Journal of Statistics Applications & Probability
*An International Journal*

http://dx.doi.org/10.18576/jsap/100115

# On Minimum Expected Search Time Algorithm for 3-Dimensional Randomly Located Target

*Mohamed A. Kassem*[1], *Abd Al-Aziz Hosni El-Bagoury*[1,*], *Sundus Naji AL-Aziz*[2] *and W. A. Afifi*[1,3]

[1]Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt
[2]Department of Mathematics, College of Science, Princess Nourah bint Abdulrahman University, Riyadh, KSA
[3]Department of Mathematics and Statistics, College of Science, Taibah University, Yanbu, Kingdom of Saudi Arabia

Received: 24 Feb. 2020, Revised: 21 Mar. 2020, Accepted: 31 Mar. 2020
Published online: 1 Mar. 2021

**Abstract:** This article involves a new search model for a randomly located target; it looks into greater depth at search theory, the proposed model constructs the search for a three-dimensional randomly located target in a known region, using two searchers. The search region is divided into two subzones. The central issue is to calculate the expected time of detecting the target in case of SST (symmetric Search Technique) and AST (Asymmetric Search Technique), by using trivariate known distribution. This work introduces a statistical technique as an optimization problem. The crux of the matter is to obtain the optimal search path that minimizes the expected time of detecting the lost target. As a result, the optimal values will demonstrate the applicability of this technique.

**Keywords:** Search Theory,lost submarine, 3-D search algorithm, optimal search, minimizing the expected time, SST, AST, probability theory

## 1. Introduction

The study of search theory has many practical aspects. It started with the Second World War when U. S. navy worked in its antisubmarine warfare operations research group. The main purpose was to study the German submarine threaten in the Atlantic. Nowadays, search theory has become an important field of research in many applications. The linear search was the beginning in these applications, like searching for a defective unit in a large linear electric system, El-Rayes [1], Mohamed, and Abou Gabal [2–4]. Sometimes, the search problems impose the use of more than one searcher such as, for instance, in searching for a valuable target or a serious target. Yang [5, 6, 8] and Gao [7] investigated an optimization techniques to minimize the value of detecting time. Previously, the coordinated search techniques were studied on the line in both cases, symmetric and asymmetric distributions [9–11]. In addition, Thomas [12] illustrated the coordinated search on the circle with a known radius, and the target is equally likely to be anywhere on its circumference. Mohamed, Abou Gabal, and El-Hadidy [13, 14] studied coordinated techniques in the plane when the target has symmetric and asymmetric distributions, and the optimal search plan was illustrated. Recently, El-Hadidy and El-Bagoury [15] introduced a search model in the three dimensional space that determines a located target in a 3-D known zone by a single searcher, also, El-Hadidy and El-Bagoury [16] developed the technique to be a coordinated search technique that finds a 3-dimensional randomly located target by two searchers. More recently, Caraballo, Teamah, and El-Bagoury [17] suggested a modern model that facilities searching procedures using a quartile 3-dimensional statistical analysis, for more different kinds of search plans, [21–23].

In this paper, we introduce a new search model in 3-dimensions by using two searchers in both SST and AST. The contents are organized as follows: The correct statement of our problem is stated with the design of the searching framework, and the statement with the searching technique are, respectively analyzed in sections 2. While the expected time of detecting the target in case of SST and AST are illustrated in section 3. An optimal condition ensuring finite expected time of searching will be described in section 4. The effectiveness of this strategy will be illustrated in Section 5 by analyzing an application from the real world. Finally, some conclusions and comments about future research are included in the Conclusions section.

