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Constructing New Solutions for Some Types of Two-Mode Nonlinear Equations

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Abstract: In this work we investigated further solutions of new family of nonlinear two-mode equations. We considered the two-mode Sharma-Tasso-Olver (TMSTO) equation and two-mode fourth-order Burgers (TMBE-4th). These models describe the propagation of two different wave modes simultaneously. We used the (G'/G)-expansion method and obtained more new solutions.

Keywords: Two-mode Sharma-Tasso-Olver, Two-mode fourth-order Burgers, (G'/G)-expansion method

1 Introduction

Two-mode nonlinear equations are second-order partial differential equations (PDE) in time. They represent the propagation of two-wave modes in the same direction simultaneously [1]. This new type of equations have been observed based on the fact that most of the nonlinear equations are defined by first-order PDE in time. They describe unidirectional waves where these equations model one right-moving for $x > 0$. Such equations is the well-known KdV. Other equations model two left-right-moving as in Boussinesq which is defined by second-order PDE in time. Therefore, Korsunsky [1] developed this phenomena and he was able to identify and derive two-mode KdV (TMKdV) as a nonlinear PDE of second order in time. Further, different types of solitary wave solutions are obtained to this TMKdV in [2,3,4,5,6,7,8,9].

Recently, some two-modes nonlinear equations have been established and studied. Wazwaz [10] obtained multiple kink solutions of the two-mode Sharma-Tasso-Olver (TMSTO) equation and two-mode fourth-order Burgers (TMBE-4th) by using the simplified Hirotas method. In [11,12,13], the simplified bilinear is used to study the two-mode coupled Burgers equation, the two-mode coupled modified Korteweg-de Vries and the two-mode coupled Korteweg-de Vries.

Many researchers have put their efforts into action to develop nonlinear dynamics during the last few decades. The studies of nonlinear wave phenomena have taken more than half a century to reproduce interesting and exciting descriptions on their formation and propagation [14]. One of their active contributions is the nonlinear plasma theory which is considered as the most important frontier for the fundamental understanding of proximal space of the earth and a rich testing ground for application of innovative mathematical methods. Most investigated among the solitons in plasma are the ion-acoustic solitary waves/solitons. Other examples of these applications are the one-dimensional gas flow, longitudinal wave propagation on a moving thread line, and electromagnetic transmission line.

The motivation of this work is to revisit the TMSTO and the TMBE-4th to extract more new solutions by using (G'/G)-expansion method [15,16,17,18]. These two-mode equations are defined respectively as

$$0 = u_{tt} - s^2 u_{xx} + \mu\left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right)\{(u^3 + 3uu_x)_x\} + \mu\left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right)u_{xxx}, \quad (1)$$

and

$$0 = u_{tt} - s^2 u_{xx} + \gamma\left(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x}\right)u_{xxxx} + \gamma\left(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x}\right)\{(u^4 + 4uu_{xx} + 6u^2 u_x + 3u_x^3)_x\}, \quad (2)$$

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where $u(x,t)$ is the field function and it represents the height of the water's free surface above a flat bottom, $-\infty < x,t < \infty$. The coefficients α and β are the non-linearity and the dispersion variables respectively such that $|\alpha| \leq 1$, $|\beta| \leq 1$ and s is a positive integer that is related to the phase velocities.

We should point here that the above two equations are established in [10] and soliton solutions are obtained under the condition $\alpha = \beta = 1$. In this work, we obtain new soliton solutions to the TMSTO for arbitrary values of α and β, while in the case of TMBE-4th we require a more general constraint which is $\alpha = \beta$.

2 Survey of (G'/G)-expansion method

Consider the following nonlinear partial differential equation:

$$P(u, u_t, u_x, u_{tt}, u_{xt}, \ldots) = 0, \tag{3}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. By the wave variable $\zeta = x - ct$ the PDE (3) is then transformed to an ordinary differential equation (ODE)

$$P(u, -cu', u', c^2u'', -cu'', u'', \ldots) = 0, \tag{4}$$

where $u = u(\zeta)$. We write the solution of the ODE (4) as a polynomial in (G'/G) as follows [15,16,17,18]

$$u(\zeta) = a_m(\frac{G'}{G})^m + \ldots, \tag{5}$$

where $G = G(\zeta)$ is the solution of

$$G'' + \lambda G' + \mu G = 0. \tag{6}$$

The coefficients $a_0, a_1, \ldots, a_m$ and the parameters λ, μ are constants to be determined later, provided that $a_m \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the ODE (4).

