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A Study on the k-Jacobsthal and k-Jacobsthal Lucas Quaternions and Octonions

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Abstract: In this paper, we first consider the Jacobsthal and Jacobsthal Lucas quaternions and octonions. By making use of definitions of these sequences, we derive some novel and interesting properties and relations for the Jacobsthal and Jacobsthal Lucas quaternions and octonions. As universal formulae for these sequences, we obtain the binet formulas. Finally, we introduce a new class of octonions of Jacobstal and Jacobstal Lucas sequences. We then investigate several new properties including Catalan identities, D′ocagene′s identities and binet-like formulas.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, Quatenion algebras, Octonions algebras

1 Introduction

Intensive research activities in such an topic as special number sequences, which includes Fibonacci, Lucas, Pell and Jacobsthal sequences, are primarily motivated by their importance in combinatorics, computer algorithms, biological settings and several fields of mathematics (see [1-3, 5-12] and also cited references therein). Akkus *et al.* [1] proposed the split Fibonacci and Lucas octonions and provided their some properties and relations. Aydın [2] introduced bicomplex Fibonacci quaternions and gave some algebraic properties of bicomplex Fibonacci quaternions which connected with bicomplex numbers and Fibonacci numbers including Binet's formula, Cassini's identity, Catalan's identity for these quaternions. Aydın *et al.* [3] investigated various relations between the Jacobsthal quaternions connected with Jacobsthal and Jacobsthal-Lucas numbers and provided Binet formulas and Cassini identities for these quaternions. Bilgici *et al.* [5] introduced the Fibonacci and Lucas sedenions and presented generating functions and Binet formulas for the Fibonacci and Lucas sedenions. Catarino [6] considered the modified Pell and the modified k-Pell quaternions and octonions and gave some properties involving the mentioned sequences, including the Binet-style formulas and the ordinary generating functions. Cimen et al. [7] defined the Jacobsthal octonions and the Jacobsthal-Lucas octonions and provided diverse relations between Jacobsthal octonions and Jacobsthal-Lucas octonions. Halici [8] investigated the complex Fibonacci quaternions and presented the generating function, Binet formula and matrix representations for the aforesaid sequences. Szynal-Liana *et al.* [10] introduced the Jacobsthal quaternions and the Jacobsthal-Lucas quaternions and derived some of their properties. Savin [11] determined multifarious properties of Fibonacci octonions and then considered the generalized Fibonacci-Lucas octonions with several basic properties. Uygun *et al*. [13] developed some formulas for k-Jacobsthal numbers and k-Jacobsthal Lucas numbers.

The usual Jacobsthal numbers J_n and Jacobsthal Lucas numbers JL_n are defined by ($cf.$ [3, 7, 10, 12, 13])

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0\,,\, J_1 = 1 \tag{1}$$
$$JL_n = JL_{n-1} + 2JL_{n-2}, JL_0 = 2, JL_1 = 1. \tag{2}$$

In view of (1) and (2), the following properties given for Jacobsthal and Jacobsthal Lucas numbers play important roles to obtain the main results of this paper (*cf.* [3, 7, 10, 12, 13]):

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Proposition 1. *Each of the following formulas holds true:*

$$
\begin{aligned}
JL_n &= J_{n+1} + 2J_{n-1} && (3)\\
J_n + JL_n &= 2J_{n+1} && (4)\\
J_{n+1} &= J_n + 2J_{n-1} && (5)\\
JL_n &= J_{n+1} + 2J_{n-1}\\
J_{n+1} &= J_n + 2J_{n-1}\\
J_{n+2} + 2J_{n-2} &= JL_{n+1}\\
3J_n + JL_n &= 2^{n+1}\\
JL_{n+1} + JL_n &= 3.2^n\\
9J_n &= JL_{n+1} + 2JL_{n-1}\\
JL_{n+1} &= 2JL_n - 3(-1)^n.
\end{aligned}
$$