* Corresponding author e-mail: azizhel2013@yahoo.com

## 2. Mathimatical formulation

The disappearance of submarines became an ubiquitous problem facing any army. It is not easy to find the best ways to detect lost submarines in the depth of seas or oceans. On November 15, 2017, the Argentine Navy spokesperson announced that the ARA San Juan submarine had been disappeared. Nearly 24 countries around the world participated with efforts to find ARA San Juan submarine or any of the survivors. This paper is concerned with the problem of finding a lost submarine in a 3-dimensional known region. We have two searchers trying to detect the target in the minimal expected time. The target position is unknown, but the searchers know its probability distribution. Our scenario is more applicable and more effective in real world search scenarios. The main purpose here is obtaining the optimal search plan that minimizes the expected value of the time to detect the lost submarine, assuming trivariate standard normal distribution in case of symmetric search technique. Most of the processes are followed (SST) symmetric search technique, now the question is what will we do if the two zones are not equal, also is what will we do if the nature of one is more complex than the other coral reef rocks. So we have to add another technique (AST), asymmetric search technique, to facilitate the search process. A 3-dimensional zone and divided by two planes as indicated in Figures 1 and 2. The planes intersecting the initial point $(0,0,0)$ and the target is randomly located in the space of search. There are two searchers that looks for the target from the initial point using a continuous path where distance equals time.

### 2.1 *In case of (SST)*

(I) Start searching from the initial point $(0,0,0)$.
(II) Search inside the first cubic, $C_1$, and its tracks following these steps:
  (i) Move $d_1$ unit length towards point $p_1$
  (ii) Turn 90° clockwise and move $d_1$ unit length.
  (iii) Turn 90° clockwise, move $d_1$ units, extending the search process to another $d_1$ unit length, along the same direction.
  (iv) Turn 90° clockwise, moving $d_1$ unit length.
  (v) Turn 90° clockwise, moving forward $d_1$ unit length towards the initial point $(0,0,0)$.
(III) If the submarine is not detected, we should carry the searching in the second cube, $C_2$, and its tracks. The searcher will move to the second point, $p_2$, with $d_2$ unit length, and repeat the searching process with $d_2$ unit length, if the submarine is still not detected, the searcher will retrace the same steps as far as $d_3$ to check the next cube and its tracks, and so on until the submarine is detected as illustrated in Fig. 1. The searcher should use a sensor in his searching; the sensor range will be equal to $d_i$, $i = 1,2,3....n$.

The second searcher:

In the second zone, the second searcher will search as follows:

(I) Start searching from the initial point $(0,0,0)$.
(II) Search inside the first cubic, $C'_1$, and its tracks following these steps:
  (i) Move $d_1$ unit length towards the point $p'_1$
  (ii) Turn 90° counter clockwise and move $d_1$ unit length.
  (iii) Turn 90° counter clockwise, move $d_1$ units, extending the search process to another $d_1$ unit length, along the same direction.
  (iv) Turn 90° counter clockwise, moving $d_1$ unit length.
  (v) Turn 90° counter clockwise, moving forward by $d_1$ unit length towards the initial point $(0,0,0)$ .
(III) If the submarine is not detected, we should carry the searching in the second cube, $C'_2$, and its tracks. The searcher will move to the second point, $p'_2$, with $d_2$ unit length, and repeat the searching process with $d_2$ unit length, if the submarine is still not detected, the searcher will retrace the same steps as far as $d_3$ to check the next cube, $C'_3$, and its tracks, and so on until the submarine is detected as illustrated in Fig. 1. The searcher should use a sensor, the sensor range will be equal to $d_i, i = 1,2,3....n$.

