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Numerical Solutions for Nonlocal Problem of Partial Differential Equations with Deviated Boundary Conditions

A.M.A. El-Sayed[1], A. Elsaid[2], S. Helal[2]

[1]*Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt.*
[2]*Department of Mathematics and Engineering Physics, Faculty of Engineering, Mansoura University, Mansoura, Egypt.*
amasayed@hotmail.com (**A.M.A. El-Sayed**), a_elsaid@mans.edu.eg (**A. Elsaid**), shmhelal@yahoo.com (**S. Helal**)

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Abstract: In this work, we propose a model of nonlocal partial differential equation (PDE) with deviated type function in the boundary condition. This model is solved numerically by finite difference method (FDM) using variable space grid (VSG) technique. The results obtained by this method are in a good agreement with the solution of the corresponding rectangular domain problem. Also, we investigated the stability analysis of problem technique by using von-Neumann method.

Keywords: Nonlocal partial differential equation; Deviated boundary condition; Variable space grid method.

1. Introduction

Nonlocal boundary value problems for PDEs have been utilized by researchers to model numerous processes in different fields of applied sciences (see, for example, [1-7]). Numerical methods and theory of solutions for this type of problems are carried out in [8-11]. Problems of nonlocal type include chemical diffusion, thermo-elasticity, heat conduction processes, population dynamics, inverse problems, control theory and certain biological processes [11-20]. These physical phenomena are modeled by nonclassical parabolic, elliptic [21], hyperbolic [22] or hyperbolic-parabolic equation [23].

Nonlocal boundary value problems can be classified according to what is being unknown, initial condition or boundary conditions. The purpose of this paper is to propose a new nonlocal PDE with deviated boundary conditions of the following form

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(x,t),\quad 0<x<1,\quad t\geq 0, \tag{1.1}$$

subject to the initial and boundary conditions

$$u(x,0)=u_0(x),\qquad 0<x<1, \tag{1.2}$$

$$u(\psi_1(t),t)=u_1(t),\qquad t\geq 0, \tag{1.3}$$

$$u(1-\psi_2(t),\, t)=u_2(t),\quad t\geq 0, \tag{1.4}$$

where $\psi_i(t):[0,T]\to[0,T],\ \ i=1,2$ and $\psi_i(t)$ are deviated functions satisfying $\psi_i(t)\leq t$.

The solution behavior of this problem is numerically simulated using the VSG scheme that we construct based on the explicit FDM of the proposed PDE. The idea of this technique is suggested by Murray and Landis in [24] and used by other authors to solve one dimension Stefan problem as in [25, 26].

The paper is organized as follows: The numerical technique used to solve the problem is presented in section 2. In section 3, we discuss the stability analysis for the numerical technique. Some numerical results are presented in section 4. Finally, a conclusion of this work is given in section 5.

2. Numerical technique

The VSG is applied to find the numerical solution for problem (1.1)-(1.4). In this technique, the number of space intervals between the two boundaries is kept fixed and we denote it by *N*. Thus, as the domain size changes, the number of nodes remains fixed but the size of space intervals changes. By tracking particular grid lines for the *j*th grid point we have

$$\left.\frac{\partial u}{\partial t}\right|_j = \left.\frac{\partial u}{\partial x}\right|_t \left.\frac{d\,x}{d\,t}\right|_j + \left.\frac{\partial u}{\partial t}\right|_x \qquad (2.1)$$

The first step is to create a mesh with grid points (x_j^n, t_n) where $j = 0, 1, \ldots, N$ and $n = 0, 1, \ldots, m$ denote the indices in the x and t directions, respectively. Then we have the mesh constants $x_j^n = \psi_1(t_n) + j\,h_n$, $h_n = \frac{1-\psi^*(t_n)}{N}$, where $\psi^*(t) = \psi_1(t) + \psi_2(t)$ and for final time t_f, $k = \frac{t_f}{m}$.