Quaternions and octonions have been increasingly used in multifarious areas such as quantum physics, group theory, computer sciences, differential geometry, geotatics and color image processing (*cf.* [1-4, 6-12]).The quaternion numbers or the bicomplex numbers are defined by ($cf.$ [2, 3, 6, 8-10, 12])

$$C_2 = (q = q_1 + iq_2 + jq_3 + ijq_4 \mid q_1.q_2.q_3.q_4 \in R) \quad (6)$$

where i, j and ij satisfy the conditions

$$i^2 = -1, j^2 = -1, ij = ji.$$

Moreover, three different conjugations can operate on bicomplex numbers as follows

$$
\begin{aligned}
q &= q_1 + iq_2 + jq_3 + ijq_4 = (q_1 - iq_2) + j(q_3 + iq_4),\ q \in C_2\\
q_i^* &= q - iq + jq - ijq = (q_1 - iq_2) + j(q_3 - iq_4),\\
q_j^* &= q + iq - jq - ijq = (q_1 + iq_2) - j(q_3 + iq_4),\\
q_{ij}^* &= q - iq - jq + ijq = (q_1 - iq_2) - j(q_3 - iq_4)
\end{aligned}
$$

and properties of conjugation, for $q_{1,}q_2 \in C_2$ and $\lambda, \mu \in R$

$$
\begin{aligned}
(q^*)^* &= q,\\
(q_1.q_2)^* &= q_2^*.q_1^*,\\
(q_1 + q_2)^* &= q_1^* + q_2^*,\\
(\lambda q_1)^* &= \lambda q_1^*,\\
(\lambda q_1 + \mu q_2)^* &= \lambda q_1^* + \mu q_2^*.
\end{aligned}
$$

The norms of the bicomplex numbers are defined as

$$
\begin{aligned}
N_{q_i} &= \|q \times q_i\| = \sqrt{|\, q_1^2 + q_2^2 - q_3^2 - q_4^2 + 2j(q_1.q_3 + q_2.q_4)},\\
N_{q_j} &= \|\, q \times q_j \,\| = \sqrt{|\, q_1^2 - q_2^2 + q_3^2 - q_4^2 + 2i(q_1.q_2 + q_3 q_4)},\\
N_{q_{ij}} &= \|\, q \times q_{ij} \,\| = \sqrt{|\, q_1^2 + q_2^2 + q_3^2 + q_4^2 + 2ij(q_1.q_4 - q_2.q_3)}.
\end{aligned}
$$

Let $\mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ be the generalized octonion algebra over $\mathbb{R}$ with basis $\{1, e_1, e_2, \cdots, e_7\}$. This algebra is an eight-dimentional non-commutative and non-associative algebra. Let $x \in \mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ with $x = x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7$ and its conjugate $\overline{x} = x_0 - x_1e_1 - x_2e_2 - x_3e_3 - x_4e_4 - x_5e_5 - x_6e_6 - x_7e_7$. The norm of x is $n(x) = x\overline{x} = x_0^2 + \alpha x_1^2 + \beta x_2^2 + \alpha\beta x_3^2 + \gamma x_4^2 + \alpha\gamma x_5^2 + \beta\gamma x_6^2 + \alpha\beta\gamma x_7^2$. Also, the multiplication table for the basis of $\mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ can be found in the references [11]. For $x \in \mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, if $n(x) = 0$ if and only if $x = 0$, then the octonion algebra $\mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is called a division algebra. Otherwise, $\mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$ is called a split algebra. In the special cases of $\mathbf{O}_{\mathbb{R}}(\alpha,\beta,\gamma)$, the octonion algebra $\mathbf{O}_{\mathbb{R}}(1,1,1)$ is a division algebra which will be used in this paper and the octonion algebra $\mathbf{O}_{\mathbb{R}}(1,1,-1)$ is a split algebra, (see [1, 4, 6, 7, 11] for more details). In this paper, we study on the Jacobstal and Jacobstal Lucas quaternions and octonions. We derive some new and interesting properties and relations for the Jacobstal and Jacobstal Lucas quaternions and octonions. The generating functions of the foregoing numbers are acquired. Finally, we introduce the k-Jacobstal octonions and k-Jacobstal Lucas octonions. We then invertigate several properties.