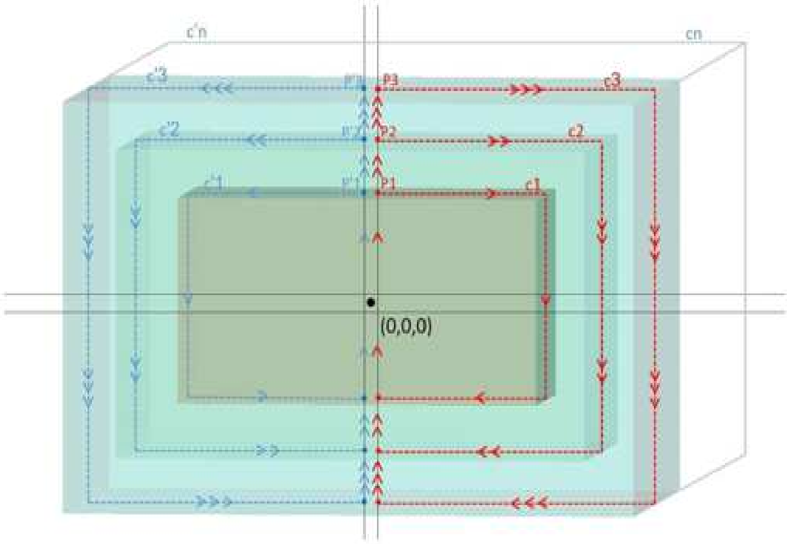


**Fig. 1:** SST

## 2.2 *In case of (AST):*

The first searcher will follow the same steps in case of (SST).
The second searcher will move as follows:

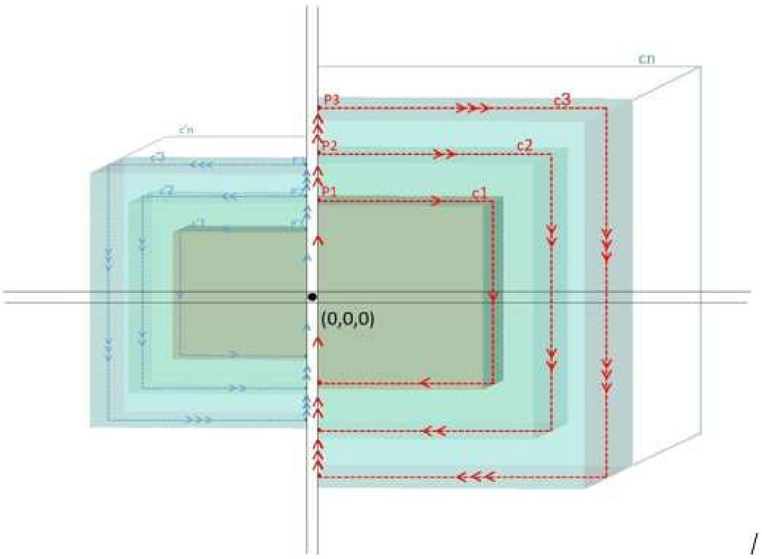


**Fig. 2:** AST

(I) Start searching from the initial point $(0,0,0)$.
(II) Search inside the first cubic and its tracks following these steps:
(i) Move $0.5\,d_1$ unit length towards the point $p'_1$
(ii) Turn $90^\circ$ counter clockwise and move $0.5d_1$ unit length.
(iii) Turn $90^\circ$ counter clockwise, move $0.5\,d_1$ units, extending the search process to another $0.5\,d_1$ unit length, along the same direction.

(iv) Turn 90° counter clockwise, moving $0.5d_1$ unit length.
(v) Turn 90° counterclockwise, moving forward $0.5d_1$ unit length towards the initial point $(0,0,0)$.

(III) If the submarine is not detected, we should carry the searching in the second cube, $C'_2$, and its tracks. The searcher will move to the second point, $p'_2$, with $0.5d_2$ unit length and repeat the searching process with $0.5d_2$ unit length, if the submarine still not detected, the searcher will retrace the same steps as far as $0.5d_3$ to check the next cube, $C'_3$, and its tracks, and so on until the submarine is detected as illustrated in Fig. 2. The searcher should use a sensor, the sensor range will be equal to $d_i, i = 1,2,3\ldots,n$.