Now, if we let

$$Y = Y(\zeta) = \frac{G'}{G}, \tag{7}$$

then by the help of (6) we get

$$\begin{aligned} Y' &= \frac{GG'' - G'^2}{G^2} = \frac{G(-\lambda G' - \mu G) - G'^2}{G^2} \\ &= -\lambda Y - \mu - Y^2 \end{aligned} \tag{8}$$

or, equivalently

$$Y' = -Y^2 - \lambda Y - \mu. \tag{9}$$

By result (9) and implicit differentiation, one can derive the following two formulas

$$Y'' = 2Y^3 + 3\lambda Y^2 + (2\mu + \lambda^2)Y + \lambda\mu, \tag{10}$$

$$\begin{aligned} Y''' &= -6Y^4 - 12\lambda Y^3 - (7\lambda^2 + 8\mu)Y^2 \\ &- (\lambda^3 + 8\lambda\mu)Y - (\lambda^2\mu + 2\mu^2). \end{aligned} \tag{11}$$

Combining equations (5), (7) and (9-11), it results in a polynomial of powers of Y. Then, collecting all terms of same order of Y and equating to zero, yields a set of algebraic equations for $a_0, a_1, \ldots, a_m, \lambda$, and μ.

It is known that the solution of equation (6) is a linear combination of sinh and cosh or of sine and cosine, respectively, if $\triangle = \lambda^2 - 4\mu > 0$ or $\triangle < 0$. Without lost of generality, we consider the first case and therefore

$$G(\zeta) = \left(A \sinh(\frac{\sqrt{\triangle}\,\zeta}{2}) + B \cosh(\frac{\sqrt{\triangle}\,\zeta}{2})\right) e^{-\frac{\lambda\zeta}{2}}. \tag{12}$$

3 Two-mode Sharma-Tasso-Olver (TMSTO)

The TMSTO equation is given by:

$$\begin{aligned} 0 &= u_{tt} - s^2 u_{xx} + \gamma(\frac{\partial}{\partial t} - \alpha s \frac{\partial}{\partial x})\{(u^3 + 3uu_x)_x\} \\ &+ \gamma(\frac{\partial}{\partial t} - \beta s \frac{\partial}{\partial x})u_{xxx}, \end{aligned} \tag{13}$$

By using the wave variable $\zeta = x - ct$, equation (13) is then reduced in a simplified form to the ODE:

$$(c^2 - s^2)u - \gamma(c + \alpha s)(u^3 + 3uu') - \gamma(c + \beta s)u'' = 0 \tag{14}$$

By the proposed method, the solution of equation (14) is

$$u(\zeta) = \sum_{i=1}^{m} a_i(\frac{G'}{G})^i, \tag{15}$$

then, we require the following two secondary equations

$$u^3(\zeta) = a_m^3(\frac{G'}{G})^{3m} + \ldots \tag{16}$$

and

$$u''(\zeta) = m(m+1)a_m(\frac{G'}{G})^{m+2} + \ldots. \tag{17}$$

Considering the homogeneous balance between u^3 and u'' in equation (14), based on (16) and (17), we require that $3m = m + 2$. Thus $m = 1$, and therefore we can rewrite equation (15) as

$$u(\zeta) = a_1(\frac{G'}{G}) + a_0 = a_1 Y + a_0. \tag{18}$$

By the analysis given in the preceding section, we reach to the following main relations

$$u'(\zeta) = a_1(-Y^2 - \lambda Y - \mu), \tag{19}$$

$$u''(\zeta) = a_1(2Y^3 + 3\lambda Y^2 + (2\mu + \lambda^2)Y + \lambda\mu), \tag{20}$$

$$u^3(\zeta) = a_1^3 Y^3 + 3a_0 a_1^2 Y^2 + 3a_0^2 a_1 Y + a_0^3. \tag{21}$$

Substituting equations (19)-(21) into equation (14) and collecting all terms with the same power of Y together and equating each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for a_0, a_1, λ, c and μ.