2 Jacobsthal and Jacobsthal Lucas Quaternions and Octonions

2.1 Bicomplex Jacobsthal and Bicomplex Jacobsthal Lucas Numbers

We will start with by giving the definitions of classical Jacobsthal quaternions and Jacobsthal Lucas quaternions, and some properties of them.

Bicomplex Jacobsthal and bicomplex Jacobsthal Lucas numbers BJ_N and BJL_N (or so called Jacobsthal quternions and Jacobsthal Lucas quaternions) are defined ($cf.$ [3, 7, 10, 12]) by the basis $\{1, i, j, ij\}$

$$C_2^J = \{BJ_n = J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3} | J_n\} \quad (7)$$

$$i^2 = -1,\ j^2 = -1,\ ij = ji.$$

$$C_2^{JL} = \{BJL_n = JL_n + iJL_{n+1} + jJL_{n+2} + ijJL_{n+3} | JL_n\} \quad (8)$$

$$i^2 = -1, j^2 = -1, ij = ji$$

Starting from $n = 0$, the first few bicomplex Jacobstal and bicomplex Jacobstal Lucas numbers can be written respectively as; $BJ_0 = i + 3j + 5ij$, $BJ_1 = 1 + 3i + 5j + 11ij$, $BJ_2 = 3 + 5i + 11j + 17ij, \cdots$ and $BJL_0 = 2 + i + 5j + 7ij$, $BJL_1 = 1 + 5i + 7j + 14ij$, $BJL_2 = 5 + 7i + 17j + 31ij$ and so on, see [8].

Let

$$BJ_N = J_n + iJ_n + jJ_n + ijJ_n$$

and

$$BJ_M = J_m + iJ_m + jJ_m + ijJ_m$$

be two bicomplex Jacobstal numbers. The addition, subtraction and multiplication of these numbers are given by

$$BJ_n + BJ_m = (J_n \pm J_m) + (J_n \pm J_m)i + (J_n \pm J_m)j + (J_n \pm J_m)ij$$

and

$$\begin{aligned} BJ_n \times BJ_m = &\left(J_nJ_m + J_{n+1}J_{m+1} + J_{n+2}J_{m+2} + J_{n+3}J_{m+3}\right) \\ &+\left(J_nJ_{m+1} + J_{n+1}J_m - J_{n+2}J_{m+3} - J_{n+3}J_{m+2}\right) i \\ &+\left(J_nJ_{m+2} + J_{n+2}J_m - J_{n+1}J_{m+3} - J_{n+3}J_{m+1}\right) j \\ &+\left(J_nJ_{m+3} + J_{n+3}J_m + J_{n+1}J_{m+2} + J_{n+2}J_{m+1}\right) ij. \end{aligned}$$

We now give the Binet formula for BJ_n and BJL_n.

Theorem 1.*(cf. [3,12])Let BJ_n and BJL_n be bicomplex Jacobstal and Jacobstal Lucas numbers, respectively. For $n \geq 0$, the Binet formulas for these numbers are given as;*

$$BJ_n = \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta}$$

and

$$BJL_n = \overline{\alpha}\alpha^n + \overline{\beta}\beta^n.$$

*Proof.*By using the Binet formulas for Jacobstal and Jacobstal Lucas numbers,

$$\overline{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3 \text{ and } \overline{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3,$$

we find the results clearly as

$$\begin{aligned} BJ_n &= J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3} \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} + i\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} + j\frac{\alpha^{n+2} + \beta^{n+2}}{\alpha - \beta} + ij\frac{\alpha^{n+3} + \beta^{n+3}}{\alpha - \beta} \\ &= \frac{\alpha^n\left(1 + i\alpha + j\alpha^2 + ij\alpha^3\right)}{\alpha - \beta} + \frac{\beta^n\left(1 + i\beta + j\beta^2 + ij\beta^3\right)}{\alpha - \beta} \\ &= \frac{\overline{\alpha}\alpha^n - \overline{\beta}\beta^n}{\alpha - \beta} \end{aligned}$$

and

$$\begin{aligned} BJL_N &= BJL_n + iBJL_{n+1} + jBJL_{n+2} + ijBJL_{n+3} \\ &= \left(\alpha^n + \beta^n\right) + i\left(\alpha^{n+1} + \beta^{n+1}\right) + j\left(\alpha^{n+2} + \beta^{n+2}\right) + ij\left(\alpha^{n+3} + \beta^{n+3}\right) \\ &= \alpha^n\left(1 + i\alpha + j\alpha^2 + ij\alpha^3\right) + \beta^n\left(1 + i\beta + j\beta^2 + ij\beta^3\right) \\ &= \overline{\alpha}\alpha^n + \overline{\beta}\beta^n, \end{aligned}$$

which give the desired results. We state the following theorem.