## 3. Finite expected value

### 3.1 In case of (SST):

**Theorem 3.1.** By considering that the submarine has symmetric trivariate known distribution, the expected time of SST to detect the submarine is given by:

$$12\sum_{i=1}^{n}\left\{(d_i)\left[\xi_i - \xi_{i-1}\right]\right\} \tag{1}$$

**Proof.** Supposing the search path and the submarine has symmetric trivariate known distribution, for the first searcher, if the target lies in the first cubic, $C_1$, and its track, then $D(\Gamma_1) = 6d_1$, and if the target lies in the space between $C_1$ and $C_2$, then $D(\Gamma_2) = 6d_1 + 6d_2$ . Moreover, if the target lies in the space between and then $D(\Gamma_3) = 6d_1 + 6d_2 + 6d_3$, etc. Since SST is used, the second searcher will follow the same steps except for the search direction, which will be opposite, sometimes, if the target lies in the first cubic, $C'_1$, and its track, then $D(\Gamma'_1) = 6d_1$ , if the target lies in the space between $C'_1$ and $C'_2$, then $D(\Gamma'_2) = 6d_1 + 6d_2$. Morever, if the target lies between $C'_2$ and $C'_3$, then $D(\Gamma'_3) = 6d_1 + 6d_2 + 6d_3$ etc. Let

$$\xi_i = \int_{-d_i}^{d_i}\int_{-d_i}^{d_i}\int_{-d_i}^{d_i} f(x,y,z)dxdydz,$$

$$\xi_{i-1} = \int_{-d_{i-1}}^{d_{i-1}}\int_{-d_{i-1}}^{d_{i-1}}\int_{-d_{i-1}}^{d_{i-1}} f(x,y,z)dxdydz,$$

$$\xi_1 = \int_{-d_1}^{d_1}\int_{-d_1}^{d_1}\int_{-d_1}^{d_1} f(x,y,z)dxdydz,$$

$$\xi_2 = \int_{-d_2}^{d_2}\int_{-d_2}^{d_2}\int_{-d_2}^{d_2} f(x,y,z)dxdyd,$$

$$\xi_3 = \int_{-d_3}^{d_3}\int_{-d_3}^{d_3}\int_{-d_3}^{d_3} f(x,y,z)dxdydz, \text{ and}$$

$$\xi_n = \int_{-d_n}^{d_n}\int_{-d_n}^{d_n}\int_{-d_n}^{d_n} f(x,y,z)dxdydz,$$

$$\xi_{n-1} = \int_{-d_{n-1}}^{d_{n-1}}\int_{-d_{n-1}}^{d_{n-1}}\int_{-d_{n-1}}^{d_{n-1}} f(x,y,z)dxdydz,$$

$$\eta_i = \int_{-l_i}^{l_i}\int_{-l_i}^{l_i}\int_{-l_i}^{l_i} f(x,y,z)dxdydz,$$

$$\eta_{i-1} = \int_{-l_{i-1}}^{l_{i-1}}\int_{-l_{i-1}}^{l_{i-1}}\int_{-l_{i-1}}^{l_{i-1}} f(x,y,z)dxdydz.$$

So,

$$\begin{aligned} D(\Gamma,\Lambda,F) = &\left\{6d_1\left[\xi_1\right]\right\} + \{(6d_1+6d_2)\left[\xi_2-\xi_1\right]\right\} + \left\{(6d_1+6d_2+6d_3)\left[\xi_3-\xi_2\right]\right\} + \cdots \\ &+ \left\{(6d_1+6d_2+6d_3+\cdots+6d_n)\left[\xi_n-\xi_{n-1}\right]\right\} + \left\{6d_1\left[\xi_1\right]\right\} + \left\{(6d_1+6d_2)\left[\xi_2-\xi_1\right]\right\} \\ &+ \left\{(6d_1+6d_2+6d_3)\left[\xi_3-\xi_2\right]\right\} + \cdots + \left\{(6d_1+6d_2+6d_3+\cdots+6d_n)\left[\xi_n-\xi_{n-1}\right]\right\}. \end{aligned}$$

