q)[\mu-\nu]_q}{q^{-2\nu}[\nu+\mu+1]_q[2\nu+1]_q}x^{\nu+\mu+1}$$
$$+x^{\mu+1}\left(\frac{q^{\nu-\mu}[\mu]_q-[\nu]_q}{x}H_\nu^{(3)}(\lambda x;q^2)+\frac{\lambda}{q}H_{\nu+1}^{(3)}(\lambda x;q^2)\right). \tag{63}$$

*In particular,*

$$\int x^{\nu+1}H_\nu^{(3)}(\lambda x;q^2)\,d_qx=\frac{x^{\nu+1}}{\lambda}H_{\nu+1}^{(3)}(\lambda x;q^2), \tag{64}$$

$$\int x^{1-\nu}H_\nu^{(3)}(\lambda x;q^2)\,d_qx=\frac{-q^{\nu}}{\lambda}x^{1-\nu}H_{\nu-1}^{(3)}(\frac{\lambda x}{q};q^2)+\lambda^{\nu-1}C_\nu(q)x, \tag{65}$$

*and*

$$C_\nu(q)\int x^{\nu}J_\nu^{(3)}(\lambda x(1-q);q^2)\,d_qx=$$
$$\frac{q^{\nu}x}{\lambda^{1+\nu}}\left(J_\nu^{(3)}(\lambda x(1-q);q^2)D_{q^{-1}}H_\nu^{(3)}(\lambda x;q^2)-H_\nu^{(3)}(\lambda x;q^2)D_{q^{-1}}J_\nu^{(3)}(\lambda x(1-q);q^2)\right), \tag{66}$$

*where $A_{\mu,\nu}$ and $C_\nu(q)$ are defined in (34) and (37), respectively.*

*Proof.*The $q$-Struve function $H_\nu^{(3)}(\lambda x;q^2)$ is defined in (I.19) and satisfies the $q$-difference equation (I.20). Thus,

$$g(x)=\frac{\lambda^{\nu+1}x^{\nu-1}(1+q)^{-\nu+1}}{q^{\nu+1}\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\nu+\frac{1}{2})}\qquad \textit{and}\quad f(x)=x.$$

Therefore,

$$\int f(x)h(x)g(x)\,d_qx=\left(\frac{\lambda}{q}\right)^{\nu+1}C_\nu(q)\int x^{\nu}h(x)\,d_qx. \tag{67}$$

For $h(x)=x^{\mu}$, Equation (67) will be

$$\int f(x)h(x)g(x)\,d_qx=\frac{\lambda^{\nu+1}C_\nu(q)}{q^{\nu+1}[\nu+\mu+1]_q}x^{\nu+\mu+1}. \tag{68}$$

Substituting with the $q$-integral in (68) into Equation (22) gives the $q$-integral (62). The proof of (63) follows by using [19, Eq.(3.9)] (with $x$ is replaced by $\frac{x}{q}$ ) to obtain

$$D_{q^{-1}}H_\nu^{(3)}(\lambda x;q^2)-\frac{q^{1-\nu}[\nu]_q}{x}H_\nu^{(3)}(\lambda x;q^2)=C_{\nu+1}(q)\lambda^{\nu+1}(\frac{x}{q})^{\nu}-q^{-\nu}\lambda H_{\nu+1}^{(3)}(\lambda x;q^2),$$

and substituting into Equation (62). Substituting with $\nu=\mu$ yields (64). Substituting with $\mu=-\nu$, into (62) and using [19, Eq.(3.8)](with $x$ is replaced by $\frac{x}{q}$ and multiplying with $x^{1-\nu}$ ) to obtain

$$\frac{1}{q}D_{q^{-1}}H_\nu^{(3)}(\lambda x;q^2)+\frac{[\nu]}{x}H_\nu^{(3)}(\lambda x;q^2)=\frac{\lambda}{q}H_{\nu-1}^{(3)}(\frac{\lambda x}{q};q^2),$$

we get (65). Let $h(x)=H_\nu^{(3)}(\lambda x;q^2)$, then

$$\frac{1}{q}D_{q^{-1}}D_qh(x)+\frac{1}{qx}D_{q^{-1}}h(x)+\left(q^{-\nu-1}\lambda^2-\frac{q^{-\nu}[\nu]_q^2}{x^2}\right)h(x)=\frac{\lambda^{\nu+1}C_\nu(q)}{q^{\nu+1}}x^{\nu-1}. \tag{69}$$

Substituting with (69) into (13), we get (66) and completes the proof.

**Theorem 13.** *For $\Re(\nu) > \frac{-1}{2}$. Let $\sin(\cdot;q)$ and $\cos(\cdot;q)$ be the functions defined in (I.10) and (I.11), respectively. Set*

$$\hat{A}_{n,\nu} = q^{\frac{1}{2}(3n+\nu)} A_{n,\nu},$$
$$O_{n,\nu}(x) := \frac{q^{\frac{1}{2}(n+\nu+3)}[n]}{x}\sin(q^{\frac{-1}{2}(\nu+n+1)}x;q) + \cos(q^{\frac{-1}{2}(\nu+n+2)}x;q),$$
$$\tilde{O}_{n,\nu}(x) := \frac{q^{\frac{1}{2}(n+\nu)}[n]}{x}\cos(q^{\frac{-1}{2}(\nu+n+1)}x;q) - \sin(q^{\frac{-1}{2}(\nu+n+2)}x;q).$$

*Then*

$$\int \left([2n+1]_q x^n \cos(q^{\frac{-1}{2}(n+\nu+2)}x;q) + q^{\frac{3}{2}} x^{n-1}\hat{A}_{n,\nu}\sin(q^{\frac{-1}{2}(n+\nu+1)}x;q)\right) H_\nu^{(3)}(x;q^2)\, d_qx$$
$$= x^{n+1} O_{n,\nu}(x) H_\nu^{(3)}(\frac{x}{q};q^2) - q^{\frac{1}{2}(n+\nu+1)} x^n \sin(q^{\frac{-1}{2}(n+\nu+3)}x;q) D_{q^{-1}} H_\nu^{(3)}(x;q^2)$$
$$+ \frac{q^{n-\nu} C_\nu(q) x^{n+\nu+2}}{[n+\nu+2]_q}\, {}_2\phi_2(0, q^{\nu+n+2}; q^3, q^{\nu+n+4}; q^2, q^{\nu+m+1}(1-q)^2x^2),$$

*and*

$$\int \left(-[2n+1]_q x^n \sin(q^{\frac{-1}{2}(n+\nu+2)}x;q) + x^{n-1}\hat{A}_{n,\nu}\cos(q^{\frac{-1}{2}(n+\nu+1)}x;q)\right) H_\nu^{(3)}(x;q^2)\, d_qx$$
$$= x^{n+1}\tilde{O}_{n,\nu}(x) H_\nu^{(3)}(\frac{x}{q};q^2) - q^{n+\frac{\nu}{2}-1} x^n \cos(q^{\frac{-1}{2}(n+\nu+3)}x;q) D_{q^{-1}} H_\nu^{(3)}(x;q^2)$$
$$+ \frac{q^{\frac{-1}{2}(\nu-3n+2)} C_\nu(q) x^{\nu+n+1}}{[\nu+n+1]_q}\, {}_2\phi_2(0, q^{\nu+n+1}; q^3, q^{\nu+n+3}; q^2, q^{-n-\nu}(1-q)^2x^2),$$

*where $A_{\mu,\nu}$ and $C_\nu(q)$ are defined in (34) and (37), respectively.*

*Proof.* Substituting with

$$h(x) = x^n \sin(q^{\frac{-1}{2}(n+\nu+1)}x;q) \quad \text{and} \quad h(x) = x^n \cos(q^{\frac{-1}{2}(n+\nu+1)}x;q),$$

respectively, with $\lambda = 1$ into Equation (22), we get the desired results.