Then, we can write

$$\frac{d\,x_j^n}{d\,t} = \frac{x_j^{n+1} - x_j^n}{k}$$

$$= \frac{\psi_1(t_{n+1}) + jh_{n+1} - \psi_1(t_n) - jh_n}{k}$$

$$= \frac{\frac{j}{N}\left(\left(1-\psi^*(t_{n+1})\right) - \left(1-\psi^*(t_n)\right)\right)}{k} + \frac{\psi_1(t_{n+1}) - \psi_1(t_n)}{k}$$

$$= \frac{1}{\left(1-\psi^*(t_n)\right)}\left(\left(1 - x_j^n - \psi_2(t_n)\right)\dot{\psi}_1\big|_{t=t_n} - \left(x_j^n - \psi_1(t_n)\right)\dot{\psi}_2\big|_{t=t_n}\right) \qquad (2.2)$$

Then, equation (1.1) after substitute equation (2.1) and the FDM of $\frac{dx}{dt}$ takes the form

$$\frac{\partial u}{\partial t} = \frac{\frac{\partial u}{\partial x}}{\left(1-\psi^*\right)}\left(\left(1 - x - \psi_2\right)\dot{\psi}_1 - \left(x - \psi_1\right)\dot{\psi}_2\right) + \frac{\partial^2 u}{\partial x^2} + f(x, t), \; 0 < x < 1, \qquad (2.3)$$

subject to the conditions (1.2)-(1.4). Hence, the following explicit finite difference scheme for the heat equation (2.3)

$$u_j^{n+1} = u_j^n + k\,\frac{u_{j+1}^n - u_{j-1}^n}{2h_{n+1}}\,\frac{1}{\left(1-\psi^*(t_n)\right)}\left(\left(1 - x_j^n - \psi_2(t_n)\right)\dot{\psi}_1 - \left(x_j^n - \psi_1(t_n)\right)\dot{\psi}_2\right) + \frac{k}{h_n^2}\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) + k\,f(x_j^n, t_n)$$

$$0 < x < 1, \quad t \geq 0, \qquad (2.4)$$

with the initial and boundary condition

$$u\left(x_j, 0\right) = u_0(x_j) \qquad 0 < x < 1, \quad (2.5)$$

$$u(\psi_1(t_n), t_n) = u_1(t_n), \qquad t > 0, \quad (2.6)$$

$$u(1-\psi_2(t_n),\, t_n) = u_2(t_n), \quad t > 0, \quad (2.7)$$

3. Stability condition

By using von-Neumann's method [27] to simulate the stability analysis of equation (2.4), we seek a solution of the finite difference equations having the form

$$u_j^n = e^{i\,j\,\beta h_n}\, e^{n\lambda k}, \qquad (3.1)$$

where $i = \sqrt{-1}$, the behavior of this solution is examined as $t \to \infty$ or as $n \to \infty$. On substituting the equation (3.1) into (2.4), we obtain

$$e^{i\,j\,\beta h_n} e^{(n+1)\lambda k} = e^{i\,j\,\beta h_n} e^{n\lambda k} + \frac{k}{2h_n(1-\psi^*(t_n))}\left((1 - x_j^n - \psi_2(t_n))\,\dot{\psi}_1(t_n) - (x_j^n - \psi_1(t_n))\,\dot{\psi}_2(t_n)\right) * \left(e^{i\,(j+1)\beta h_n} e^{n\lambda k} - e^{i\,(j-1)\beta h_n} e^{n\lambda k}\right) + \frac{k}{h_n^2}\left(e^{i\,(j+1)\beta h_n} e^{n\lambda k} - 2e^{i\,j\,\beta h_n} e^{n\lambda k} + e^{i\,(j-1)\beta h_n} e^{n\lambda k}\right) + k\,f(x_j^n, t_n) \qquad (3.2)$$

which yields,

$$e^{\lambda k} = 1 + \frac{k\,(e^{i\,\beta h_n} - e^{-i\,\beta h_n})}{2h_n\,(1-\psi^*(t_n))}\left((1 - x_j^n - \psi_2(t_n))\,\dot{\psi}_1(t_n) - (x_j^n - \psi_1(t_n))\,\dot{\psi}_2(t_n)\right) + \frac{k}{h_n^2}(e^{i\,\beta h_n} - 2 + e^{-i\,\beta h_n}) + k\,f(x_j^n, t_n) \qquad (3.3)$$