The expected value of detecting the submarine for the two searchers will be:

$$D(\Gamma,\Lambda,F) = 6\sum_{i=1}^{n}[(d_i)\{\xi_i-\xi_{n-1}\} + (d_i)\{\xi_i-\xi_{i-1}\}] = 12\sum_{i=1}^{n}\left[(d_i)\{\xi_i-\xi_{i-1}\}\right].$$

### 3.2 In case of (AST)

**Theorem 3.2.** In general, If the submarine has a trivariate asymmetric known distribution, then the expected time of AST to detect it will be:

$$\Omega(\Gamma,\Lambda,F)=6\sum_{i=1}^{n}\left[(d_i)\left\{\eta_i-\eta_{i-1}\right\}+(l_i)\left\{\xi_i-\xi_{i-1}\right\}\right].$$

**Proof.** By the same steps in Theorem 3.1.

We suppose in (AST) that the traveled distances for the second searcher will be half the traveled distances for the first, so from theorem 3.1, we conclude that

$$\begin{aligned}\Omega(\Gamma,\Lambda,F)=&\left\{6d_1\left[\xi_1\right]\right\}+\{(6d_1+6d_2)\left[\xi_2-\xi_1\right]\right\}+\left\{(6d_1+6d_2+6d_3)\left[\xi_3-\xi_2\right]\right\}+\cdots\\&+\left\{(6d_1+6d_2+6d_3+\cdots+6d_n)[\xi_n-\xi_{n-1}]\right\}+\left\{3d_1\left[\xi_1\right]\right\}+\left\{(3d_1+3d_2)\left[\xi_2-\xi_1\right]\right\}\\&+\left\{(3d_1+3d_2+3d_3)\left[\xi_3-\xi_2\right]\right\}+\cdots+\left\{(3d_1+3d_2+3d_3+\cdots+3d_n)\left[\xi_n-\xi_n\right]\right\}.\end{aligned}$$

The expected value of detecting the submarine for the two searchers will be:

$$6\sum_{i=1}^{n}\left[(d_i)\left\{\xi_i-\xi_{i-1}\right\}\right]$$

## 4. Optimal search path

### 4.1 OSST

**Theorem 4.1.** If $F(x,y,z)$ is a trivarite standard normal distribution function and has a density function $f(x,y,z)$, supposing $\Gamma,\Lambda\in Q$ are optimal search paths in both sides, then the necessary conditions that give the OSST of $d_i^*, i=1,2,\cdots,n$ is proved by solving the following equation:

$$[d_i-d_{i+1}]\frac{\partial}{\partial d_i}\xi_i+\xi_i=0. \tag{2}$$

**Proof.** From (1), we have

$$12\sum_{i=1}^{n}\left\{(d_i)\left[\xi_i-\xi_{i-1}\right]\right\}.$$

Differentianting partially with respect to $d_1$, we get

$$\frac{\partial D(\Gamma,\Lambda,F)}{\partial d_1}=12d_1\left\{\frac{\partial}{\partial d_1}\xi_1\right\}+12\left\{\xi_1\right\}-12d_2\left\{\frac{\partial}{\partial d_1}\xi_1\right\}=0.$$

Also,

$$\frac{\partial D(\Gamma,\Lambda,F)}{\partial d_2}=d_2\left\{\frac{\partial}{\partial d_2}\xi_2\right\}+\left\{\xi_2\right\}-d_3\left\{\frac{\partial}{\partial d_2}\xi_2\right\}=0.$$

By mathematical induction, we obtain

$$\frac{\partial D(\Gamma,\Lambda,F)}{\partial d_i}=d_i\left\{\frac{\partial}{\partial d_i}\xi_i\right\}+\left\{\xi_i\right\}-d_{i+1}\left\{\frac{\partial}{\partial d_i}\xi_i\right\}=\left\{[d_i-d_{i+1}]\frac{\partial}{\partial d_i}\xi_i\right\}+\left\{\xi_i\right\}=0.$$