**Theorem 14.** *Let $\Re(\nu) > \frac{-1}{2}$, $\lambda \in \mathbb{C}$, $x \in \mathbb{C}$ and $\Re(\mu+\nu) > -1$. If $H_\nu^{(2)}(\cdot;q^2)$ is the $q$-Struve function defined in (I.18), then*

$$\int \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(A_{\mu,\nu}x^{-2} + q^\mu\lambda^2\right) H_\nu^{(2)}(\lambda x|q^2)\, d_qx$$
$$= \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(\frac{q^{-\mu}[\mu]_q}{x} H_\nu^{(2)}(\lambda x|q^2) - \frac{1}{q} D_{q^{-1}} H_\nu^{(2)}(\lambda x|q^2)\right)$$
$$+ \left(\frac{\lambda}{q}\right)^{\nu+1} C_\nu(q) \int \frac{x^{\nu+\mu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\, d_qx. \tag{70}$$

*In particular,*

$$\int \frac{x^{\nu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty} H_\nu^{(2)}(\lambda x|q^2)\, d_qx = \frac{q x^{\nu+1}}{\lambda(-x^2\lambda^2(1-q)^2;q^2)_\infty} H_{\nu+1}^{(2)}(\lambda x|q^2)$$
$$- \frac{\lambda^{\nu-1}}{q^{2\nu+1}} C_\nu(q)\left(\frac{x^{2\nu+1}}{[2\nu+1]_q(-x^2\lambda^2(1-q)^2;q^2)_\infty} - \int \frac{x^{2\nu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\, d_qx\right), \tag{71}$$

$$\int \frac{x^{1-\nu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty} H_\nu^{(2)}(\lambda x|q^2)\, d_qx = \frac{x^{1-\nu}}{q\lambda(-x^2\lambda^2(1-q)^2;q^2)_\infty} H_{\nu-1}^{(2)}(\lambda x|q^2)$$
$$+ \frac{1}{q}\lambda^{\nu-1} C_\nu(q) \int \frac{1}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\, d_qx, \tag{72}$$

*and*

$$C_\nu(q)\int \frac{x^\nu}{(-x^2\lambda^2(1-q)^2;q^2)_\infty} J_\nu^{(2)}(\lambda x|q^2)\,d_qx = \frac{\lambda^{-1-\nu}q^\nu x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(J_\nu^{(2)}(\lambda x|q^2)D_{q^{-1}}H_\nu^{(2)}(\lambda x|q^2) - H_\nu^{(2)}(\lambda x|q^2)D_{q^{-1}}J_\nu^{(2)}(\lambda x|q^2)\right), \tag{73}$$

*where $A_{\mu,\nu}$ and $C_\nu(q)$ are defined in (34) and (37), respectively.*

*Proof.*The $q$-Struve function $H_\nu^{(2)}(\lambda x;q^2)$ is defined in (I.18) and satisfies the $q$-difference equation (I.21). Thus,

$$g(x) = \frac{\lambda^{\nu+1}C_\nu(q)}{q^\nu}x^{\nu-1} \qquad \text{and} \quad f(x) = \frac{x}{(-x^2\lambda^2(1-q)^2;q^2)_\infty},$$

with $h(x) = x^\mu$, we get

$$\int f(x)h(x)g(x)\,d_qx = \frac{\lambda^{\nu+1}C_\nu(q)}{q^\nu}\int \frac{x^{\nu+\mu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\,d_qx.$$

Substituting into Equation (24) gives the $q$-integral (70). Using [19, Eq.(39)] (with $x$ is replaced by $\frac{x}{q}$ ) to obtain

$$D_{q^{-1}}H_\nu^{(2)}(\lambda x|q^2) - \frac{q^{1-\nu}[\nu]_q}{x}H_\nu^{(2)}(\lambda x|q^2) = \frac{\lambda^{\nu+1}(1+q)^{-\nu}x^\nu}{q^\nu\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\nu+\frac{3}{2})} - \lambda q^{\nu+2}H_{\nu+1}^{(2)}(\lambda x|q^2). \tag{74}$$

Equation (70) can be represented as

$$\begin{aligned}&\int \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(A_{\mu,\nu}x^{-2}+q^\mu\lambda^2\right)H_\nu^{(2)}(\lambda x|q^2)\,d_qx \\ &= \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(\frac{q^{-\mu}[\mu]_q - q^{1-\nu}[\nu]}{x}H_\nu^{(2)}(\lambda x|q^2) + q^{\nu+1}\lambda H_{\nu+1}^{(2)}(\lambda x|q^2)\right) \\ &- \left(\frac{\lambda}{q}\right)^{\nu+1}C_\nu(q)\left(\frac{x^{\nu+\mu+1}}{[2\nu+1]_q(-x^2\lambda^2(1-q)^2;q^2)_\infty} - \int \frac{x^{\nu+\mu}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\,d_qx\right).\end{aligned} \tag{75}$$

The proof of (71) follows by substituting with $\nu = \mu$ into (75). Substituting with $\mu = -\nu$ into Equation (75), using (74), and [19](with $x$ is replaced by $\frac{x}{q}$ )

$$\frac{[\nu]_q}{x}H_\nu^{(2)}(\lambda x|q^2) + \frac{1}{q}D_{q^{-1}}H_\nu^{(2)}(\lambda x|q^2) = \lambda q^{-(\nu+1)}H_{\nu-1}^{(2)}(\lambda x|q^2),$$

to obtain

$$\lambda q^{\nu+1}x^{1-\nu}H_{\nu+1}^{(2)}(\lambda x|q^2) - \frac{q^{-\nu}[2\nu]}{x^\nu}H_\nu^{(2)}(\lambda x|q^2) = \frac{\lambda^{\nu+1}C_\nu(q)x}{q^{\nu+1}[2\nu+1]_q} + \lambda q^{-\nu-1}x^{1-\nu}H_{\nu-1}^{(2)}(\lambda x|q^2),$$

we get (72). To prove (73), let $h(x) = H_\nu^{(2)}(\lambda x|q^2)$. Then

$$\frac{1}{q}D_{q^{-1}}D_qh(x) + \left(\frac{1}{x} - q\lambda^2x(1-q)\right)D_qh(x) + \left(q\lambda^2 - \frac{q^{1-\nu}[\nu]_q^2}{x^2}\right)h(x) = \frac{\lambda^{\nu+1}C_\nu(q)}{q^\nu}x^{\nu-1}. \tag{76}$$

Substituting with (76) into (15), we get (73) and completes the proof.

*Remark.*For $\Re(\nu) > -\frac{1}{2}$, $\lambda \in \mathbb{C}$, $|\lambda x(1-q)| < 1$ and $\Re(\mu+\nu) > -1$, Equation (70) can be represented as

$$\begin{aligned}&\int \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(A_{\mu,\nu}x^{-2}+q^\mu\lambda^2\right)H_\nu^{(2)}(\lambda x|q^2)\,d_qx \\ &= \frac{x^{\mu+1}}{(-x^2\lambda^2(1-q)^2;q^2)_\infty}\left(\frac{q^{-\mu}[\mu]_q}{x}H_\nu^{(2)}(\lambda x|q^2) - \frac{1}{q}D_{q^{-1}}H_\nu^{(2)}(\lambda x|q^2)\right) \\ &+ \frac{\lambda^{\nu+1}C_\nu(q)x^{\nu+\mu+1}}{q^{\nu+1}[\nu+\mu+1]_q}\,{}_2\phi_1(0,q^{\nu+\mu+1};q^{\nu+\mu+3};q^2,-x^2\lambda^2(1-q)^2).\end{aligned}$$

**Theorem 15.** *For $\Re(\nu) > -\frac{1}{2}$. Let $Sin_q(\cdot)$ and $Cos_q(\cdot)$ be the functions defined in (I.8) and (I.9), respectively. Set*

$$B_{n,\nu}(x) := \frac{q^{-n}[n]_q}{x} Sin_q(q^{n+\frac{1}{2}}x) + q^{\frac{3}{2}} Cos_q(q^{n+\frac{1}{2}}x),$$
$$\tilde{B}_{n,\nu}(x) := \frac{q^{-n}[n]_q}{x} Cos_q(q^{n+\frac{1}{2}}x) - q^{\frac{3}{2}} Sin_q(q^{n+\frac{1}{2}}x).$$

*Then*

$$\begin{aligned}&\int \left(x^{n-1}A_{n,\nu} Sin_q(q^{n+\frac{3}{2}}x) + x^n C_{n,\nu}(x) Cos_q(q^{n+\frac{3}{2}}x)\right) \frac{H_\nu^{(2)}(x|q^2)}{(-x^2(1-q)^2;q^2)_\infty} d_qx \\ &= \frac{x^{n+1}}{(-x^2(1-q)^2;q^2)_\infty}\left(B_{n,\nu}(x)H_\nu^{(2)}(x|q^2) - q Sin_q(q^{n+\frac{1}{2}}x) D_{q^{-1}} H_\nu^{(2)}(x|q^2)\right) \\ &+ q^{1-\nu} C_\nu(q) \int \frac{x^{\nu+n} Sin_q(q^{n+\frac{1}{2}}x)}{(-x^2(1-q)^2;q^2)_\infty} d_qx,\end{aligned}$$

*and*

$$\begin{aligned}&\int \left(x^{n-1}A_{n,\nu} Cos_q(q^{n+\frac{3}{2}}x) - x^n C_{n,\nu}(x) Sin_q(q^{n+\frac{3}{2}}x)\right) \frac{H_\nu^{(2)}(x|q^2)}{(-x^2(1-q)^2;q^2)_\infty} d_qx \\ &= \frac{x^{n+1}}{(-x^2(1-q)^2;q^2)_\infty}\left(\tilde{B}_{n,\nu}(x)H_\nu^{(2)}(x|q^2) - q Cos_q(q^{n+\frac{1}{2}}x) D_{q^{-1}} H_\nu^{(2)}(x|q^2)\right) \\ &+ q^{1-\nu} C_\nu(q) \int \frac{x^{\nu+n} Cos_q(q^{n+\frac{1}{2}}x)}{(-x^2(1-q)^2;q^2)_\infty} d_qx,\end{aligned}$$

*where $A_{n,\nu}$, $C_{n,\nu}(x)$, and $C_\nu(q)$ are defined in (34), (35), and (37), respectively.*

*Proof.* Substituting with $h(x) = x^n \mathrm{Sin}_q(q^{n+\frac{1}{2}}x)$, and $h(x) = x^n \mathrm{Cos}_q(q^{n+\frac{1}{2}}x)$, respectively, into (24) with $\lambda = 1$, we get the desired result.