Theorem 2.*Let BJ_n and BJL_n be bicomplex Jacosthal and Jacobsthal Lucas numbers. For $n \geq 1$,*

$$BJ_{n+2} + 2BJ_{n-2} = BJL_{n+1} \text{ and } BJ_{n+1} + 2BJ_{n-1} = BJL_n.$$

*Proof.*Let

$$BJ_n = J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3}.$$

We then obtain, by Proposition 1,

$$\begin{aligned} BJ_{n+2} + 2BJ_{n-2} &= \left(J_{n+2} + 2J_{n-2}\right) + i\left(J_{n+3} + 2J_{n-1}\right) + j\left(J_{n+4} + 2J_n\right) + ij\left(J_{n+5} + 2J_{n+1}\right) \\ &= J_{n+1} + iJ_{n+2} + jJ_{n+3} + ijJ_{n+4} \\ &= BJL_{n+1} \end{aligned}$$

and

$$\begin{aligned} BJ_{n+1} + 2BJ_{n-1} &= \left(J_{n+1} + 2J_{n-1}\right) + i\left(J_{n+2} + 2J_n\right) + j\left(J_{n+3} + 2J_{n+1}\right) + ij\left(J_{n+4} + 2J_{n+2}\right) \\ &= J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3} \\ &= BJL_n. \end{aligned}$$

We give the following theorem.

Theorem 3.*If BJ_N bicomplex Jacobsthal number respectively, then for $n \geq 0$,*

$$BJ_{n+1} - 2BJ_{n-1} = BJ_n$$

and

$$BJ_{n+2} - 2J_{n-2} = BJ_{n+1}.$$

*Proof.*Let

$$BJ_n = J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3}.$$

Via Proposition 1, we then get

$$\begin{aligned} BJ_{n+1} - 2BJ_{n-1} &= \left(J_{n+1} - 2J_{n-1}\right) + i\left(J_{n+2} - 2J_n\right) + j\left(J_{n+3} - 2J_{n+1}\right) + ij\left(J_{n+4} - 2J_{n+2}\right) \\ &= J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3} = BJ_n \end{aligned}$$

and

$$\begin{aligned} BJ_{n+2} - 2BJ_{n-2} &= \left(J_{n+2} - 2J_{n-2}\right) + i\left(J_{n+3} - 2J_{n-1}\right) + j\left(J_{n+4} - 2J_n\right) + ij\left(J_{n+5} - 2J_{n+1}\right) \\ &= J_{n+1} + iJ_{n+2} + jJ_{n+3} + ijJ_{n+4} \\ &= BJ_{n+1}. \end{aligned}$$

We present the following theorem.

Theorem 4.*Let BJ_n and BJL_n be the bicomplex Jacobsthal and Jacobsth Lucas numbers respectively. We then give the following relations:*

$$3BJ_n + BJL_n = B2^{n+1}$$

and

$$2BJL_{n+1} + 2BJL_{n-1} = 9BJ_n.$$

*Proof.*Let

$$BJ_n = J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3}$$

and

$$BJL_n = JL_n + iJL_{n+1} + jJL_{n+2} + ijJL_{n+3}.$$

Then, we derive that

$$\begin{aligned} 3BJ_n + BJL_n &= \left(3J_n + JL_n\right) + i\left(3J_{n+1} + JL_{n+1}\right) + j\left(3J_{n+2} + JL_{n+2}\right) + ij\left(3J_{n+3} + JL_{n+3}\right) \\ &= 2^{n+1} + i2^{n+2} + j2^{n+3} + ij2^{n+4} \\ &= B2^{n+1} \end{aligned}$$

and

$$\begin{aligned} BJL_{n+1} + 2BJL_{n-1} &= \left(JL_{n+1} + 2JL_{n-1}\right) + i\left(JL_{n+2} + 2JL_n\right) \\ &\quad + j\left(JL_{n+3} + 2JL_{n+1}\right) + ij\left(JL_{n+4} + 2JL_{n+2}\right) \\ &= 9J_n + i9J_{n+1} + j9J_{n+2} + ij9J_{n+3} \\ &= 9BJ_n. \end{aligned}$$

We now state the following theorem.