We obtain nonlinear optimization problem, where

$$\min_{d_i} D(\Gamma,\Lambda,F),$$

where

$$0\le\xi_i\le 1, \forall i=1,2,\cdots,n$$

and

$$d_i-d_{i-1}\ge 0, \forall i=1,2,\cdots,n,$$

where

$$D(\Gamma,\Lambda,F)=12\sum_{i=1}^{n}\left\{(d_i)\left[\xi_i-\xi_{i-1}\right]\right\}$$

### 4.2 OAST

Our main purpose is represented in determine $d*$,$l*$ for both searchers, which gives the optimal searching paths.

**Theorem 4.2.** By considering $F(x,y,z)$, a skew distribution function which has a skew density function $f(x,y,z)$, if $\Gamma,\Lambda \in Q$ are the optimal search paths for both searchers, then the necessary conditions are:

$$\xi_i - \xi_{i-1} + d_i\left\{\frac{\partial}{\partial l_i}\left[\eta_i - \eta_{i-1}\right]\right\} = 0 \tag{3}$$

and

$$\eta_i - \eta_{i-1} + l_i\left\{\frac{\partial}{\partial d_i}\left[\xi_i - \xi_{i-1}\right\} = 0.$$

**Proof.** By the same method in Theorem 4.1, we obtain nonlinear optimization problem, where $\min_{d_i,l_i} \Omega(\Gamma,\Lambda,F)$, where

$$0 \leq \xi_i < 1,\ \ 0 \leq \eta_i < 1,\ \forall\ i = 1,2,\cdots,n,$$
$$0 \leq \xi_i + \eta_i \leq 1 \forall i = 1,2,\cdots,n$$

and

$$d_i - d_{i-1} \geq 0, l_i - l_{i-1} \geq 0, \forall i = 1,2,\cdots,n,$$

where

$$\Omega(\Gamma,\Lambda,F) = 6\Sigma_{i=1}^{n}\left\{(d_i)\left[\eta_i - \eta_{i-1}\right] + (l_i)\left[\xi_i - \xi_{i-1}\right]\right\}.$$

## 5. Application

IF $f(x,y,z) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-\frac{1}{2}(x^2+y^2+z^2)}$, where $-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty$ is a trivarite normal distribution, also, using the spherical coordinates $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$ by substituting in (1) we obtain

$$D(\Gamma,\Lambda,F) = \frac{6}{\sqrt{2\pi}}\Sigma_{i=1}^{n}[d_i]\left\{\sqrt{\tfrac{\pi}{2}}Erf(\tfrac{d_i}{\sqrt{2}}) - d_i e^{-\frac{1}{2}d_i^2} + \sqrt{\tfrac{\pi}{2}}Erf(\tfrac{d_{i-1}}{\sqrt{2}}) - d_{i-1}e^{\frac{1}{2}d_{i-1}^2}\right\}.$$

Then the above nonlinear optimization problem will be

$$\min_{d_i} = \frac{6}{\sqrt{2\pi}}\Sigma_{i=1}^{n}[d_i]\left\{\sqrt{\tfrac{\pi}{2}}Erf(\tfrac{d_i}{\sqrt{2}}) - d_i e^{-\frac{1}{2}d_i^2} + \sqrt{\tfrac{\pi}{2}}Erf(\tfrac{d_{i-1}}{\sqrt{2}}) - d_{i-1}e^{\frac{1}{2}d_{i-1}^2}\right\}$$

subject to

$$d_i - d_{i-1} \geq 0, \forall i = 1,2,\cdots,n,\ \sqrt{\frac{\pi}{2}}Erf(\frac{d_i}{\sqrt{2}}) - d_i e^{\frac{1}{2}d_i^2} - \frac{6}{\sqrt{2\pi}} \leq 0.$$