**Theorem 16.** *Let $\Re\nu > \frac{-1}{2}$, $\lambda \in \mathbb{C}$, and $\Re(\mu+\nu) > -1$. If $H_\nu^{(1)}(\cdot;q^2)$ is the q-Struve function defined in (I.17), then*

$$\begin{aligned}&\int x^{\mu+1}(-x^2\lambda^2(1-q)^2;q^2)_\infty \left(A_{\mu,\nu}x^{-2} + q^{-(\mu+2)}\lambda^2\right) H_\nu^{(1)}(\lambda x|q^2)\, d_qx \\ &= x^{\mu+1}(-\frac{\lambda^2}{q^2}(1-q)^2x^2;q^2)_\infty \left(\frac{q^{-\mu}[\mu]_q}{x} H_\nu^{(1)}(\lambda x|q^2) - q^{-1} D_{q^{-1}} H_\nu^{(1)}(\lambda x|q^2)\right) \\ &+ \frac{\lambda^{\nu+1}C_\nu(q)x^{\nu+\mu+1}}{q^{\nu+1}[\nu+\mu+1]_q}\, {}_1\phi_1(q^{\nu+\mu+1};q^{\nu+\mu+3};q^2,-x^2\lambda^2(1-q)^2).\end{aligned}$$

*In particular,*

$$\begin{aligned}&\int x^{1+\nu}(-x^2\lambda^2(1-q)^2;q^2)_\infty H_\nu^{(1)}(\lambda x|q^2)\, d_qx = \frac{(qx)^{1+\nu}}{\lambda}(-\frac{\lambda^2}{q^2}(1-q)^2x^2;q^2)_\infty H_{\nu+1}^{(1)}(\frac{\lambda x}{q}|q^2) \\ &+ \frac{q\lambda^{\nu-1}C_\nu(q)x^{2\nu+1}}{[2\nu+1]_q}\left({}_1\phi_1(q^{2\nu+1};q^{2\nu+3};q^2,-x^2\lambda^2(1-q)^2) - (-\frac{\lambda^2}{q^2}(1-q)^2x^2;q^2)_\infty\right),\end{aligned}$$

$$\begin{aligned}\int x^{1-\nu}(-x^2\lambda^2(1-q)^2;q^2)_\infty H_\nu^{(1)}(\lambda x|q^2)\, d_qx &= \frac{-(qx)^{1-\nu}}{\lambda}(-\frac{\lambda^2}{q^2}(1-q)^2x^2;q^2)_\infty H_{\nu-1}^{(1)}(\frac{\lambda x}{q}|q^2) \\ &+ q^{-2\nu+1}\lambda^{\nu-1}C_\nu(q)x\, {}_1\phi_1(q;q^3;q^2,-x^2\lambda^2(1-q)^2),\end{aligned}$$

*and*

$$\begin{aligned}&C_\nu(q) \int x^\nu(-x^2\lambda^2(1-q)^2;q^2)_\infty J_\nu^{(1)}(\lambda x|q^2)\, d_qx = \\ &\frac{q^\nu x}{\lambda^{1+\nu}}(-\frac{\lambda^2}{q^2}(1-q)^2x^2;q^2)_\infty \left(J_\nu^{(1)}(\lambda x|q^2) D_{q^{-1}} H_\nu^{(1)}(\lambda x|q^2) - H_\nu^{(1)}(\lambda x|q^2) D_{q^{-1}} J_\nu^{(1)}(\lambda x|q^2)\right),\end{aligned}$$

*where $A_{n,\nu}$ and $C_\nu(q)$ are defined in (34) and (37), respectively.*

**Theorem 17.** *For $\Re(\nu) > \frac{-1}{2}$. Let $\sin_q(\cdot)$ and $\cos(\cdot)$ be the functions defined in (I.6) and (I.7), respectively. Set*

$$E_{n,\nu}(x) := \frac{q^{-n}[n]_q}{x}\sin_q(q^{-n-\frac{3}{2}}x) + q^{-(n+\frac{3}{2})}\cos_q(q^{-n-\frac{3}{2}}x),$$
$$\tilde{E}_{n,\nu}(x) := \frac{q^{-n}[n]_q}{x}\cos_q(q^{-n-\frac{3}{2}}x) - q^{-(n+\frac{3}{2})}\sin_q(q^{-n-\frac{3}{2}}x).$$

*Then*

$$\begin{aligned}&\int(-x^2(1-q)^2;q^2)_\infty\Big(A_{n,\nu}x^{n-1}\sin_q(q^{-n-\frac{3}{2}}x) + K_{n,\nu}x^n\cos_q(q^{-n-\frac{3}{2}}x)\Big)H_\nu^{(1)}(x|q^2)\,d_qx\\ &= x^{n+1}(-\frac{x^2}{q^2}(1-q)^2;q^2)_\infty\Big(E_{n,\nu}(x)H_\nu^{(1)}(\frac{x}{q}|q^2) - q^{-(n+1)}\sin_q(q^{-n-\frac{3}{2}}x)D_{q^{-1}}H_\nu^{(1)}(x|q^2)\Big)\\ &+ q^{-\nu-1}C_\nu(q)\int x^{\nu+n}(-x^2(1-q)^2;q^2)_\infty\sin_q(q^{-n-\frac{1}{2}}x)\,d_qx,\end{aligned}$$

*and*

$$\begin{aligned}&\int(-x^2(1-q)^2;q^2)_\infty\Big(A_{n,\nu}x^{n-1}\cos_q(q^{-n-\frac{3}{2}}x) - K_{n,\nu}x^n\sin_q(q^{-n-\frac{3}{2}}x)\Big)H_\nu^{(1)}(x|q^2)\,d_qx\\ &= x^{n+1}(-\frac{x^2}{q^2}(1-q)^2;q^2)_\infty\Big(\tilde{E}_{n,\nu}(x)H_\nu^{(1)}(\frac{x}{q}|q^2) - q^{-(n+1)}\cos_q(q^{-n-\frac{3}{2}}x)D_{q^{-1}}H_\nu^{(1)}(x|q^2)\Big)\\ &+ q^{-\nu-1}C_\nu(q)\int x^{\nu+n}(-x^2(1-q)^2;q^2)_\infty\cos_q(q^{-n-\frac{1}{2}}x)\,d_qx,\end{aligned}$$

*where $A_{n,\nu}$, $K_{n,\nu}(x)$, and $C_\nu(q)$ are defined in (34), (36), and (37), respectively.*

## 4 Applications to $q$-special functions

In this section, we apply the Lagrangian method we introduced in section 2 to the $q$-hypergeometric functions, the $q$-Laguerre and the big $q$-Laguerre polynomials, Stieltjes-Wigert polynomials, the discrete $q$-Hermite I and II polynomials, and the little and big $q$-Legendre polynomials.