Theorem 5.*For $n \geq 1$, we have the following identities*

$$BJ_n + BJL_n = 2BJ_{n+1},$$

$$BJL_{n+1} + BJL_n = 3B2^n$$

and

$$BJL_{n+1} - 2BJL_n = -3(-1)^n(1 + i + + j + ij).$$

Proof. Take

$$BJ_n = J_n + iJ_{n+1} + jJ_{n+2} + ijJ_{n+3}$$

and

$$BJL_n = JL_n + iJL_{n+1} + jJL_{n+2} + ijJL_{n+3}.$$

So, by Proposition 1, we compute that

$$\begin{aligned} BJ_n + BJL_n &= (J_n + JL_n) + i(J_{n+1} + JL_{n+1}) + j(J_{n+2} + JL_{n+2}) + ij(J_{n+3} + JL_{n+3}) \\ &= 2J_{n+1} + i2J_{n+2} + j2J_{n+3} + ijJ_{n+4} \\ &= 2BJ_{n+1} \end{aligned}$$

and

$$\begin{aligned} BJL_{n+1} + BJL_n &= (JL_{n+1} + JL_n) + i(JL_{n+2} + JL_{n+1}) + j(JL_{n+3} + JL_{n+2}) + ij(JL_{n+4} + JL_{n+3}) \\ &= 3\left(2^n + i2^{n+1} + j2^{n+2} + ij2^{n+3}\right) \\ &= 3B2^n. \end{aligned}$$

Also, we easily get

$$\begin{aligned} BJL_{n+1} - 2BJL_n &= (JL_{n+1} - 2JL_n) + i(JL_{n+2} - 2JL_{n+1}) \\ &\quad + j(JL_{n+3} - 2JL_{n+2}) + ij(JL_{n+4} - 2JL_{n+3}) \\ &= -(-1)^n(3 + 3i + 3j + 3ij) \\ &= -3(-1)^n(1 + i + j + ij). \end{aligned}$$

2.2 *Jacobsthal Octonions and Jacobsthal Lucas Octonions*

In this part, we investigate some properties of Jacobsthal octonions and Jacobsthal Lucas octonions. The Jacobsthal octonions $\widehat{J}_n$ and Jacobsthal Lucas octonions $\widehat{j}L_n$, are given respectively by the following recurrence relations (see [7]):

$$\begin{aligned} \widehat{J}_n &= \sum_{s=0}^{7} J_{n+s}e_s \qquad (9) \\ &= J_ne_0 + J_{n+1}e_1 + J_{n+2}e_2 + J_{n+3}e_3 \\ &\quad + J_{n+4}e_4 + J_{n+5}e_5 + J_{n+6}e_6 + J_{n+7}e_7 \end{aligned}$$

and

$$\begin{aligned} \widehat{j}L_n &= \sum_{s=0}^{7} j_{n+s}e_s \qquad (10) \\ &= JL_ne_0 + JL_{n+1}e_1 + JL_{n+2}e_2 + JL_{n+3}e_3 \\ &= JL_{n+4}e_4 + JL_{n+5}e_5 + JL_{n+6}e_6 + JL_{n+7}e_7. \end{aligned}$$

We here give Binet formulas for the Jacobsthal octonions $\widehat{J}_n$ and Jacobsthal Lucas octonions $\widehat{j}L_n$.