If $D(\Gamma,\Lambda,F)$ is a convex function, using the necessary conditions of Kuhn-Tucker, we get

$$\frac{6}{\sqrt{2\pi}}\Sigma_{i=1}^{n}\left\{\sqrt{\frac{\pi}{2}}Erf(\frac{d_i}{\sqrt{2}}) - d_i e^{-\frac{1}{2}d_i^2} - \sqrt{\frac{\pi}{2}}Erf(\frac{d_{i-1}}{\sqrt{2}}) - d_{i-1}e^{-\frac{1}{2}d_{i-1}^2} + d_i\left(e^{-d_{i-1}^2} + e^{-\frac{1}{2}d_{i-1}^2} + d_i^2 e^{-\frac{1}{2}d_{i-1}^2}\right)\right\} + \rho_1(1) + \rho_2\left(e^{-d_{i-1}^2} + e^{-\frac{1}{2}d_{i-1}^2} + d_i^2 e^{-\frac{1}{2}d_{i-1}^2}\right) = 0 \text{and} \tag{4}$$

$$\rho_1(d_i - d_{i-1}) = 0, \tag{5}$$

$$\rho_2\left(\sqrt{\frac{\pi}{2}}Erf(\frac{d_i}{\sqrt{2}}) - d_i e^{\frac{1}{2}d_i^2} - \frac{6}{\sqrt{2\pi}}\right) = 0. \tag{6}$$

By considering $d_0 \cong 0$, we can calculate the optimal values of $d_i^*, i = 1, 2, \cdots, n$ after solving the following equation numerically,

$$\Sigma_{i=1}^{n}\left\{\sqrt{\frac{\pi}{2}}Erf(\frac{d_i}{\sqrt{2}}) - d_i e^{-\frac{1}{2}d_i^2} + \sqrt{\frac{\pi}{2}}Erf(\frac{d_{i-1}}{\sqrt{2}}) - d_{i-1}e^{\frac{1}{2}d_{i-1}^2} + d_i\left(d_i^2 e^{-\frac{1}{2}d_i^2}\right)\right\}. \tag{7}$$

A. H. EL-Bagoury et al, [ [11], [12], [13]] solved these equations in the applications and obtained the optimal values for all used techniques, here, we developed the technique where we use two searchers instead of coordinated search, after solving equation (7), we get $d_1 = 0.1320199880 \times 10^{-5}$ unit length and $d_2 = 0.1327075136 \times 10^{-5}$ unit length, $d_3 = 0.2450857348 \times 10^{-4}$ unit length, $d_4 = 0.3017646106 \times 10^{-4}$ unit length.

## 6. Conclusions and future work

We have designed a modern search technique in the 3-D space, calculated the expected value of the time and obtained the optimal search path where minimizing the expected value of the time to detect the lost target, the accuracy of the technique is introduced in a numerical example, in future work, we will investigate an interesting search problem, optimal search strategy for a randomly moving target in 3-D by using four searchers. In particular, we plan to analyze a medical application to benefit from the results in the current manuscript in the discovery of diseases or tumors.

## Acknowledgement

The authors are grateful to the anonymous referee, the editorial manger and the editor in-chief for the careful checking of the details and for helpful comments that improved this paper.

## Conflict of Interest

The authors declare that they have no conflict of interest.