### 4.1 The q-hypergeometric functions

**Theorem 18.** *Let $a,b,c$ be complex numbers, $c \neq 0$, $\alpha = \frac{[a]_q[b]_q}{[c]_q}$, $\beta = \frac{[a+b+1]_q}{[c]_q}$ and*

$$T_\alpha(x) := \frac{\alpha}{(1+\alpha(1-q)x)} - \frac{[a+b+1]_q}{q^{c+1}(1-q^{a+b-c}x)}.$$

*Then*

$$\int x^{c-1}(x;q)_{a+b-c}\,{}_2\phi_1(q^a,q^b;q^c;q,x)\,d_qx = \frac{x^c}{[c]_q}(\frac{x}{q};q)_{a+b+1-c}\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q}), \tag{77}$$

$$\begin{aligned}&\int x^c(x;q)_{a+b-c}\,{}_2\phi_1(q^a,q^b;q^c;q,x)\,d_qx = \frac{qx^c}{[a+1]_q[b+1]_q}(\frac{x}{q};q)_{a+b+1-c}\times\\ &\left(\left(1+\frac{[a]_q[b]_qx}{[c]_q}\right){}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q}) - {}_2\phi_1(q^a,q^b;q^c;q,x)\right),\end{aligned} \tag{78}$$

$$\begin{aligned}&\int x^c(x;q)_{a+b+1-c}(-\alpha(1-q)x;q)_\infty T_\alpha(x)\,{}_2\phi_1(q^a,q^b;q^c;q,x)\,d_qx =\\ &\frac{x^c}{q^c}(\frac{x}{q};q)_{a+b+1-c}(-\alpha(1-q)x;q)_\infty\left({}_2\phi_1(q^a,q^b;q^c;q,x) - {}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q})\right),\end{aligned} \tag{79}$$

*and*

$$\int \frac{x^c(q(\beta-\alpha)x;q)_\infty (x;q)_{a+b+1-c}}{(q^{-1}\beta x;q)_\infty}\,{}_2\phi_1(q^a,q^b;q^c;q,x)\,d_qx =$$
$$\frac{q^{1-c}(1-q)}{q\alpha+\beta(1-q)}x^c\frac{(\beta-\alpha)x;q)_\infty(\frac{x}{q};q)_{a+b+1-c}}{(q^{-1}\beta x;q)_\infty}\left({}_2\phi_1(q^a,q^b;q^c;q,x)-(1-q)(1-\frac{\beta}{q}x){}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q})\right). \quad (80)$$

*Proof.*The $q$-hypergeometric functions ${}_2\phi_1(q^a,q^b;q^c;q,x)$ satisfies the second-order $q$-difference equation (I.22). By comparing (I.22) with (7), we get

$$p(x)=\frac{(1-q^c)-(1-q^{a+b+1})\frac{x}{q}}{x(1-q)(q^c-q^{a+b}x)},\quad r(x)=-\frac{(1-q^a)(1-q^b)}{x(1-q)^2(q^c-q^{a+b}x)}.$$

If $f(x)$ satisfies (9), then

$$f(x)=x^c(x;q)_{a+b+1-c}.$$

The proof of (77), follows by substituting with $h(x)=1$ into Equation (13), and using

$$D_{q^{-1}}\,{}_2\phi_1(q^a,q^b;q^c;q,x)=\frac{(1-q^a)(1-q^b)}{(1-q^c)(1-q)}\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q}). \quad (81)$$

Substituting with $h(x)=x$ into (13), we get

$$\int x^{c-1}(x;q)_{a+b-c}\left([c]_q-\frac{x}{q}[a+1]_q[b+1]_q\right){}_2\phi_1(q^a,q^b;q^c;q,x)\,d_qx =$$
$$x^c(\frac{x}{q};q)_{a+b+1-c}\left({}_2\phi_1(q^a,q^b;q^c;q,x)-\frac{[a]_q[b]_qx}{[c]_q}\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,\frac{x}{q})\right). \quad (82)$$

Solving (77) with (82) we get (78). Equation (79) follows by taking $h(x)$ to be a solution of

$$\frac{(1-q^c)}{x(1-q)(q^c-q^{a+b}x)}D_{q^{-1}}h(x)-\frac{(1-q^a)(1-q^b)}{x(1-q)^2(q^c-q^{a+b}x)}h(x)=0,$$

I.e.

$$h(x)=(-\alpha(1-q)x;q)_\infty,\qquad \alpha=\frac{[a]_q[b]_q}{[c]_q}.$$

Finally, Equation (80) follows by taking $h(x)$ to be a solution of

$$\frac{(1-q^c)-(1-q^{a+b+1})\frac{x}{q}}{x(1-q)(q^c-q^{a+b}x)}D_{q^{-1}}h(x)-\frac{(1-q^a)(1-q^b)}{x(1-q)^2(q^c-q^{a+b}x)}h(x)=0.$$

Thus,

$$h(x)=\frac{((\beta-\alpha)x;q)_\infty}{(\beta x;q)_\infty}\qquad \beta=\frac{[a+b+1]_q}{[c]_q}.$$

## *4.2 The big q-Laguerre polynomials*

**Theorem 19.** *Let $a,b,c$ be complex numbers. If $p_n(x;a,b;q)$ is the big $q$-Laguerre polynomial of degree $n$ defined in (I.23), then*

$$\int \frac{x^m(\frac{x}{b},\frac{x}{a};q)_\infty}{(x;q)_\infty}\left(\frac{q[m]_q[m-1]_q}{x^2}-\frac{[m]_q((a+b)-q^mab)}{abx(1-q)}+\frac{[m-n]_q}{qab(1-q)}\right)p_n(x;a,b;q)\,d_qx$$
$$=\frac{x^m(\frac{x}{bq},\frac{x}{aq};q)_\infty}{(x;q)_\infty}\left(\frac{q[m]_qp_n(x;a,b;q)}{x}-\frac{q^{m-n+1}[n]_q}{(1-qa)(1-qb)}p_{n-1}(x;aq,bq;q)\right). \quad (83)$$

*In particular,*

$$\int \frac{(\frac{x}{b},\frac{x}{a};q)_\infty}{(x;q)_\infty} p_n(x;a,b;q)\,d_qx = \frac{abq^2(1-q)(\frac{x}{bq},\frac{x}{aq};q)_\infty}{(1-aq)(1-bq)(x;q)_\infty} p_{n-1}(x;aq,bq;q), \tag{84}$$

*and*

$$\int \frac{x^n(\frac{x}{b},\frac{x}{a};q)_\infty}{(qx;q)_\infty}\left(\frac{[n-1]_q}{x^2} - \frac{(a+b-qab)}{abqx(1-q)(1-x)}\right) p_n(x;a,b;q)\,d_qx = x^n \frac{(\frac{x}{bq},\frac{x}{aq};q)_\infty}{(x;q)_\infty}\left(\frac{p_n(x;a,b;q)}{x} - \frac{p_{n-1}(x;aq,bq;q)}{(1-aq)(1-bq)}\right). \tag{85}$$

*Proof.*The big $q$-Laguerre polynomial of degree $n$ is defined in (I.23) and satisfies the second order $q$-difference equation (I.24). By comparing (I.24) with (7), we get

$$p(x) = \frac{x-q(a+b-qab)}{abq^2(1-q)(1-x)} \qquad r(x) = -\frac{q^{-n-1}[n]_q}{ab(1-q)(1-x)}.$$

Thus $f(x) = \frac{(\frac{x}{b},\frac{x}{a};q)_\infty}{(qx;q)_\infty}$. The proof of (83) follows by substituting with $h(x) = x^m$ into (8) and using [20, Eq.(3.11.7)] (with $x$ is replaced by $\frac{x}{q}$)

$$D_{q^{-1}} p_n(x;a,b;q) = \frac{q^{1-n}[n]_q}{(1-aq)(1-bq)} p_{n-1}(x;aq,bq;q). \tag{86}$$

Substituting with $m=0$, into (83) gives (84). Substituting with $m=n$, into (83) gives (85).

*Remark.*Equation (84) is equivalent to [20, Eq.(3.11.9)] (with $n$ is replaced by $n-1$ )

$$D_q\left(w(x;aq,bq;q)p_{n-1}(x;aq,bq;q)\right) = \frac{(1-aq)(1-bq)}{abq^2(1-q)} w(x;a,b;q)p_n(x;a,b;q),$$

where $w(x;q) = \frac{(\frac{x}{b},\frac{x}{a};q)_\infty}{(x;q)_\infty}$ .

**Proposition 3.***If $n$ and $m$ are non-negative integers, then*

$$\int \frac{(\frac{x}{b},\frac{x}{a};q)_\infty}{(x;q)_\infty} p_m(x;a,b;q)p_n(x;a,b;q)\,d_qx = \frac{abq^2(1-q)(\frac{x}{bq},\frac{x}{aq};q)_\infty}{(1-aq)(1-bq)[n-m]_q(x;q)_\infty}$$
$$\left(-q^{n-m}[m]_q p_{m-1}(x;aq,bq;q)p_n(x;a,b;q) + [n]_q p_m(x;a,b;q)p_{n-1}(x;aq,bq;q)\right).$$

*Proof.*The proof follows by substituting with $h(x) = p_m(x;a,b;q)$ and $y(x) = p_n(x;a,b;q)$ into (8).