Theorem 6. *Let $\widehat{J}_n$ be the Jacobsthal octonion and $\widehat{j}L_n$ Jacobsthal Lucas octonion. We then have*

$$\sum_{s=0}^{7} \widehat{J}_{n+s}e_s = \frac{\alpha^n \overset{\approx}{\alpha} - \beta^n \overset{\approx}{\beta}}{\alpha - \beta}$$

and

$$\sum_{s=0}^{7} \widehat{j}L_{n+s}e_s = \alpha^n \overset{\approx}{\alpha} + \beta^n \overset{\approx}{\beta}.$$

Proof. By using the Binet formulas for Jacobsthal and Jacobsthal Lucas numbers, we readily derive

$$\begin{aligned} \sum_{s=0}^{7} \widehat{J}_{n+s}e_s &= \sum_{s=0}^{7} \left(\frac{\alpha^{n+s} - \beta^{n+s}}{\alpha - \beta}\right) e_s \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} e_0 + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} e_1 + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} e_2 + \ldots + \frac{\alpha^{n+7} - \beta^{n+7}}{\alpha - \beta} \\ &= \frac{\alpha^n\left(e_{0+}\alpha e_1 + \alpha^2 e_2 + \ldots + \alpha^7 e_7\right) - \beta^n\left(e_{0+}\beta e_1 + \beta^2 e_{2+\ldots+}\beta^7 e_7\right)}{\alpha - \beta} \\ &= \frac{\alpha^n \overset{\approx}{\alpha} - \beta^n \overset{\approx}{\beta}}{\alpha - \beta} \end{aligned}$$

and also

$$\begin{aligned} \sum_{s=0}^{7} \widehat{j}L_{n+s}e_s &= \sum_{s=0}^{7} \left(\alpha^{n+s} + \beta^{n+s}\right) e_s \\ &= (\alpha^n + \beta^n)e_0 + \left(\alpha^{n+1} + \beta^{n+1}\right)e_1 + \left(\alpha^{n+2} + \beta^{n+2}\right)e_2 + \ldots + \left(\alpha^{n+7} + \beta^{n+7}\right)e_7 \\ &= \alpha^n\left(e_0 + \alpha e_1 + \alpha^2 e_2 + \ldots + \alpha^7 e_7\right) + \beta^n\left(e_0 + \beta e_1 + \beta^2 e_2 + \ldots + \beta^7 e_7\right) \\ &= \alpha^n \overset{\approx}{\alpha} + \beta^n \overset{\approx}{\beta}. \end{aligned}$$

We give the following theorem.

Theorem 7. *We have*

$$\widehat{J}_{n+1} + 2\widehat{J}_{n-1} = \widehat{j}L_n \text{ and } 3\widehat{j}_n + \widehat{j}L_n = 2^{n+1}\left(2^{s+1} - 1\right).$$

Proof. By using the properties of Jacobsthal numbers and Jacobsthal Lucas numbers in Proposition 1, one can easily get the desired results with some basic computations. So we omit them.

We now state the following theorem.

Theorem 8. *Let $\widehat{J}_n$ and $\widehat{j}L_n$ Jacobsthal and Jacobsthal Lucas octonion numbers, respectively:*

$$\widehat{J}_{n+2} + 2\widehat{J}_{n-2} = \widehat{j}L_{n+1} \text{ and } \widehat{J}_{n+2-}2\widehat{J}_{n-2} = \widehat{J}_{n+1}.$$

Proof. The proof of this theorem just follows from Proposition 1 and Eqs. (9) and (10).

Theorem 9. *Let $\widehat{J}_n$ be n-th Jacobsthal octonion numbers. Then for $n \geq 0$, we get*

$$\widehat{J}_{n+1} - 2\widehat{J}_{n-1} = \widehat{J}_{n+1}$$

and

$$\widehat{J}_{n+1} + 2\widehat{J}_n = \widehat{J}_{n+2}.$$

Proof. The proof of this theorem just follows from Proposition 1.

3 k-Jacobsthal Octonions and k-Jacobsthal Lucas Octonions

In this section, we perform to introduce the k-Jacobstal octonions and k-Jacobstal Lucas octonions and then analyze some properties.
The k-Jacobstal numbers and k-Jacobstal Lucas numbers are defined by the following recurrence relations

$$J_{k,n} = kJ_{k,n-1} + 2J_{k,n-2} \text{ with initial values } J_{k,0} = 0 \text{ and } J_{k,1} = 1 \text{ for } n \geqq 2 \qquad (11)$$

and

$$JL_{k,n} = kJL_{k,n-1} + 2JL_{k,n-2} \text{ with initial values } JL_{k,0} = 2 \text{ and } JL_{k,1} = k \text{ for } n \geqq 2 \qquad (12)$$

for any real number k, see [12, 13] for more details.