$$\begin{aligned}
\left(\frac{G'}{G}\right)^3 = (Y^3) : 0 &= -2a_1c\gamma + 3a_1^2c\gamma - a_1^3c\gamma + 3a_1^2s\alpha\gamma \\
&- a_1^3s\alpha\gamma - 2a_1s\beta\gamma \\
\left(\frac{G'}{G}\right)^2 = (Y^2) : 0 &= 3a_0a_1c\gamma - 3a_0a_1^2c\gamma + 3a_0a_1s\alpha\gamma \\
&- 3a_0a_1^2s\alpha\gamma - 3a_1c\gamma\lambda + 3a_1^2c\gamma\lambda \\
&+ 3a_1^2s\alpha\gamma\lambda - 3a_1s\beta\gamma\lambda \\
\left(\frac{G'}{G}\right)^1 = (Y^1) : 0 &= a_1c^2 - a_1s^2 - 3a_0^2a_1c\gamma - 3a_0^2a_1s\alpha\gamma \\
&+ 3a_0a_1c\gamma\lambda + 3a_0a_1s\alpha\gamma\lambda - a_1c\gamma\lambda^2 \\
&- a_1s\beta\gamma\lambda^2 - 2a_1c\gamma\mu + 3a_1^2c\gamma\mu \\
&+ 3a_1^2s\alpha\gamma\mu - 2a_1s\beta\gamma\mu \\
\left(\frac{G'}{G}\right)^0 = (Y^0) : 0 &= a_0c^2 - a_0s^2 - a_0^3c\gamma - a_0^3s\alpha\gamma \\
&+ 3a_0a_1c\gamma\mu + 3a_0a_1s\alpha\gamma\mu \\
&- a_1c\gamma\lambda\mu - a_1s\beta\gamma\lambda\mu. \qquad (22)
\end{aligned}$$

Solving the above system gives two solutions:

The first solution is:

$$u_1(x,t) = \frac{M_1\left(A + B\ \tanh\left[\frac{(x-ct)\Delta_2}{2\sqrt{2}}\right]\right)}{\left(B + A\ \tanh\left[\frac{(x-ct)\Delta_2}{2\sqrt{2}}\right]\right)}. \qquad (23)$$

The second solution is:

$$u_2(x,t) = \frac{M_2\left(A + B\ \tanh\left[\frac{(x-ct)\Delta_3}{2\sqrt{2}}\right]\right)}{\left(B + A\ \tanh\left[\frac{(x-ct)\Delta_3}{2\sqrt{2}}\right]\right)}. \qquad (24)$$

Where

$$M_1 = \frac{(3c + 3s\alpha - \Delta_1)\Delta_2}{4\sqrt{2}(c + s\alpha)},$$

$$M_2 = \frac{(3c + 3s\alpha + \Delta_1)\Delta_3}{4\sqrt{2}(c + s\alpha)},$$

and

$$\Delta_1 = \sqrt{c + s\alpha}\sqrt{c + 9s\alpha - 8s\beta}$$

$$\Delta_2 = \sqrt{\frac{(c^2 - s^2)(5c + 9s\alpha - 4s\beta + 3\Delta_1)}{(c + s\beta)^2\gamma}}$$

$$\Delta_3 = \sqrt{\frac{(c^2 - s^2)(5c + 9s\alpha - 4s\beta - 3\Delta_1)}{(c + s\beta)^2\gamma}}. \qquad (25)$$

The provided figures represent Kink and singular-Kink of the TMSTO derived from equation (23) for some assigned values of the free parameters involved in the solution.