## *4.3 The q-Laguerre polynomials*

**Theorem 20.***If $L_n^\alpha(x;q)$ is the q-Laguerre polynomial of degree n defined in* (I.25) *and* $\mu_n = 1 - \frac{\ln(1+q-q^n)}{\ln q}$, *then*

$$\int \frac{x^{m+\alpha}}{(-x;q)_\infty}\left(\frac{(1-q^{m+\alpha})[m]_q}{x} + q^{m+\alpha}([n]_q - [m]_q)\right) L_n^\alpha(x;q)\,d_qx = \frac{x^{m+\alpha+1}}{(-x;q)_\infty}\left(\frac{(1-q^m)}{x} L_n^\alpha(x;q) + q^{\alpha+m} L_{n-1}^{\alpha+1}(x;q)\right), \tag{87}$$

$$\int \frac{x^\alpha(q^{\alpha+2}(1-q^n)x;q)_\infty}{(-qx;q)_\infty} L_n^\alpha(x;q)\,d_qx = \frac{x^{\alpha+1}(q^{\alpha+1}(1-q^n)x;q)_\infty}{q^{\alpha+1}(-x;q)_\infty}\left(\frac{L_{n-1}^{\alpha+1}(x;q)}{[n]_q} + (1-q)L_n^\alpha(x;q)\right), \tag{88}$$

*and*

$$\sum_{k=0}^{\infty}\left(\frac{q^{\alpha+1}}{1+q-q^n}\right)^k\left(\frac{q^2(1-q^{n-1})(1+a)}{1+q-q^n} + q^{-\alpha}(1-q^{\alpha+1})\right)(-qa;q)_k L_n^\alpha(q^k a;q)$$
$$= q^{-\alpha} L_n^\alpha(a;q) - \frac{qa}{1-q^n} L_{n-1}^{\alpha+1}(a;q). \tag{89}$$

*Proof.*The $q$-Laguerre polynomial of degree $n$ is defined in (I.25) and satisfies the second order $q$-difference equation (I.26). By comparing (I.26) with (7), we obtain

$$p(x) = \frac{1-q^{\alpha+1}(1+x)}{q^{\alpha+1}x(1+x)(1-q)}, \qquad r(x) = \frac{[n]_q}{x(1-q)(1+x)}.$$

Hence $f(x) = \dfrac{x^{\alpha+1}}{(-qx;q)_\infty}$ is a solution of Equation (9). The proof of (87) follows by substituting with $h(x) = x^m$ into (8), and using [20, Eq.(3.21.8)] (with $x$ is replaced by $\frac{x}{q}$ )

$$D_{q^{-1}} L_n^\alpha(x;q) = \frac{-q^{\alpha+1}}{(1-q)} L_{n-1}^{\alpha+1}(x;q). \tag{90}$$

Equation (88) follows by taking $h(x)$ as the solution of

$$\frac{1}{q^{\alpha+1}(1-q)x(1+x)} D_{q^{-1}} h(x) + \frac{[n]_q}{x(1-q)(1+x)} h(x) = 0,$$

which gives $h(x) = (q^{\alpha+1}(1-q^n)x;q)_\infty$. Taking $h(x)$ as the solution of

$$\frac{-1}{(1-q)(1+x)} D_{q^{-1}} h(x) + \frac{[n]_q}{x(1-q)(1+x)} h(x) = 0,$$

which gives $h(x) = x^{\mu_n}$, $\mu_n = 1 - \frac{\ln(1+q-q^n)}{\ln q}$. Substituting with $h(x)$ into (8), we get

$$\int \frac{x^{\alpha+\mu_n-1}}{(-qx;q)_\infty}\left(q[\mu_n-1]_q + \frac{[\alpha+1]_q}{q^\alpha(1+x)}\right) L_n^\alpha(x;q)\, d_qx = \frac{x^{\alpha+\mu_n+1}}{(-x;q)_\infty}\left(\frac{L_n^\alpha(x;q)}{q^\alpha x} + \frac{q^{\mu_n} L_{n-1}^{\alpha+1}(x;q)}{1-q_n^\mu}\right), \tag{91}$$

Hence (89) follows by using (1).

*Remark.*It is worth noting that

$$\int \frac{x^\alpha}{(-x;q)_\infty} L_n^\alpha(x;q)\, d_qx = \frac{x^{\alpha+1}}{[n]_q(-x;q)_\infty} L_{n-1}^{\alpha+1}(x;q),$$

which is the case $m = 0$ in (87) is equivalent to [20, Eq.(3.21.10)] (with $\alpha$ is replaced by $\alpha+1$ and $n$ is replaced by $n-1$ )

$$D_q\left(w(x;\alpha+1;q) L_{n-1}^{\alpha+1}(x;q)\right) = [n]_q w(x;\alpha;q) L_n^\alpha(x;q),$$

where $w(x;\alpha;q) = \dfrac{x^\alpha}{(-x;q)_\infty}$.

**Proposition 4.***If $m$ and $n$ are non-negative integers, then*

$$([n]_q - [m]_q) \int \frac{x^\alpha}{(-x;q)_\infty} L_n^\alpha(x;q) L_m^\alpha(x;q)\, d_qx = \frac{x^{\alpha+1}}{(-x;q)_\infty}\left(L_m^\alpha(x;q) L_{n-1}^{\alpha+1}(x;q) - L_{m-1}^{\alpha+1}(x;q) L_n^\alpha(x;q)\right).$$

*Proof.*The proof follows by substituting with $h(x) = L_m^\alpha(x;q)$ and $y(x) = L_n^\alpha(x;q)$ into (8).

If we substitute with $h(x) = L_m^\alpha(x;q)$ and $y(x) = L_n^\alpha(x;q)$ into (20), we get

$$\int_0^\infty \frac{x^\alpha}{(-x;q)_\infty} L_n^\alpha(x;q) L_m^\alpha(x;q)\, dx = 0, \text{if} \quad m \neq n,$$

which is the Orthogonality relation , see [20, Eq.(3.21.2)].

### 4.4 The Stieltjes-Wigert polynomials

**Theorem 21.** *If $S_n(x;q)$ is the Stieltjes-Wigert polynomial of degree n defined in (I.27), $c=\frac{1}{2\ln q}$ and $a>0$, then*

$$\sum_{k=0}^{\infty} q^{\frac{k^2+k}{2}} a^k S_n(q^k a;q) = \frac{S_{n-1}(qa;q)}{(1-q^n)}, \tag{92}$$

$$\sum_{k=0}^{\infty} q^{\frac{k^2+k}{2}} q^{mk} a^k \left(q^{2m}[n-m]_q + \frac{[m]_q}{q^k a}\right) S_n(q^k a;q) = \frac{[m]_q}{a} S_n(a;q) + \frac{q^m S_{n-1}(qa;q)}{1-q}, \tag{93}$$

$$\sum_{k=0}^{\infty} q^{\frac{k^2+k}{2}} a^k \left(-q^n + (1-q^n)q^{k+1}a\right) \frac{S_n(q^k a;q)}{(qa;q)_{k+1}} = (S_{n-1}(qa;q) - S_n(a;q), \tag{94}$$

$$\sum_{k=0}^{\infty} q^{\frac{k^2+k}{2}} a^k \frac{S_n(q^k a;q)}{(-q^{n+1}a;q)_{k+1}} = S_{n-1}(qa;q) + q^n S_n(a;q), \tag{95}$$

*and*

$$\sum_{k=0}^{\infty} \left(\frac{q}{1-q^n+q}\right)^k q^{\frac{k^2+k}{2}} a^k \left(\frac{q(1-q^{n-1})}{1-q^n+q} + \frac{q^{-k-1}}{a}\right) S_n(q^k a;q) = \frac{S_n(a;q)}{qa} + \frac{1}{1-q^n} S_{n-1}(qa;q). \tag{96}$$

*Proof.* The Stieltjes-Wigert polynomial is defined in (I.27) and satisfies the second-order $q$-difference equation (I.28). By comparing (I.28) with (7), we get

$$p(x) = \frac{1-qx}{qx^2(1-q)}, \qquad r(x) = \frac{[n]_q}{x^2(1-q)}.$$

Thus $f(x) = x^{\frac{3}{2}} e^{c\ln^2 x}$, ($c=\frac{1}{2\ln q}$) is a solution of Equation (9). Substituting with $h(x)=\frac{1-q}{[n]_q}$ into (8), and using [20, Eq.(3.27.7)] (with $x$ is replaced by $\frac{x}{q}$ )

$$D_{q^{-1}} S_n(x;q) = \frac{-q}{1-q} S_{n-1}(qx;q), \tag{97}$$

we get

$$\int x^{\frac{-1}{2}} e^{c\ln^2 x} S_n(x;q)\, d_q x = \frac{x^{\frac{3}{2}} e^{c\ln^2 \frac{x}{q}}}{\sqrt{q}[n]_q} S_{n-1}(qx;q), \tag{98}$$

and (92) follows from (1). Substituting with $h(x)=x^m$ into Equation (8), and using (97) and (1) yields (93). The proof of (94) follows by taking $h(x)$ as a solution of

$$\frac{1}{qx^2(1-q)} D_{q^{-1}} h(x) + \frac{1}{x^2(1-q)^2} h(x) = 0.$$

I.e. $h(x)=(qx;q)_\infty$ and using (1). Equation (95) follows from (1) by taking $h(x)$ as a solution of

$$\frac{1}{qx^2(1-q)} D_{q^{-1}} h(x) - \frac{q^n}{x^2(1-q)^2} h(x) = 0,$$

i.e, $h(x) = (-q^{n+1}x;q)_\infty$. Equation (96) follows by substituting in (1) with the solution of

$$\frac{-1}{x(1-q)} D_{q^{-1}} h(x) + \frac{[n]_q}{x^2(1-q)} h(x) = 0,$$

which is $h(x)=x^{\alpha_n}$, $\alpha_n = 1 - \dfrac{\ln(1+q-q^n)}{\ln q}$.