So,

$$e^{\lambda k} = 1 - \frac{4k}{h_n^2}\sin^2(\beta h_n/2) + \frac{i\,k\,\sin(\beta h_n)}{h_n\,(1-\psi^*(t_n))}\left((1 - x_j^n - \psi_2(t_n))\,\dot{\psi}_1(t_n) - (x_j^n - \psi_1(t_n))(\dot{\psi}_2(t_n))\right), \qquad (3.4)$$

To ensure convergence, we require $\left|e^{\lambda k}\right| \leq 1$. Applying the condition of convergent and after some manipulation, we get

$$\frac{4k^2}{h_n^2(1-\psi^*(t_n))}\Big((1-x_j^n-\psi_2(t_n))\dot{\psi}_1(t_n)$$
$$-(x_j^n-\psi_1(t_n))\dot{\psi}_2(t_n)\Big)^2$$
$$*\sin^2(\beta h_n/2)(1-\sin^2(\beta h_n/2))$$
$$-\frac{8}{h_n^2}\sin^2(\beta h_n/2)+\frac{16k^2}{h_n^4}\sin^4(\beta h_n/2)\le 0. \quad (3.5)$$

Thus, the condition of stability is given by

$$k\le\frac{2h_n^2}{4+(\dot{\psi}_1^2+\dot{\psi}_2^2)\,h_n^2} \quad (3.6)$$

4. Numerical examples

In this section, results of some numerical simulations of the proposed model are presented. In the following tables u_{app} and u_{ex} of denote the approximate solution of the nonlocal model and exact solution of the corresponding rectangular domain problem, respectively.

Example 1

In the model problem of equations (1.1)-(1.4), we consider

$$f(x,t)=-(x^2+2)\,e^{-t}, \quad 0<x<1, \quad t\ge 0,$$
$$\psi_1(t)=\psi_2(t)=\beta t, \quad \beta\in[0,1],$$
$$u_1(t)=0, \quad u_2(t)=e^{-t},$$

The domain of this example is shown in Fig. 1

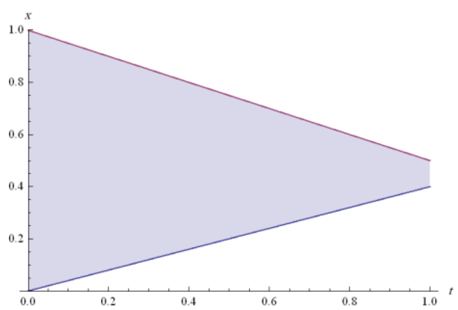

Fig. 1 The domain of example1

for which the exact solution of the corresponding problem on rectangular domain $\psi_1(t)=\psi_1(t)=0$ is given by

$$u(x,t)=x^2e^{-t}.$$

The following tables illustrate the numerical results for different values of β . The results in the following tables are calculated at node

$$x_j^n=\psi_1(t_n)+j\,\frac{1-\psi^*(t_n)}{N}.$$

Table 1. Numerical results of example 1 as $\beta\to 0$.

		$\beta=0.01$, $t_f=0.9$, $k=0.004$	$\beta=0.0001$, $t_f=0.9$, $k=0.004$	$\beta=0.00001$, $t_f=0.9$, $k=0.004$
N	$u_{ex.}$	u_{app}	u_{app}	u_{app}
1	0.004081	0.005519	0.004263	0.004252
2	0.016327	0.018880	0.016654	0.016634
3	0.036737	0.040085	0.037170	0.037143
4	0.065311	0.069135	0.065810	0.065779
5	0.102049	0.106030	0.102572	0.102541
6	0.146951	0.150771	0.147456	0.147426
7	0.200017	0.203359	0.200461	0.200435
8	0.261247	0.263793	0.261587	0.261567
9	0.3306413	0.332073	0.330833	0.330821
10	0.408199	0.408199	0.408199	0.408199

Table 2. Numerical results of example 1 at different value of β .

