Sohag Journal of Mathematics
An International Journal

http://dx.doi.org/10.18576/sjm/040102

Life Time Distributions: Derived from some Minimum Guarantee Distribution

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Received: 22 Nov 2016, Revised: 10 Dec. 2016, Accepted: 17 Dec. 2016
Published online: 1 Jan. 2017

Abstract: A transformation through some minimum guarantee distribution to get a new life time distribution is proposed. It is derived for the baseline distribution as exponential distribution. The new distribution, thus obtained have been shown better fit to a real data set in comparison to some existing distributions in terms of log- likelihood, *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *KS* (Kolmogorov- Smirnov) test statistics criterion.

Keywords: Life Time Distribution, Maximum Likelihood Estimator and Hazard Rate Function.
AMS Subject Classification: 62F15, 62N05.

1 Introduction

It is the era of generalizing/ transforming any available distribution in order to cover most of the real situations. In statistical literature, there are several methods to propose new distribution by the use of some available distributions, called baseline distributions. For example, Gupta et al. [1] proposed the cumulative distribution function (cdf) $G_1(x)$ of new distribution corresponding to the cdf $F(x)$ of a baseline distribution as, $G_1(x) = \{F(x)\}^{\alpha}$, where, $\alpha > 0$ is the shape parameter of the proposed one. Another idea of generalizing a baseline distribution is to transmute it by using quadratic rank transmutation map (QRTM) (see, Shaw and Buckley [2]). If $G_2(x)$ be the cdf of transmuted distribution corresponding to the baseline distribution having cdf $F(x)$, then $G_2(x) = (1+\lambda)F(x) - \lambda\{F(x)\}^2$, where $|\lambda| \leq 1$. Various generalizations has been introduced based on QRTM. For example, transmuted extreme value distribution (see, Aryal and Tsokos [5]), transmuted inverse Weibull distribution (see, Khan et al. [12]), transmuted modified Weibull distribution (see, Khan and King [11]), transmuted log-logistic distribution (see, Aryal [9]) and many more.

Recently, some authors transforms a baseline distribution in order to get new distributions, which is flexible and they have proved their applicability to some available real data. It also not suffers the excess of any additional parameters, hence it is parsimonious as well in terms of computation as it never contains any new parameter other than the parameter(s) involved in the baseline distribution. For example, Kumar et al. [14,15] introduced DUS and SS transformation to transform any available baseline distribution.

We propose a way to derive the new life time distributions from any available minimum guarantee distribution as follows; let $d(z)$ be the pdf of some minimum guarantee distribution of minimum guarantee as 1 unit, i.e. $z > 1$ and if $k(y)$ be the pdf of $y = \frac{1}{z}$, then it is given by $k(y) = d(z)\,|\frac{\partial z}{\partial y}|; 0 < y < 1$. Further, if $F(x)$ and $f(x)$ be the cdf and pdf of some baseline distribution, then the pdf $g(x)$ of new life time distribution on the basis of $F(x)$ and $f(x)$ is proposed as follows;

$$g(x) = k(F(x))\,f(x) \quad ; \quad x > 0 \tag{1}$$

We will call the transformation (1) as MG transformation for frequently used purpose in present article or elsewhere.

Any minimum guarantee distribution of arbitrary minimum guarantee of $\sigma > 0$ units (i.e. $z > \sigma > 0$) may be considered. The only change will be that the transformation $y = \frac{\sigma}{z}$ should be chosen instead of the transformation $y = \frac{1}{z}$ and accordingly other calculations will be changed.

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2 A Particular Case of (1)

Let $d(z)$ be the pdf of Exp (1, 1)- distribution, so that

$$d(z) = e^{z-1} \quad ; \quad z > 1 \tag{2}$$

Now, if $y = \frac{1}{z}$, then its pdf $k(y)$ is obtained as follows;

$$k(y) = \frac{e^{1-\frac{1}{y}}}{y^2} \quad ; \qquad 0 < y < 1 \tag{3}$$

Using (3), we propose the pdf g(x) of the new life time distribution corresponding to the baseline distribution having pdf $f(x)$ and cdf $F(x)$ as follows,