**Proposition 5.** *Let $n$ and $m$ be non-negative integers. Then*

$$([n]_q-[m]_q)\int x^{-\frac{1}{2}}e^{c\ln^2 x}S_m(x;q)S_n(x;q)\,d_qx=q^{-\frac{1}{2}}x^{\frac{3}{2}}e^{c\ln^2\frac{x}{q}}\left(S_m(x;q)S_{n-1}(qx;q)-S_{m-1}(qx;q)S_n(x;q)\right).$$

*Proof.* The proof follows by substituting with $h(x)=S_m(x;q)$ and $y(x)=S_n(x;q)$ into (8).

**Theorem 22.** *Let $n$ and $m$ be non-negative integers. If $S_n(x;q)$ is the Stieltjes-Wigert polynomial of degree $n$ defined in (I.27), then*

$$\int\frac{x^m}{(-x,\frac{-q}{x};q)_\infty}\left(q^{2m}[n-m]_q+\frac{[m]_q}{x}\right)S_n(x;q)\,d_qx=\frac{x^{m+2}}{q(\frac{-x}{q},\frac{-q^2}{x};q)_\infty}\left(\frac{(1-q^m)S_n(x;q)}{x}+q^mS_{n-1}(qx;q)\right),\tag{99}$$

$$\int\frac{(q^2x;q)_\infty}{(-x,\frac{-q}{x};q)_\infty}\left(-q^{n-1}+(1+q^n)x\right)S_n(x;q)\,d_qx=\frac{(1-q)x^2(qx;q)_\infty}{q^2(\frac{-x}{q},\frac{-q^2}{x};q)_\infty}\left(S_{n-1}(qx;q)-S_n(x;q)\right),\tag{100}$$

*and*

$$\int\frac{(-q^{n+2}x;q)_\infty}{(-x,\frac{-q}{x};q)_\infty}S_n(x;q)\,d_qx=\frac{(1-q)x^2(-q^{n+1}x;q)_\infty}{q(\frac{-x}{q},\frac{-q^2}{x};q)_\infty}\left(S_{n-1}(qx;q)+q^nS_n(x;q)\right).\tag{101}$$

*Proof.* From (I.28), we get $f(x)=\frac{x^2}{(-x,\frac{-q}{x};q)_\infty}$ is a solution of (9). Substituting with $h(x)=x^m$ into (8), and using (97) yields (99). The proof of (100) follows by taking $h(x)$ as a solution of

$$\frac{1}{qx^2(1-q)}D_{q^{-1}}h(x)+\frac{1}{x^2(1-q)^2}h(x)=0,$$

which gives $h(x)=(qx;q)_\infty$. Equation (101) follows by taking $h(x)$ as a solution of

$$\frac{1}{qx^2(1-q)}D_{q^{-1}}h(x)-\frac{q^n}{x^2(1-q)^2}h(x)=0,$$

i.e., $h(x)=(-q^{n+1}x;q)_\infty$.

*Remark.* The case $m=0$ in (99) is

$$\int\frac{S_n(x;q)}{(-x,\frac{-q}{x};q)_\infty}\,d_qx=\frac{x^2}{q[n]_q(\frac{-x}{q},\frac{-q^2}{x};q)_\infty}S_{n-1}(qx;q),$$

equivalent to [20, Eq.(3.27.9)] (with $x$ is replaced by $qx$ and $n$ is replaced by $n-1$ )

$$D_q\left(w(qx;q)S_{n-1}(qx;q)\right)=q^{-1}[n]_qw(x;q)S_n(x;q),$$

where $w(x;q)=\dfrac{1}{(-x,\frac{-q}{x};q)_\infty}$.

**Corollary 6.** *Let $n$ and $m$ be non-negative integers. Then*

$$\int_0^\infty\frac{1}{(-x,\frac{-q}{x};q)_\infty}S_m(x;q)S_n(x;q)dx=0\quad if\quad m\neq n,\tag{102}$$

*which is consistent with the orthogonality relation [20, Eq.(3.27.2)].*

*Proof.* The proof follows by taking $h(x)=S_m(x;q)$, we get

$$([n]_q-[m]_q)\int\frac{1}{(-x,\frac{-q}{x};q)_\infty}S_m(x;q)S_n(x;q)dx=\frac{x^2}{q(\frac{-x}{q},\frac{-q^2}{x};q)_\infty}\left(S_m(x;q)S_{n-1}(qx;q)-S_{m-1}(qx;q)S_n(x;q)\right).\tag{103}$$

Applying Theorem 3 gives (102).

## 4.5 The discrete q-Hermite I polynomials

**Theorem 23.** *If $h_n(x;q)$ is the discrete q-Hermite I polynomial of degree n defined in (I.29), then*

$$\int x^m(q^2x^2;q^2)_\infty\left(\frac{[m]_q[m-1]_q}{x^2}-\frac{[m-n]_q}{1-q}\right)h_n(x;q)\,d_qx=x^m(x^2;q^2)_\infty\left(\frac{[m]_qh_n(x;q)}{x}-q^{m-1}[n]_qh_{n-1}(\frac{x}{q};q)\right). \quad (104)$$

*Proof.* The discrete $q$-Hermite I polynomial of degree $n$ is defined in (I.29) and satisfies the second-order $q$-difference equation (I.30). By comparing (I.30) with (7), we get

$$p(x)=-\frac{x}{1-q},\qquad r(x)=\frac{q^{1-n}[n]_q}{1-q}.$$

Then we get $f(x)=(q^2x^2;q^2)_\infty$ is a solution of Equation (9). Substituting with $h(x)=x^m$ into Equation (8), and using [20, Eq.(3.28.7)] (with $x$ is replaced by $\frac{x}{q}$ )

$$D_{q^{-1}}h_n(x;q)=[n]_qh_{n-1}(\frac{x}{q};q), \quad (105)$$

we get (104).

*Example 2.* (i)If $m=n$ in (104), then

$$\int x^{n-2}(q^2x^2;q^2)_\infty h_n(x;q)\,d_qx=\frac{x^n(x^2;q^2)_\infty}{[n-1]_q}\left(\frac{h_n(\frac{x}{q};q)}{x}-q^{n-1}h_{n-1}(\frac{x}{q};q)\right),(n\neq 1). \quad (106)$$

(ii)If we calculate the definite $q$-integral from $-1$ to 1 in (106), we obtain

$$\int_{-1}^{1}x^{n-2}(q^2x^2;q^2)_\infty h_n(x;q)\,d_qx=0,$$

which is expected since $(h_n(x;q))$ is orthogonal on $[-1,1]$.

**Proposition 6.** *If $h_n(x;q)$ is the discrete q-Hermite I polynomial of degree n, and $m\in\mathbb{N}_0$, then*

$$\begin{aligned}&\left(q^{-n}[n]_q-q^{-m}[m]_q\right)\int(q^2x^2;q^2)_\infty h_n(x;q)h_m(x;q)\,d_qx\\&=\frac{1}{q}(x^2;q^2)_\infty\left((1-q^m)h_{m-1}(\frac{x}{q};q)h_n(x;q)-(1-q^n)h_{n-1}(\frac{x}{q};q)h_m(x;q)\right).\end{aligned}$$

*Proof.* The proof follows by substituting with $h(x)=h_m(x;q)$ and $y(x)=h_n(x;q)$ into (8).

