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Population Growth Modeling via Rayleigh-Caputo Fractional Derivative

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Abstract: This article is concerned with the prediction of population growth using the logistic growth model in the case when the carrying capacity K for the population tends to infinity. A new fractional approach is introduced based on so what called "Rayleigh distribution". This approach produces a minimal error in estimation compared to the logistic growth model. In this paper, it is shown that the classical logistic model is not appropriate when the carrying capacity K tends to infinity, like for the Indian or Chinese population for instance. A fractional model that would be appropriate in such a case is proposed.

Keywords: Exponential growth, logistic growth, ψ-Caputo fractional derivative, optimization, initial value problems

1 Introduction

Fractional calculus was introduced more than three centuries ago. The existence and uniqueness theory of the solution were studied in the earliest days. Such theory can be found in studies done by authors of [1] – [7].

Over years, several researchers have proven that the classical approach of differential equations may fail in modeling of some complex phenomena. However, applications of fraction differential equation in such cases are often proven efficient. The efficiency of fractional differential in science branches such as epidemiology, Chemistry, finance, physics, and a few just to mention can be found in studies referred in [8] – [14].

In this work, we contribute to what has been done so far in the fields of fractional differential equations by showing that the classical logistic model and exponential model may fail to predict a population of very a large size, while carrying capacity approaching infinity. The Chinese population was used as an evidence to support the proposed approach. Hence, we proved that the ψ-Caputo model introduced by Almeida et al. [15], with the Rayleigh kernel is a suitable approach for such population modeling.

2 Preliminaries

Definition 2.1. [16] Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $g : [0,+\infty] \to \mathbb{R}$ is defined as

$$\left({}_{RL}I_{0^+}^{\alpha} g\right)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds.$$

Provided that the right hand side of the integral is point-wise defined on $(0,+\infty)$ and Γ is the well-known Gamma function defined by $\Gamma(\upsilon) = \int_0^\infty e^{-t} t^{\upsilon-1} dt, \forall \upsilon > 0$.

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Definition 2.2. [16] The Caputo derivative of order α for a function $g : [0, +\infty] \to \mathbb{R}$ is defined as

$$\left({}_{C}D_{0^+}^{\alpha} g\right)(t) = \begin{cases} \int_0^t \frac{(t-s)^{n-\alpha-1} g^{(n)}(s)}{\Gamma(n-\alpha)} ds, & n-1 < \alpha < n, \alpha \in R, \\ g^{(n)}(t), & \alpha \in N. \end{cases}$$

Where $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α.

Definition 2.3. [17] Let $\alpha > 0$, $g \in L^1[a, b]$ and $\psi \in C^1[a, b]$ be an increasing function with $\psi'(x) \neq 0, \forall x \in [a, b]$ then $I_{0^+}^{\alpha,\psi} g(t)$ denotes the fractional integral of g w.r.t, and it is given by

$$I_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{\alpha-1} g(s) ds$$

Definition 2.4. [17] Let $\alpha > 0$, $g, \psi \in C^n[a, b]$ with ψ be an increasing function and $\psi'(x) \neq 0, \forall x \in [a, b]$ then ${}_{C}D_{0^+}^{\alpha,\psi} g(t)$ denotes the fractional derivative of g w.r.tψ, and it is given by

$${}_{C}D_{0^+}^{\alpha,\psi} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} \left(\frac{1}{\psi'(s)} \frac{d}{dt}\right)^n g(s) ds$$

Definition 2.5. [17] The random variable x is said to be Rayleigh distributed with parameter σ^2 if

$$f_x(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, x \geq 0 \\ 0 \quad \text{otherwise} \end{cases}$$

Some properties of Rayleigh distribution:

1. Cumulative distribution function (CDF) is given as $F(x) = 1 - e^{-\left(\frac{x^2}{2\sigma^2}\right)}$.
2. Mean $\bar{x}_R = \sigma \sqrt{\frac{\pi}{2}}$.
3. Variance $\sigma_R^2 = \frac{4-\pi}{2} \sigma^2$.