$$g(x) = \frac{e^{1-\frac{1}{F(x)}}}{(F(x))^2} f(x) \quad ; \quad 0 < x < \infty \tag{4}$$

and the cdf $G(x)$ corresponding to the pdf $g(x)$ is given by,

$$G(x) = e^{1-\frac{1}{F(x)}} \tag{5}$$

3 A Life Time Distribution Using (4)

In the present article, we propose the pdf $g(x)$ of a new lifetime distribution obtained by using (4) corresponding to the baseline distribution as Exp (θ)- distribution having pdf

$$f(x) = \theta\, e^{-\theta x} \quad ; \quad 0 < x < \infty, \theta > 0 \tag{6}$$

and cdf

$$F(x) = 1 - e^{-\theta x} \tag{7}$$

as follows,

$$g(x) = \frac{e^{-\frac{e^{-\theta x}}{1-e^{-\theta x}}}}{(1-e^{-\theta x})^2}\, \theta\, e^{-\theta x} \quad ; \quad 0 < x < \infty, \theta > 0 \tag{8}$$

The cdf and hazard rate function correponding to the pdf $g(x)$ are given by

$$G(x) = e^{-\frac{e^{-\theta x}}{1-e^{-\theta x}}} \tag{9}$$

and

$$h(x) = \frac{\theta\, e^{-\theta x}}{(1-e^{-\theta x})^2 \left(e^{\frac{e^{-\theta x}}{1-e^{-\theta x}}} - 1\right)} \tag{10}$$

respectively. We will denote the distribution having pdf $g(x)$ as given in (8) by $\mathrm{MG}_{Exp}(\theta)$-distribution.

The plots of pdf, cdf and hazard rate function of the new life time distribution for different values of the parameter θ have been shown in Figures 1, 2 and 3 respectively.
Further, mean, median and mode values of $\mathrm{MG}_{Exp}(\theta)$-distribution for different values of θ are as shown in Figure (4).

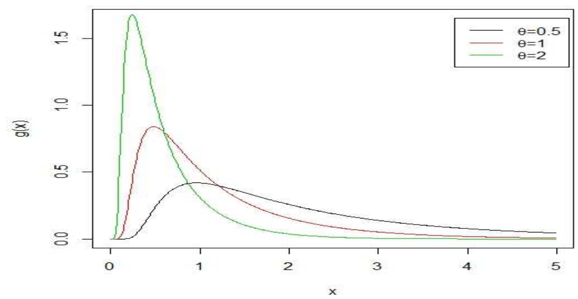

Fig. 1: Plots of pdf of $\mathrm{MG}_{Exp}(\theta)$-distribution for different values of θ

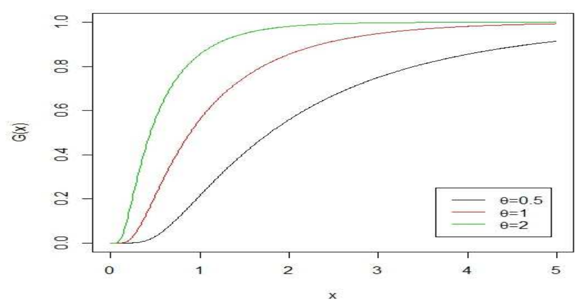

Fig. 2: Plots of cdf of $\mathrm{MG}_{Exp}(\theta)$-distribution for different values of θ

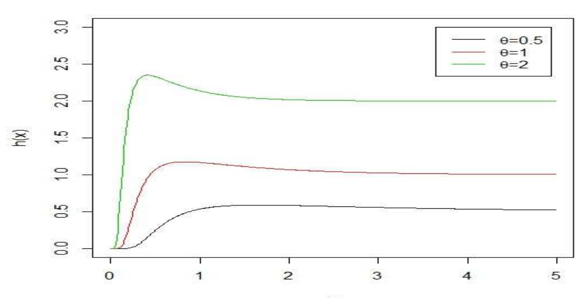

Fig. 3: Plots of hazard rate function of $\mathrm{MG}_{Exp}(\theta)$-distribution for different values of θ