## References

[1] El-Rayes A. B., M. A. Abd-Elmoneim, A. Mohamed and H. Fergani, On the generalized linear search problem, Delta Journal, **6**, 1-10, (1993).
[2] Mohamed A. and H. M. Abou Gabal, Generalized optimal search paths for a randomly located target, Annual Conference (Cairo), ISSR, Math. Statistics Part, **35**, 17-29, (2000).
[3] Mohamed A. and H. M. Abou Gabal, Linear search with multiple searchers for a randomly moving target. International Conference for Statistics, Computer Science and Its Application, Egypt, 115-124, (2003).
[4] Mohamed A. and H. M. Abou Gabal, Multiplicative linear search problem, Egyptian Statistical Journal, **48**, 34-45, (2004).
[5] Yang Z., M. Awasthi, M. Ghosh and N. Mi, A Fresh Perspective on Total Cost of Ownership Models for Flash Storage in Datacenters, 10.1109/CloudCom.2016.0049, (2016a).
[6] Yang Z., J. Tai, J. Bhimani, J. Wang, N. Mi and B. Sheng, GReM: Dynamic SSD resource allocation in virtualized storage systems with heterogeneous IO workloads, 2016 IEEE 35th International Performance Computing and Communications Conference (IPCCC), 1-8, (2016b).
[7] Gao H., Z. Yang, J. Bhimani, T. Wang, J. Wang, N. Mi and B. Sheng, AutoPath: harnessing parallel execution paths for efficient resource allocation in multi-stage big data frameworks, 10.1109/ICCCN.2017.8038381, (2017).
[8] Yang Z., J. Wang, D. Evans and N. Mi, Auto replica: automatic data replica manager in distributed caching and data processing systems, 10.1109/PCC.2016.7820664, (2016c).
[9] Reyniers D., Coordinated search for an object on the line, European Journal of Operation Research, **95**, 663-670, (1996).
[10] Reyniers D., Coordinated two searchers for an object hidden on an interval, Journal of Operational Research Society, **46**, 1386-1392, (1995).
[11] Mohamed A., H. Abou Gabal and W. Afifi, On the coordinated search problem, International Journal of Pure and Applied Mathematics, **5**, 627-636, (2007).
[12] Thomas L., Finding your kids when they are lost, Journal of Operational Research Society, **43**, 637-639, (1992).
[13] Mohamed A., H. M. Abou Gabal and M. A. El-Hadidy, Coordinated search for a randomly located target on the plane, European Journal of Pure and Applied Mathematics, **2**, 97-111, (2009).

[14] Mohamed A., H. A. Fergany and M. A. El-Hadidy, On the coordinated search problem on the plane, Journal of the School of Business Administration, Istanbul University, **41**, 80-102, (2012).
[15] El-Hadidy M. A. and A. H. El-Bagoury, Optimal search strategy for a three-dimensional randomly located target, Int. J. Operational Research, **29**, 115-126, (2017).
[16] El-Hadidy M. A. and A. H. El-Bagoury, 3-dimensional coordinated search technique for a randomly located target, Int. J. Computing Science and Mathematics, **9**, 258-272, (2018).
[17] Caraballo T., A.-E. A.M. Teamah and A. H. El-Bagoury, Minimizing the expected time to detect a randomly located target using 3-D search technique, Communications in Statistics-Theory and Methods, DOI: 10.1080/03610926.2019.1588323, (2019).
[18] Mohamed A., M. Kassem and M. A. El-Hadidy, Multiplicative linear search for a Brownian target motion, International Journal of Mathematics in Operational Research, **10**, 137-149, (2017a).
[19] Mohamed A. and M. A. El-Hadidy, Existence of a periodic search strategy for a parabolic spiral target motion in the plane, Afrika Matematika Journal, **24**, 145-160, (2013).
[20] Mohamed A., M. Kassem and M. A. El-Hadidy, $M$-states search problem for a lost target with multiple sensors, International Journal of Mathematics in Operational Research, **10**, 104-135, (2017b).
[21] Teamah A. A. M., W. A. Afifi, J. Gani Dar, A. H. El-Bagoury and S. N. Al-Aziz, Optimal discrete search for a randomly moving covid19, Journal of Statistics Applications & Probability, **9** (3), 473-481, (2020).
[22] Afifi W. A. and A. H. El-Bagoury, Optimal multiplicative generalized linear search plan for a disctete randomly located target. INformation Sciences Letters, **10**(1), 153-158, (2021).
[23] Afifi W. A., A. H. El- Bagoury and S. N. Alaziz, A novel search for a multi searchers random walk, Journal of Applied Mathematics and Information Sciences, **14**(1),115-122, (2020).