## 4.6 The discrete q-Hermite II polynomials

**Theorem 24.** *If $\widetilde{h}_n(x;q)$ is the discrete q-Hermite II polynomial of degree n defined in (I.31), then*

$$\int\frac{\widetilde{h}_n(x;q)}{(-x^2;q^2)_\infty}\,d_qx=\frac{-q^{1-n}(1-q)}{(-x^2;q^2)_\infty}\widetilde{h}_{n-1}(x;q), \quad (107)$$

*and*

$$\int\frac{x^m}{(-x^2;q^2)_\infty}\left(\frac{[m]_q[m-1]_q}{x^2}+\frac{q^{2m-1}[n-m]_q}{1-q}\right)\widetilde{h}_n(x;q)\,d_qx=\frac{x^m}{(-x^2;q^2)_\infty}\left(\frac{[m]_q}{x}\widetilde{h}_n(x;q)-q^{m-n}[n]_q\widetilde{h}_{n-1}(x;q)\right). \quad (108)$$

*Proof.*The discrete $q$-Hermite II polynomial of degree $n$ is defined in (I.31)and satisfies the second-order $q$-difference equation (I.32). By comparing (I.32) with (7), we get

$$p(x) = -\frac{x}{1-q}, \qquad r(x) = \frac{[n]_q}{1-q}.$$

Then, $f(x) = \frac{1}{(-x^2;q^2)_\infty}$ is a solution of Equation (16). Substituting with $h(x) = \frac{(1-q)}{[n]_q}$ into (15), and using [20, Eq.(3.29.7)] (with $x$ is replaced by $\frac{x}{q}$ )

$$D_{q^{-1}}\widetilde{h}_n(x;q) = q^{1-n}[n]_q\widetilde{h}_{n-1}(x;q), \tag{109}$$

we get (107). Finally, substituting with $h(x) = x^m$ into (15), and using (109) yeilds (108).

*Remark.*It is worth noting that (107) is equivalent to

$$D_q\left(w(x;q)\widetilde{h}_{n-1}(x;q)\right) = \frac{q^{n-1}}{1-q}w(x;q)\widetilde{h}_n(x;q),$$

where $w(x;q) = \frac{1}{(-x^2;q^2)}$, see [20, Eq.(3.29.9)]. Moreover, for any $c > 0$,

$$\int_{-c}^{c} \frac{\widetilde{h}_n(x;q)}{(-x^2;q^2)_\infty}\,d_qx = \begin{cases} 0, & \text{if } n \text{ is odd;} \\ \frac{-2q^{1-n}(1-q)}{(-c^2;q^2)_\infty}\widetilde{h}_{n-1}(c;q), & \text{if } n \text{ is even.} \end{cases}$$

**Proposition 7.***If $\widetilde{h}_n(x;q)$ is the discrete $q$-Hermite II polynomial of degree $n$, and $m \in \mathbb{N}_0$, then*

$$\begin{aligned}&([n]_q - [m]_q)\int \frac{1}{(-x^2;q^2)_\infty}\widetilde{h}_n(x;q)\widetilde{h}_m(x;q)\,d_qx \\ &= \frac{q}{(-x^2;q^2)_\infty}\left(q^{-m}(1-q^m)\widetilde{h}_{m-1}(x;q)\widetilde{h}_n(x;q) - q^{-n}(1-q^n)\widetilde{h}_{n-1}(x;q)\widetilde{h}_m(x;q)\right).\end{aligned}$$

*Proof.*The proof follows by substituting with $\widetilde{h}_m(x;q)$ and $y(x) = \widetilde{h}_n(x;q)$ into (15).

**Proposition 8.***For $n \in \mathbb{N}$,*

$$\int p_n(x|q)\,d_qx = \frac{x}{q}(1-x)\,{}_2\phi_1(q^{-n+1}, q^{n+2}; q^2; q, x), \tag{110}$$

$$\int p_n(x;c;q)\,d_qx = \frac{q^{n-1}(q-x)(cq-x)}{[n]_q[n+1]_q(1-q)}\,{}_3\phi_2(q^{-n}, q^{n+1}, x; q; cq; q, q). \tag{111}$$

*Proof.*By comparing (I.34) with (7), we get

$$p(x) = \frac{qx+x-1}{qx(qx-1)}, \qquad r(x) = \frac{[n]_q[n+1]_q}{q^n x(1-qx)}.$$

Then $f(x) = x(1-qx)$ is a solution of Equation (9). From Equation (17), we get

$$\frac{1}{q}D_{q^{-1}}D_qh(x) + \frac{qx+x-1}{qx(qx-1)}D_{q^{-1}}h(x) + \frac{[n]_q[n+1]_q}{q^n x(1-qx)}h(x) = \frac{1}{x(1-qx)}.$$

Then $h(x) = \frac{q^n}{[n]_q[n+1]_q}$. From

$$D_{q^{-1}}p_n(x|q) = -q^{-n}[n]_q[n+1]_q\,{}_2\phi_1(q^{1-n}, q^{n+2}; q^2; q, x). \tag{112}$$

we obtain (110). Now, we prove (111). Comparing (I.36) with (7), we get

$$p(x) = \frac{x(1+q)-q(c+1)}{q^2(x-1)(x-c)}, \quad r(x) = -\frac{[n]_q[n+1]_q}{q^{1+n}(x-1)(x-c)}.$$

Then $f(x) = (1-x)(1-\frac{x}{c})$ is a solution of (9). From (17), we obtain

$$\frac{1}{q}D_{q^{-1}}D_q h(x) + \frac{x(1+q)-q(c+1)}{q^2(x-1)(x-c)}D_{q^{-1}}h(x) - \frac{[n]_q[n+1]_q}{q^{1+n}(x-1)(x-c)}h(x) = \frac{1}{(1-x)(1-\frac{x}{c})}.$$

Hence, $h(x) = \dfrac{cq^{n+1}}{[n]_q[n+1]_q}$. Thus, (111) follows from the $q$-difference equation

$$D_{q^{-1}}p_n(x;c;q) = \frac{-1}{1-q}\,{}_3\phi_2(q^{-n},q^{n+1},x;q;cq;q,q).$$

## Appendix

Jackson's three $q$-analogs of the Bessel functions, [9,12], are defined by

$$J_\nu^{(1)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n}{(q,q^{\nu+1};q)_n}(z/2)^{2n+\nu}, \quad |z|<2, \tag{I.1}$$

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+\nu)}}{(q,q^{\nu+1};q)_n}(z/2)^{2n+\nu}, \quad z\in\mathbb{C}, \tag{I.2}$$

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q,q^{\nu+1};q)_n}z^{2n+\nu}, \quad z\in\mathbb{C}. \tag{I.3}$$

Hahn introduced the identity, See [21],

$$J_\nu^{(2)}(z;q) = (-z^2/4;q)_\infty J_\nu^{(1)}(z;q), \quad |\,z\,|<2, \quad \nu>-1. \tag{I.4}$$

We use the simpler notation

$$J_\nu^{(2)}(\lambda x|q^2) := J_\nu^{(2)}(2\lambda x(1-q);q^2). \tag{I.5}$$

There are three known $q$-analogs of the trigonometric functions, $\{\sin_q z, \cos_q z\}$, $\{\mathrm{Sin}_q z, \mathrm{Cos}_q z\}$ and $\{\sin(z;q), \cos(z;q)\}$. Each set of $q$-analogs is related to one of the three $q$-analogs of Bessel functions.

The functions $\sin_q z$ and $\cos_q z$, see [19], are defined for $|z| < \frac{1}{1-q}$ by

$$\sin_q z = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(z)^{1/2}J_{1/2}^{(1)}(2z;q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{[2n+1]_q!}z^{2n+1}, \tag{I.6}$$

$$\cos_q z = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(z)^{1/2}J_{-1/2}^{(1)}(2z;q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n}{[2n]_q!}z^{2n}. \tag{I.7}$$

The functions $\mathrm{Sin}_q z$ and $\mathrm{Cos}_q z$ , see [19], are defined for $z\in\mathbb{C}$ by

$$\mathrm{Sin}_q z = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(z)^{1/2}J_{1/2}^{(2)}(2z;q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+n}}{[2n+1]_q!}z^{2n+1}, \tag{I.8}$$

$$\mathrm{Cos}_q z = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(z)^{1/2}J^{(2)}_{-1/2}(2z;q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2-n}}{[2n]_q!}z^{2n}. \tag{I.9}$$

The functions $\sin(z;q)$ and $\cos(z;q)$, see [19], are defined for $z\in\mathbb{C}$ by

$$\sin(z;q) = \Gamma_q(1/2)(z(1-q))^{1/2}J^{(3)}_{1/2}(z(1-q);q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{\Gamma_q(2n+2)}z^{2n+1}, \tag{I.10}$$

$$\cos(z;q) = \Gamma_q(1/2)(zq^{-1/2}(1-q))^{1/2}J^{(3)}_{-1/2}(z(1-q)/\sqrt{q};q^2) = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{\Gamma_q(2n+1)}z^{2n}. \tag{I.11}$$

The $q$-trigonometric functions satisfy the $q$-difference equations

$$D_q \sin_q z = \cos_q(z), \qquad D_q \cos_q z = -\sin_q(z). \tag{I.12}$$

$$D_q \mathrm{Sin}_q z = \mathrm{Cos}_q(qz), \qquad D_q \mathrm{Cos}_q z = -\mathrm{Sin}_q(qz). \tag{I.13}$$

$$D_q \sin(z;q) = \cos(q^{\frac{1}{2}}z;q), \qquad D_q \cos(z;q) = -q^{\frac{1}{2}}\sin(q^{\frac{1}{2}}z;q). \tag{I.14}$$