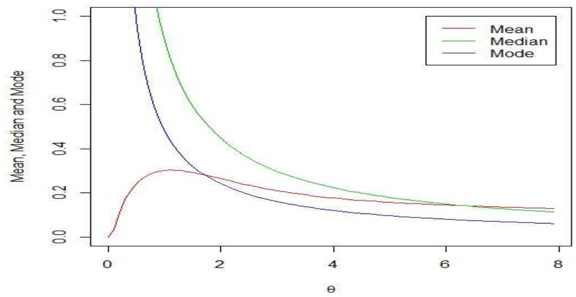

Fig. 4: Mean, Median and Mode of $\mathrm{MG}_{Exp}(\theta)$-distribution for different values of θ

4 Generation of a random sample from $g(x)$

For a known value of θ, a quantity x is said to be a random observation generated from the pdf $g(x)$ if $G(x) = u$, i.e. if

$$x = -\frac{1}{\theta}\,\ln\left(-\frac{\ln u}{1-\ln u}\right) \tag{11}$$

where u is an arbitrary continuous uniform point over (0, 1).

5 Estimation of parameter θ of $g(x)$

Let $\underline{\mathbf{X}} = (X_1, X_2, \ldots, X_n)$ be a random sample of size n from the $\mathrm{MG}_{Exp}(\theta)$-distribution having pdf $g(x)$, then the likelihood function of θ given $\underline{\mathbf{X}}$ is obtained as follows,

$$\begin{aligned} L &= \prod_{i=1}^{n} g(x_i) \\ &= \theta^n\, e^{-\theta\sum\limits_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{e^{-\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}}}}{(1-e^{-\theta x_i})^2} \end{aligned} \tag{12}$$

The logarithm of the likelihood function is given by,

$$\ln L = n\ln\theta - \theta\sum_{i=1}^{n} x_i - \sum_{i=1}^{n}\left\{\frac{e^{-\theta x_i}}{1-e^{-\theta x_i}} + 2\ln(1-e^{-\theta x_i})\right\} \tag{13}$$

Therefore, the maximum likelihood estimator (MLE) of the parameter θ is the solution of the following normal equation (obtained by equating the partial derivative of $\ln L$ w. r. t. θ to zero), i.e.

$$\frac{n}{\theta} - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{x_i e^{-\theta x_i}}{(1-e^{-\theta x_i})^2}\left\{1 - 2e^{-\theta x_i}\right\} = 0 \tag{14}$$

Above equation is not solvable analytically for θ. We propose Newton-Raphson method for its numerical solution.

6 Application

$\mathrm{MG}_{Exp}(\theta)$-distribution has been fitted to a real life time dataset to tests their goodness of fit. In order to compare $\mathrm{MG}_{Exp}(\theta)$-distribution with some well known distributions like Lindley and exponential distributions; -2 $\ln L$, *AIC* (Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *KS* test statistics (Kolmogorov-Smirnov test statistics) for the considered real life time dataset have been computed. The formulas for computing *AIC*, *BIC*, and *KS* tests statistics are as follows:

$$AIC = -2\ln L + 2k, BIC = -2\ln L + k\ln n \text{ and}$$
$$D = \sup_x |F_n(x) - F(x)|$$

where k the number of parameters, n is the sample size and $F_n(x)$ is the empirical CDF. The best distribution corresponds to lower $-2\ln L$, *AIC*, *BIC*, and *KS* statistics values.

We consider a real dataset which represents the relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark [3]. This dataset was considered by Shanker et al. [4] to compare goodness of fit on it among Lindley and exponential distributions and they found that Lindley distribution fits better than $\mathrm{Exp}(\theta)$- distribution in terms of some goodness of fit criteria.

Here, $-2\ln L$ *AIC*, *BIC*, and *KS* tests statistics values for $\mathrm{MG}_{Exp}(\theta)$-distribution have been calculated and their values are shown in the following Table. The values of the considered fitting criteria for Lindley and exponential distributions have been extracted from Shanker et al. [4] and they are also shown in the following comparative Table.