Lemma 2.1. [15]: Let $\alpha > 0$, n a natural number such that $\alpha \in (n-1, n)$ if $g, \psi \in C^n[a, b]$, then

$$I_{0^+}^{\alpha,\psi} \left({}_{C}D_{0^+}^{\alpha.\psi} g\right)(t) = g(t) - \sum_{i=0}^{n-1} \frac{\left(\frac{1}{\psi'(s)} \frac{d}{dt}\right)^i g(0)}{i!} (\psi(t) - \psi(0))^i$$

Lemma 2.2.[18]: error rate of estimate:
Consider an experimental data vector of size n, obtained from a time-dependent process such that the i^{th} value is denoted by $y_i = y(t_i)$. If the original data is predicted by a function $\hat{y}$, in a way that the i^{th} fitted data value is denoted by $\hat{y} = \hat{y}(t_i)$, the root-mean-square deviation is computed by

$$RMSD = \sqrt{\frac{\sum_{t=0}^{n} (y(t) - \hat{y}(t))^2}{n}}$$

The percentage of error that occurs while using predicted data instead of original data is given by the ratio

$$ER = \sqrt{\frac{\sum_{t=0}^{n} (y(t) - \hat{y}(t))^2}{n}} \Bigg/ \sqrt{\frac{\sum_{t=0}^{n} (y(t))^2}{n}}$$

The rate ER evaluates the magnitude of the error that occurs when the estimated values are used in place of the original values.

3 Population Growth Modeling when the Carrying Capacity $K \to +\infty$

The theory of population growth modeling using a logistic equation was introduced by an economist named Malthus T.R [19] – [21]. In fact, given an initial population that can grow, which is with a reproduction capability, the theoretical expectation would be that the population size will approach infinity as the time increases indefinitely. However, Malthus has proven in his theory that a population growth tends to stabilize at some point when the time approaches infinity. He then proposed a model of population growth, called logistic growth model is given by

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{N(t)}{K}\right) \tag{1}$$

where r is the growth rate, and K is the carrying capacity which represents the maximum value that the population size may reach. Hence, at that size, the population growth stabilizes. A general solution for the classical logistic model defined by (1) is

$$N_c(t) = \frac{K}{1 + \left(\frac{K - N_0}{N_0}\right) e^{-rt}} \tag{2}$$

Where $N_0 = N(0)$ is the initial size of the population at $t = 0$.
The carrying capacity of a population is calculated or estimated with accuracy using non-linear optimization routine on the existing sample sizes over a given time interval. From (1) , it is obvious that the carrying capacity has high impact on the model. Some researchers focused in the past on the population growth behavior based on the carrying capacity dynamic (see [19] [23] [24]). Considering the human population growth modeling, we will observe that there exist some countries that try to control their population growth. On the other hand, a large size population data from countries such as China and India produces a very large value of carrying capacity $K \to +\infty$. The effect of birth control in China on the model is as follows. The size of population $N(t)$ is much smaller than what would be expected theoretically. On the other hand, carrying capacity $K \to +\infty$. This leads mathematically to the relation

$$N(t) \ll K \tag{3}$$

That is

$$\frac{N(t)}{K} \to 0 \;\; as \;\; t \to +\infty \tag{4}$$

Under the conditions defined by (3) and (4), the initial logistic model (1) becomes an exponential model

$$\frac{dN(t)}{dt} = rN(t) \tag{5}$$

A general solution of the exponential model defined by (5) is

$$N(t) = N_0 e^{rt} \tag{6}$$

The solution to the exponential model defined by (6) shows that the population size will increase infinitely as the time approaches infinity. Such result means that the Chinese population for instance will grow infinitely. This doesn't neither the Malthusian theory nor what is practically known about the Chinese population. In the next section, we propose an alternative approach for the modeling of large size population with $K \to +\infty$.

4 Population Growth Modeling with Rayleigh Kernel when the Carrying Capacity $K \to +\infty$

In this section, we propose an alternative approach to model the growth of a population whose carrying capacity approaches infinity. Some concepts of fractional differential equation in particular ψ-Caputo fractional derivative are used to build a fractional differential equation model. Recalling the fractional differential theories discussed in the preliminaries section, the logistic model (1) can be converted into a fractional differential equation model with kernel function as follows:

$$_C D_{0^+}^{\alpha,R} N(t) = rN(t), N(0) = N_0 \tag{7}$$

Where $R(t) = 1 - e^{-\left(t^2/2\sigma^2\right)}$ is the Rayleigh cumulative density function. The solution of the model defined by (7) is defined as

$$\begin{aligned} N(t) &= N_0 E_\alpha\left[r(R(t) - R(0))^\alpha\right] \\ &= N_0 E_\alpha\left[r\left(1 - e^{-\left(t^2/2\sigma^2\right)}\right)^\alpha\right] \end{aligned} \tag{8}$$

Where $E_\alpha(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i+1)}$, $t \in R$ is the Mittage-Leffler function.
The Rayleigh kernel is selected and proven suitable in this study because of its shape. In fact, the Rayleigh cumulative density function has a logistic growth shape.