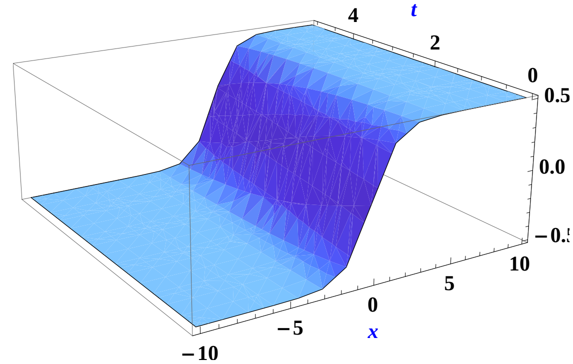

Fig. 1: Kink solution of TMSTO derived from equation (23) when $s = \frac{1}{4}$, $\lambda = \frac{1}{2}$, $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = 1$, $c = \frac{1}{2}$, $A = 0$, $B = 1$.

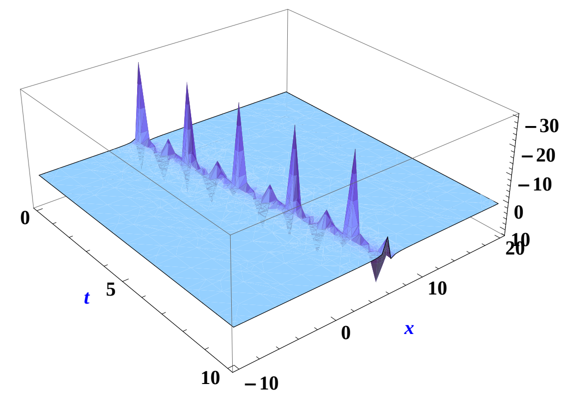

Fig. 2: Kink solution of TMSTO derived from equation (23) when $s = \frac{1}{4}$, $\lambda = \frac{1}{2}$, $\alpha = 1$, $\beta = \frac{1}{2}$, $\gamma = 1$, $c = \frac{1}{2}$, $A = -\frac{1}{2}$, $B = \frac{1}{3}$.

4 Two-mode fourth-order-Burgers equation (TMBE-4th order)

In this section we construct new solutions for TMBE-4th order:

$$\begin{aligned}
0 &= u_{tt} - s^2u_{xx} + \gamma\left(\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x}\right)u_{xxxx} \\
&+ \gamma\left(\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x}\right)\{(u^4 + 4uu_{xx} + 6u^2u_x + 3u_x^3)_x\}, \qquad (26)
\end{aligned}$$

where α, β, and s are defined earlier. Parallel to the analysis presented earlier, we use the wave variable $\zeta = x - ct$ to convert equation (26) into the ODE

$$\begin{aligned}
0 &= (c^2 - s^2)u - \gamma(c + \alpha s)(u^4 + 4uu'' + 6u^2u' + 3u'^2) \\
&- \gamma(c + \beta s)u'''. \qquad (27)
\end{aligned}$$

The solution of (27) is

$$u(\zeta) = \sum_{i=1}^{m} a_i\left(\frac{G'}{G}\right)^i. \qquad (28)$$

Also, we require the following equations

$$u^4(\zeta) = a_m^4(\frac{G'}{G})^{4m} + \ldots \quad (29)$$

$$u'''(\zeta) = m(m+1)(m+2)a_m(\frac{G'}{G})^{m+3} + \ldots. \quad (30)$$

Balancing the terms u^4 and u''' in equation (27) with the aid of equations (29) and (30), we find that $4m = m+3$. Thus $m = 1$, and accordingly, equation (35) can be rewritten as:

$$u(\zeta) = a_1(\frac{G'}{G}) + a_0 = a_1 Y + a_0. \quad (31)$$

Then we derive the following relations

$$u'^2(\zeta) = a_1^2(Y^4 + 2\lambda Y^3 + (\lambda^2+\mu)Y^2 + 2\lambda\mu Y + \mu^2), \quad (32)$$

$$\begin{aligned} u'''(\zeta) &= a_1(-6Y^4 - 12\lambda Y^3 - (7\lambda^2+8\mu)Y^2 \\ &- (\lambda^3+8\lambda\mu)Y - (\lambda^2\mu+2\mu^2)), \quad (33) \\ u^4 &= a_1^4Y^4 + 4a_0a_1^3Y^3 + 6a_0^2a_1^2Y^2 + 4a_0^3a_1Y + a_0^4. \quad (34) \end{aligned}$$