Table 1: MLEs, $-2\ln L$, *AIC*, *BIC* and *KS* test statistics of the fitted distributions for the considered data

Distributions	MLEs	$-2\ln L$	*AIC*	*BIC*	*KS*
$\mathrm{MG}_{Exp}(\theta)$- distribution	0.474991	52.17	54.16	55.17	0.25
Lindley distribution	0.816118	60.5	62.5	63.49	0.341
Exponential distribution	0.526316	65.67	67.67	68.67	0.389

From above comparative Table, it is quite clear that $\mathrm{MG}_{Exp}(\theta)$-distribution outperform Lindley and exponential distributions irrespective of the criteria and can be used effectively in the analysis of the considered data of the relief times (in minutes) of 20 patients receiving an analgesic, as $\mathrm{MG}_{Exp}(\theta)$- distribution has the lowest $-2\ln L$, *AIC*, *BIC*, and *KS* tests statistics values in comparison to those for the other two fitted models.

7 Conclusion

The present article may be fruitful to the researchers in the area of life testing experiment and the models derived by the use of the proposed methodology may produce several new life time distributions which may be flexible in the sense of parsimony in parameters as well as it may occupy different shapes of hazard rate function and can be fitted to most of the real datasets. Thus, we may recommend MG transformation to get new life time distributions with a hope that the new distribution, thus obtained may play a remarkable role in the real situations, like medical fields, engineering and technology etc.

References

[1] Gupta, R. C., Gupta, R. D. and Gupta, P. L. (1998): *Modeling failure time data by Lehman alternatives*. Communication in Statistics-Theory and Methods, 27 (4), 887-904.

[2] Shaw, W. T. and Buckley, I. R. (2007): *The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map*. Research report.

[3] Gross. A.J. and Clark, V. A. (1975): *Survival Distributions: Reliability Applications in the Biometrical Sciences*. John Wiley, New York, USA.

[4] Shanker R., Fesshaye, H. and Selvaraj, S. (2015): *On Modeling of Lifetimes Data Using Exponential and Lindley Distributions*. Biometrics & Biostatistics International Journal, 2 (5), 1-9.

[5] Aryal, G. R. and Tsokos, C. P. (2009): *On the transmuted extreme value distribution with application*. Nonlinear Analysis: Theory, Methods and Applications, 71 (12), 1401–1407,

[6] Singh, S. K., Singh, U. and Kumar, D. (2011): *Bayesian estimation of the exponentiated gamma parameter and reliability function under asymmetric loss function*. REVSTAT Statistical Journal, 9 (3), 247–260.

[7] Singh, S. K., Singh, U. and Kumar, D. (2011): *Estimation of Parameters and Reliability Function of Exponentiated Exponential Distribution: Bayesian approach Under General Entropy Loss Function*. Pakistan journal of statistics and operation research, VII (2), 119–216.

[8] Singh, S. K., Singh, U. and Kumar, D. (2012): *Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors*. Journal of Statistical Computation and Simulation, online dated 29 May, 2012, DOI:10.1080/ 009496552012. 690156, 1-12.

[9] Aryal, G. R. (2013): *Transmuted log-logistic distribution*. Journal of Statistics Applications and Probability, 2 (1), 11-20.

[10] Singh, S. K., Singh, U. and Kumar, D. (2013): *Bayesian estimation of parameters of inverse Weibull distribution*. Journal of Applied Statistics, 40 (7), 1597-1607.

[11] Khan, M. S. and King, R. (2013): *Transmuted modified Weibull distribution: A generalization of the modified Weibull probability distribution*. Europe. J. of Pure Appl. Math., 6 (1), 66–88

[12] Khan, M. S., King, R. and Hudson, I. L. (2014): *Characterisations of the transmuted inverse Weibull distribution*. ANZIAM J. 55 (EMAC-2013), C197–C217.

[13] Singh, S. K., Singh, U., Yadav, A. S. and Vishwkarma, P. K. (2015): *On the estimation of stress strength reliability parameter of inverted exponential distribution*. International Journal of Scientific World, 3 (1), 98–112.

[14] Kumar, D., Singh, U. and Singh, S. K. (2015): *A Method of Proposing New Distribution and its Application to Bladder Cancer Patients Data*. J. Stat. Appl. Pro. Lett. 2 (3), 235–245.

[15] Kumar, D., Singh, U. and Singh, S. K. (2015): *A New Distribution Using Sine Function- Its Application to Bladder Cancer Patients Data*. J. Stat. Appl. Pro. 4 (3), 417-427.

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