The third Jackson $q$-Bessel function $J^{(3)}_\nu(x;q^2)$ satisfies the second-order $q$-difference equation [15]

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{1}{qx}D_{q^{-1}}y(x) + \frac{q^{-\nu}}{(1-q)^2}\left(1-\frac{(1-q^\nu)^2}{x^2}\right)y(x) = 0. \tag{I.15}$$

The second Jackson $q$-Bessel function $J^{(2)}_\nu(\lambda x|q^2)$ satisfies the second-order $q$-difference equation [18]

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{1-q\lambda^2x^2(1-q)}{x}D_q y(x) + \frac{q\lambda^2x^2 - q^{1-\nu}[\nu]_q^2}{x^2}y(x) = 0. \tag{I.16}$$

Oraby and Mansour [19] introduced three $q$-analogs of the Bessel-Struve functions, $H^{(k)}_\nu(z;q^2)$, $(k=1,2,3)$, they are defined by

$$H^{(1)}_\nu(x|q^2) = \sum_{k=0}^{\infty}\frac{(-1)^k}{\Gamma_{q^2}(k+\frac{3}{2})\Gamma_{q^2}(k+\nu+\frac{3}{2})}\left(\frac{x}{1+q}\right)^{2k+\nu+1}, \ |x| < \frac{1}{1-q}, \tag{I.17}$$

$$H^{(2)}_\nu(x|q^2) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{2k^2+2k\nu+2k}}{\Gamma_{q^2}(k+\frac{3}{2})\Gamma_{q^2}(k+\nu+\frac{3}{2})}\left(\frac{x}{1+q}\right)^{2k+\nu+1}, \tag{I.18}$$

and

$$H^{(3)}_\nu(x;q^2) = \sum_{k=0}^{\infty}\frac{(-1)^k q^{k^2+k}}{\Gamma_{q^2}(k+\frac{3}{2})\Gamma_{q^2}(k+\nu+\frac{3}{2})}\left(\frac{x}{1+q}\right)^{2k+\nu+1}, x\in\mathbb{C}, \tag{I.19}$$

where $\Re(\nu) > -\frac{1}{2}$. The $q$-Struve function $H^{(3)}_\nu(\lambda x;q^2)$ associated with the third Jackson $q$-Bessel Equation satisfies the $q$-difference equation, see [19, Eq.(21)]

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{1}{qx}D_{q^{-1}}y(x) + \left(q^{-\nu-1}\lambda^2 - q^{-\nu}[\nu]_q^2x^{-2}\right)y(x) = \frac{\lambda^{\nu+1}x^{\nu-1}(1+q)^{-\nu+1}}{q^{\nu+1}\Gamma_{q^2}(\frac{1}{2})\Gamma_{q^2}(\nu+\frac{1}{2})}. \tag{I.20}$$

The $q$-Struve function $H^{(2)}_\nu(\lambda x|q^2)$ associated with the second Jackson $q$-Bessel Equation (I.16) satisfies the $q$-difference equation [19, Eq.(40)]

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \left(\frac{1}{x} - q\lambda^2x(1-q)\right)D_q y(x) + \left(q\lambda^2 - \frac{q^{1-\nu}[\nu]_q^2}{x^2}\right)y(x) = \frac{\lambda^{\nu+1}C_\nu(q)}{q^\nu}x^{\nu-1}. \tag{I.21}$$

The $q$-hypergeometric functions ${}_2\phi_1(q^a,q^b;q^c;q,x)$ satisfies the second-order $q$-difference equation [9]

$$\frac{1}{q}D_{q^{-1}}D_qy(x)+\frac{[c]_q-[a+b+1]_q\frac{x}{q}}{x(q^c-q^{a+b}x)}D_{q^{-1}}y(x)-\frac{[a]_q[b]_q}{x(q^c-q^{a+b}x)}y(x)=0. \tag{I.22}$$

The big $q$-Laguerre polynomial

$$p_n(x;a,b;q)={}_3\phi_2\left(\begin{matrix}q^{-n},0,x\\ aq,bq\end{matrix}\mid q;q\right) \tag{I.23}$$

satisfies the second-order $q$-difference equation, see [20, Eq.(3.11.5)],

$$\frac{1}{q}D_{q^{-1}}D_qy(x)+\frac{x-q(a+b-qab)}{abq^2(1-q)(1-x)}D_{q^{-1}}y(x)-\frac{q^{-n-1}[n]_q}{ab(1-q)(1-x)}y(x)=0. \tag{I.24}$$

The $q$-Laguerre polynomial

$$L_n^\alpha(x;q)=\frac{1}{(q;q)_n}{}_2\phi_1\left(\begin{matrix}q^{-n},-x\\ 0\end{matrix}\mid q;q^{n+\alpha+1}\right),\qquad \alpha>-1 \tag{I.25}$$

satisfies the second-order $q$-difference equation, see [20, Eq.(3.21.6)],

$$\frac{1}{q}D_{q^{-1}}D_qy(x)+\frac{1-q^{\alpha+1}(1+x)}{q^{\alpha+1}x(1+x)(1-q)}D_{q^{-1}}y(x)+\frac{[n]_q}{x(1-q)(1+x)}y(x)=0. \tag{I.26}$$

The Stieltjes and Hamburger moment problem associated with the Stieltjes-Wigert polynomials is indeterminate, and the polynomials are orthogonal to many different weight functions. For example, they are orthogonal to the weight function

$$w(x)=\frac{-c}{\sqrt{\pi}}e^{c\ln^2 x}\quad x>0\quad and\quad w(x)=\frac{x^2}{(-x,\frac{-q}{x};q)_\infty},$$

see [20,22]. The Stieltjes-Wigert polynomials

$$S_n(x;q)=\frac{1}{(q;q)_n}{}_1\phi_1\left(\begin{matrix}q^{-n}\\ 0\end{matrix}\mid q;-q^{n+1}x\right) \tag{I.27}$$

satisfies the second-order $q$-difference equation, see [20, Eq.(3.27.5)],

$$\frac{1}{q}D_{q^{-1}}D_qy(x)+\frac{1-qx}{qx^2(1-q)}D_{q^{-1}}y(x)+\frac{[n]_q}{x^2(1-q)}y(x)=0. \tag{I.28}$$

The discrete $q$-Hermite I polynomials is defined by

$$h_n(x;q)=q^{\binom{n}{2}}{}_2\phi_1\left(\begin{matrix}q^{-n},x^{-1}\\ 0\end{matrix}\mid q;-qx\right),n\in\mathbb{N}_0 \tag{I.29}$$

and satisfies the second-order $q$-difference equation, see [20, Eq.(3.28.5)],

$$\frac{1}{q}D_{q^{-1}}D_qy(x)-\frac{x}{1-q}D_{q^{-1}}y(x)+\frac{q^{1-n}[n]_q}{1-q}y(x)=0. \tag{I.30}$$

The discrete $q$-Hermite II polynomials is defined by

$$\widetilde{h}_n(x;q)=x^n\,{}_2\phi_1\left(\begin{matrix}q^{-n},q^{-n+1}\\ 0\end{matrix}\mid q^2;\frac{-q^2}{x^2}\right) \tag{I.31}$$

and satisfies the second-order $q$-difference equation, see [20, Eq.(3.29.5)],

$$\frac{1}{q}D_{q^{-1}}D_qy(x)-\frac{x}{1-q}D_qy(x)+\frac{[n]_q}{1-q}y(x)=0. \tag{I.32}$$

The little $q$-Legendre polynomials

$$p_n(x|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ q \end{matrix} \mid q; qx\right) \tag{I.33}$$

satisfies the second-order $q$-difference equation, see [20, Eq.(3.12.16)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{qx+x-1}{qx(qx-1)}D_{q^{-1}}y(x) + \frac{[n]_q[n+1]_q}{q^n x(1-qx)}y(x) = 0. \tag{I.34}$$

The big $q$-Legendre polynomials

$$p_n(x;c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}, x \\ q, cq \end{matrix} \mid q; q\right) \tag{I.35}$$

satisfies the second-order $q$-difference equation, see [20, Eq.(3.5.17)],

$$\frac{1}{q}D_{q^{-1}}D_q y(x) + \frac{x(1+q)-q(c+1)}{q^2(x-1)(x-c)}D_{q^{-1}}y(x) - \frac{[n]_q[n+1]_q}{q^{1+n}(x-1)(x-c)}y(x) = 0. \tag{I.36}$$

## 5 Conclusion

This work provides a method to obtain indefinite $q$-integrals for some $q$-special functions satisfying homogenous and nonhomogenous second-order $q$-difference equations. The results give indefinite $q$-integrals, which involve an arbitrary function and yield an infinite number of identities for any $q$-special function that obeys such a $q$-difference equation. The method in this paper is a generalization of the method introduced by Conway in [1,2].

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