Substituting (32)-(34) in (27) and using the condition $\alpha = \beta$, yields a set of algebraic equations for $a_0, a_1, \beta, \lambda, \mu, \gamma, c$ and s.

$$\begin{aligned} (\frac{G'}{G})^4 &= (Y^4) : 0 = 6a_1c\gamma - 11a_1^2c\gamma + 6a_1^3c\gamma - a_1^4c\gamma \\ &+ 6a_1s\beta\gamma - 11a_1^2s\beta\gamma + 6a_1^3s\beta\gamma - a_1^4s\beta\gamma \\ (\frac{G'}{G})^3 &= (Y^3) : 0 = -8a_0a_1c\gamma + 12a_0a_1^2c\gamma - 4a_0a_1^3c\gamma \\ &- 8a_0a_1s\beta\gamma + 12a_0a_1^2s\beta\gamma - 4a_0a_1^3s\beta\gamma \\ &+ 12a_1c\gamma\lambda - 18a_1^2c\gamma\lambda + 6a_1^3c\gamma\lambda \\ &+ 12a_1s\beta\gamma\lambda - 18a_1^2s\beta\gamma\lambda + 6a_1^3s\beta\gamma\lambda, \end{aligned}$$

and

$$\begin{aligned} (\frac{G'}{G})^2 &= (Y^2) : 0 = 6a_0^2a_1c\gamma - 6a_0^2a_1^2c\gamma + 6a_0^2a_1s\beta\gamma \\ &- 12a_0a_1c\gamma\lambda + 12a_0a_1^2c\gamma\lambda - 12a_0a_1s\beta\gamma\lambda \\ &+ 12a_0a_1^2s\beta\gamma\lambda + 7a_1c\gamma\lambda^2 - 7a_1^2c\gamma\lambda^2 \\ &+ 7a_1s\beta\gamma\lambda^2 - 7a_1^2s\beta\gamma\lambda^2 + 8a_1c\gamma\mu \\ &- 14a_1^2c\gamma\mu + 6a_1^3c\gamma\mu + 8a_1s\beta\gamma\mu - 14a_1^2s\beta\gamma\mu \\ &+ 6a_1^3s\beta\gamma\mu - 6a_0^2a_1^2s\beta\gamma, \end{aligned}$$

and

$$\begin{aligned} (\frac{G'}{G})^1 &= (Y^1) : 0 = a_1c^2 - a_1s^2 - 4a_0^3a_1c\gamma - 4a_0^3a1s\beta\gamma \\ &+ 6a_0^2a_1c\gamma\lambda + 6a_0^2a_1s\beta\gamma\lambda - 4a_0a_1c\gamma\lambda^2 \\ &- 4a_0a_1s\beta\gamma\lambda^2 + a_1c\gamma\lambda^3 + a_1s\beta\gamma\lambda^3 - 8a_0a_1c\gamma\mu \\ &+ 12a_0a_1^2c\gamma\mu - 8a_0a_1s\beta\gamma\mu + 12a_0a_1^2s\beta\gamma\mu \\ &+ 8a_1c\gamma\lambda\mu - 10a_1^2c\gamma\lambda\mu + 8a_1s\beta\gamma\lambda\mu \\ &- 10a_1^2s\beta\gamma\lambda\mu, \end{aligned}$$

$$\begin{aligned} (\frac{G'}{G})^0 &= (Y^0) : 0 = a_0c^2 - a_0s^2 - a_0^4c\gamma - a_0^4s\beta\gamma \\ &+ 6a_0^2a_1c\gamma\mu + 6a_0^2a_1s\beta\gamma\mu - 4a_0a_1c\gamma\lambda\mu \\ &- 4a_0a_1s\beta\gamma\lambda\mu + a_1c\gamma\lambda^2\mu + a_1s\beta\gamma\lambda^2\mu \\ &+ 2a_1c\gamma\mu^2 - 3a_1^2c\gamma\mu^2 + 2a_1s\beta\gamma\mu^2 \\ &- 3a_1^2s\beta\gamma\mu^2. \quad (35) \end{aligned}$$