	$\beta=0.5$, $t_f=0.9$, $k=0.00005$	$\beta=0.7$, $t_f=0.6$, $k=0.0001$	$\beta=0.9$, $t_f=0.4$, $k=0.0003$
N	u_{app}	u_{app}	u_{app}
1	0.040552	0.054380	0.064034
2	0.081029	0.010857	0.012790
3	0.012146	0.162690	0.191943
4	0.161906	0.216846	0.256455
5	0.202380	0.271154	0.321749
6	0.242927	0.325724	0.388130
7	0.283584	0.380668	0.455898
8	0.324389	0.436098	0.525357
9	0.365378	0.492126	0.596815
10	0.406589	0.548866	0.670588

Example 2

In the model problem of equations (1.1)-(1.4), we consider

$$f(x,t)=-(x^2+2)\,e^{-t}, \quad 0<x<1, \quad t\ge 0,$$

$$\psi_1(t)=(1+t)^{\gamma}-1,\quad \psi_2(t)=0,\quad \gamma\le 1,\quad t\ge 0$$
$$u_1(t)=0,\quad u_2(t)=e^{-t}\,.$$

The domain of example 2 is shown in Figure 2

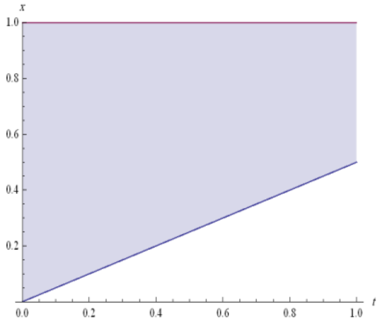

Fig. 2 The domain of example 2.

The following tables illustrate the numerical results for different values of γ.

Table 3. Numerical results of example 2 as $\gamma \to 0$.

		$\gamma = 0.01$, $t_f = 0.9$, $k = 0.004$	$\gamma = 0.0001$, $t_f = 0.9$, $k = 0.004$	$\gamma = 0.00001$, $t_f = 0.9$, $k = 0.004$
N	$u_{ex.}$	u_{app}	u_{app}	u_{app}
1	0.004081	0.004719	0.004255	0.004251
2	0.016327	0.017464	0.016640	0.016632
3	0.036737	0.038233	0.037151	0.037142
4	0.065311	0.067024	0.065789	0.065777
5	0.102049	0.103837	0.102550	0.102538
6	0.146951	0.148670	0.147435	0.147424
7	0.200017	0.201524	0.200443	0.200433
8	0.261247	0.262396	0.261573	0.261565
9	0.330641	0.331288	0.330825	0.330821
10	0.408199	0.408199	0.408199	0.408199

Table 4. Numerical results of example 2 at different value of γ.

	$\gamma=0.5$, $t_f = 0.9$, $k = 0.001$	$\gamma=0.7$, $t_f = 0.9$, $k = 0.0009$	$\gamma=0.9$, $t_f = 0.9$, $k = 0.0002$
N	u_{app}	u_{app}	u_{app}
1	0.027585	0.035577	0.040678
2	0.057986	0.072119	0.081179
3	0.091241	0.109691	0.121573
4	0.127384	0.148360	0.161923
5	0.166454	0.188191	0.202296
6	0.208487	0.229246	0.242752
7	0.253519	0.271587	0.283355
8	0.301586	0.315276	0.324166
9	0.352726	0.360372	0.365244
10	0.406976	0.406935	0.406650

Tables 1 and 3 show that as β and γ tend to zero, the results obtained for the proposed model approach the results of the corresponding problem with rectangular domain. So, we present other values of β and γ.

5. Conclusion

In this paper, we propose new a nonlocal problem of PDE with deviated boundary conditions. The computational results obtained by using VSG scheme outlined in section 2 show good agreement with the exact solution of the corresponding problem with the classical (fixed) boundary conditions. It is also shown that the numerical solution displays the expected convergence to the exact one as the mesh size is refined.

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