5 Simulation Results

In this section, we prove by the mean of simulation study of the Chinese population that the proposed method in section 4 can better fit a population with a large carrying capacity than the usual logistic and exponential models. We retrieved the Chinese population historical data from the World Bank website [25]. We used non-linear optimization routine 'lsqcurvefit' from Matlab to estimate the parameter that would best fit the classical logistic model (1), the exponential model (5) and the Rayleigh kernel fractional model (8). Lemma 2.2 is recalled for the error rate evaluation purpose. The results obtained are given below and illustrated by Fig.1.

– The classical logistic approach with a growth rate of $r = 0.0145$ and a carrying capacity of $K = 3.3198x10^{23}$. Would best fit the data by producing a total error rate of $ER = 6.27\%$.
– The classical exponential approach produced the same result as the classical logistic approach. In fact, $r = 0.0145$, minimizing the error rate to $ER = 6.27\%$.
– The Rayleigh Kernel fractional approach with a fractional order of derivative $\alpha = 0.2867$ and a rate of $3.050x10^3$, produced a minimum error rate to $ER = 3.67\%$.

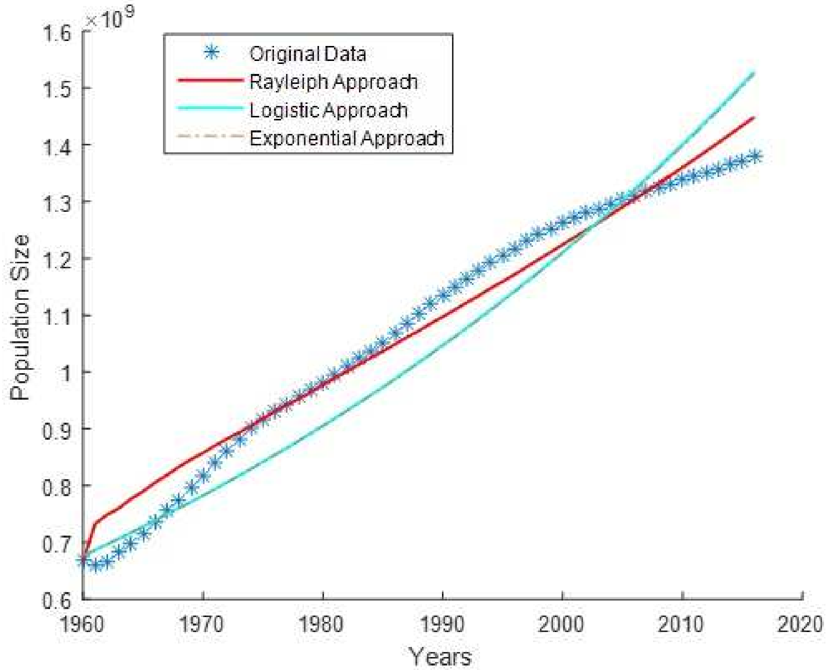

Fig. 1: Chinese population growth modeling

Remark: The difference in the error rate value from 6.27% to 3.67% seems to be little. However, this difference is highly significant as the population size is too large and evaluated in Billions.

6 Conclusion

It is difficult to give a practical meaning to a fractional order of derivative. In the physical sense, the first derivative of a trajectory equation is the speed. For the population growth, it is the speed by which the said population growth. It is also important to give a meaning to the fractional derivative order α , which we used in fractional differential equations. With this regard, if a classical problem involves a first derivative, one would expect from its counterpart fractional to be optimal with a α value that belongs to $\alpha \in (0,1) \cup (1,2)$. In fact, if the fractional approach to a problem involving a first order differential equation is optimal when $\alpha = 1$; this coincides with the classical solution. Hence the fractional approach in such case wouldn't improve anything to the classical solution.

When the carrying capacity of the population is very large and the said population is subject to a restricted growth, the logistic approach and the exponential approach coincide. However, they might not be suitable for modeling the growth of such population.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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