Solving this obtained algebraic system, gives eight solutions

$$u_1(x,t) = \frac{(A+B)\Phi\left(1+\tanh\left[\frac{1}{4}\Phi(2x+t\Delta_4)\right]\right)}{2\left(B+A\ \tanh\left[\frac{1}{4}\Phi(2x+t\Delta_4)\right]\right)}, \quad (36)$$

$$u_2(x,t) = \frac{(A+B)\Phi\left(1+\tanh\left[\frac{1}{2}\Phi(x-\frac{1}{2}t\Delta_5)\right]\right)}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x-\frac{1}{2}t\Delta_5)\right]\right)}, \quad (37)$$

$$u_3(x,t) = \frac{(A-B)\Phi\left(1-\tanh\left[\frac{1}{4}\Phi(2x+t\Delta_6)\right]\right)}{2\left(B+A\ \tanh\left[\frac{1}{4}\Phi(2x+t\Delta_6)\right]\right)}, \quad (38)$$

$$u_4(x,t) = \frac{(A-B)\Phi\left(1-\tanh\left[\frac{1}{2}\Phi(x-\frac{1}{2}t\Delta_7)\right]\right)}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x-\frac{1}{2}t\Delta_7)\right]\right)}, \quad (39)$$

$$\begin{aligned} u_5(x,t) &= \frac{1}{6}(3\lambda+\sqrt{3}\Psi) \\ &+ \frac{(-B\lambda+A\Phi)}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x+\frac{1}{6\sqrt{3}}t\Delta_8)\right]\right)} \\ &+ \frac{(-A\lambda+B\Phi)\ \tanh\left[\frac{1}{2}\Phi(x+\frac{1}{6\sqrt{3}}t\Delta_8)\right]}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x+\frac{1}{6\sqrt{3}}t\Delta_8)\right]\right)}, \quad (40) \end{aligned}$$

$$\begin{aligned} u_6(x,t) &= \frac{1}{6}(3\lambda+\sqrt{3}\Psi) \\ &+ \frac{(-B\lambda+A\Phi)}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x-\frac{1}{6\sqrt{3}}t\Delta_9)\right]\right)} \\ &+ \frac{(-A\lambda+B\Phi)\ \tanh\left[\frac{1}{2}\Phi(x-\frac{1}{6\sqrt{3}}t\Delta_9)\right]}{2\left(B+A\ \tanh\left[\frac{1}{2}\Phi(x-\frac{1}{6\sqrt{3}}t\Delta_9)\right]\right)}, \quad (41) \end{aligned}$$

$$u_7(x,t) = \frac{1}{6}(3\lambda - \sqrt{3}\Psi) + \frac{(-B\lambda + A\Phi)}{2\left(B + A\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{10})\right]\right)} + \frac{(-A\lambda + B\Phi)\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{10})\right]}{2\left(B + A\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{10})\right]\right)}, \quad (42)$$

$$u_8(x,t) = \frac{1}{6}(3\lambda - \sqrt{3}\Psi) + \frac{(-B\lambda + A\Phi)}{2\left(B + A\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{11})\right]\right)} + \frac{(-A\lambda + B\Phi)\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{11})\right]}{2\left(B + A\tanh\left[\frac{1}{2}\Phi(x - \frac{1}{6\sqrt{3}}t\Delta_{11})\right]\right)}, \quad (43)$$

where

$$\begin{aligned}
\Delta_4 &= -\gamma\Phi^3 + \sqrt{4s^2 + 4s\beta\gamma\Phi^3 + \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_5 &= \gamma\Phi^3 + \sqrt{4s^2 + 4s\beta\gamma\Phi^3 + \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_6 &= \gamma\Phi^3 + \sqrt{4s^2 - 4s\beta\gamma\Phi^3 + \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_7 &= -\gamma\Phi^3 + \sqrt{4s^2 - 4s\beta\gamma\Phi^3 + \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_8 &= \gamma\Psi^3 + \sqrt{108s^2 - 12\sqrt{3}s\beta\gamma\Psi^3 - \gamma^2(\lambda^2 - 4\mu)^3},
\end{aligned} \quad (44)$$

$$\begin{aligned}
\Delta_9 &= -\gamma\Psi^3 + \sqrt{108s^2 - 12\sqrt{3}s\beta\gamma\Psi^3 - \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_{10} &= \gamma\Psi^3 - \sqrt{108s^2 + 12\sqrt{3}s\beta\gamma\Psi^3 - \gamma^2(\lambda^2 - 4\mu)^3},\\
\Delta_{11} &= \gamma\Psi^3 + \sqrt{108s^2 + 12\sqrt{3}s\beta\gamma\Psi^3 - \gamma^2(\lambda^2 - 4\mu)^3},\\
\Phi &= \sqrt{\lambda^2 - 4\mu},\\
\Psi &= \sqrt{-\lambda^2 + 4\mu}.
\end{aligned}$$

5 Applications: Effect of α, β and s on the field function of TMSTO

In this part, we study graphically the effect of the phase velocity s, the non-linearity parameter α and the dispersion parameter β on the behavior of the field function $u(x,t)$ for the TMSTO equation. By fixing the values of the other coefficients and parameters involved in (23), we reached to the following findings.

1. By increasing the phase velocities parameter s, the field function decreases. See Figure 3.

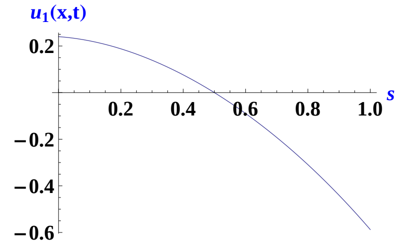

Fig. 3: Influence of the phase velocities parameter s on the field function obtained in (23) for $x = 1,\ t = 1,\ \alpha = 0.3,\ \beta = \frac{1}{4},\ c = 0.5,\ A = 0,\ B = 1,\ \lambda = 0.5,\ \gamma = 1$ and $0 < s < 1$.

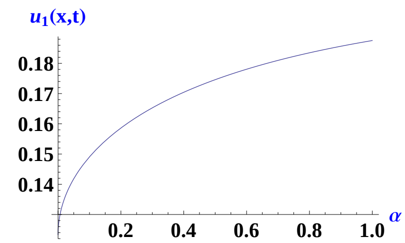

Fig. 4: Influence of the phase velocities parameter s on the field function obtained in (23) for $x = 1,\ t = 1,\ s = 0.25,\ \beta = \frac{1}{4},\ c = 0.5,\ A = 0,\ B = 1,\ \lambda = 0.5,\ \gamma = 1$ and $0 < s < 1$.

2. By increasing the non-linearity parameter α, the field function increases. See Figure 4.
3. By increasing the dispersion parameter β, the field function decreases. See Figure 5.

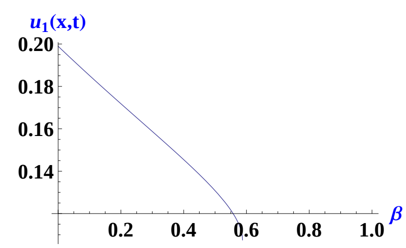

Fig. 5: Influence of the phase velocities parameter s on the field function obtained in (23) for $x = 1,\ t = 1,\ s = 0.25,\ \alpha = 0.3,\ c = 0.5,\ A = 0,\ B = 1,\ \lambda = 0.5,\ \gamma = 1$ and $0 < s < 1$.

6 Conclusion

In this paper we studied the solution of two-mode nonlinear models. The (G'/G)-expansion method is used and we obtained more new soliton solutions to the

TMSTO for arbitrary values of α and β, while in the case of TMBE-4th we require a more general condition which is $\alpha = \beta$.

As future work, we aim to establish more two-mode nonlinear equations and search for its solitary wave solutions using different ansatze methods such as: sine-cosine method, first integral method, sech-tanh method and rational trigonometric function method [19, 20, 21, 22, 23].

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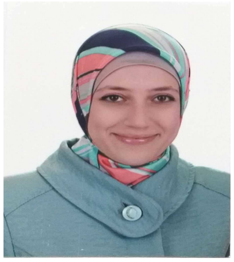

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