Definition 1. *The k-Jacobsthal octonions are defined as follows*

$$\begin{aligned}\widehat{J}_{k,n} &= \sum_{s=0}^{7} J_{k,n+s}e_s \\ &= J_{k,n}e_0 + J_{k,n+1}e_1 + J_{k,n+2}e_2 + \ldots + J_{k,n+7}e_7,\end{aligned}$$

where $J_{k,n}$ is given in (11).

We note that for $n \geq 0$

$$\widehat{J}_{k,n+2} = k\widehat{J}_{k,n+1} + 2\widehat{J}_{k,n}.$$

The binet-like formula for the k-Jacobsthal octonions is given by the following theorem.

Theorem 10. *The k-Jacobsthal octonions hold the following equality:*

$$\widehat{J}_{k,n+s} = \frac{\widehat{\widehat{\alpha}}\alpha^n - \widehat{\widehat{\beta}}\beta^n}{\alpha - \beta}, \tag{13}$$

where

$$\widehat{\widehat{\alpha}} = e_0 + \alpha e_1 + \alpha^2 e_2 + \ldots + \alpha^7 e_7 \text{ and } \widehat{\widehat{\beta}} = e_0 + \beta e_1 + \beta^2 e_2 + \ldots + \beta^7 e_7$$

with $\alpha = 1 + e_1 r_1 + e_2 r_1^2 + e_3 r_1^3 + \ldots + e_7 r_1^7$ and $\beta = 1 + e_1 r_2 + e_2 r_2^2 + e_3 r_2^3 + \ldots + e_7 r_2^7$ where r_1 and r_2 are roots of the equation $r^2 - kr - 2 = 0$.

Proof. By means of the binet formula for k-Jacobsthal numbers (*cf.* [12,13]), we easily get

$$\begin{aligned}\sum_{s=0}^{7} \widehat{J}_{k,n+s}e_s &= \sum_{s=0}^{7} \left(\frac{\alpha^{n+s} - \beta^{n+s}}{\alpha - \beta}\right) e_s \\ &= \frac{\alpha^n - \beta^n}{\alpha - \beta} e_0 + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} e_1 + \ldots + \frac{\alpha^{n+7} - \beta^{n+7}}{\alpha - \beta} e_7 \\ &= \frac{\alpha^n \left(e_0 + \alpha e_1 + \alpha^2 e_2 + \ldots + \alpha^7 e_7\right)}{\alpha - \beta} - \frac{\beta^n \left(e_0 + \beta e_1 + \beta^2 e_2 + \ldots + \beta^7 e_7\right)}{\alpha - \beta} \\ &= \frac{\alpha^n \widehat{\widehat{\alpha}} - \beta^n \widehat{\widehat{\beta}}}{\alpha - \beta}.\end{aligned}$$

We give the following theorem.

Theorem 11. *(Catalan Identity) We have*

$$\widehat{J}_{k,n-r}\widehat{J}_{k,n+r} - \left(\widehat{J}_{k,n+r}\right)^2 = -(\alpha\beta)(-2)^{n-r}\left(\widehat{J}_{k,r}\right)^2,$$

where α and β are given in Theorem 10.

Proof. We have

$$\begin{aligned}\widehat{J}_{k,n-r}\widehat{J}_{k,n+r} - \left(\widehat{J}_{k,n+r}\right)^2 &= \left(\frac{\alpha r_1^{n-r} - \beta r_2^{n-r}}{r_1 - r_2}\right) \cdot \left(\frac{\alpha r_1^{n+r} - \beta r_2^{n+r}}{r_1 - r_2}\right) - \left(\frac{\alpha r_1^{n} - \beta r_2^{n}}{r_1 - r_2}\right)^2 \\ &= \frac{\alpha^2 r_1^{2n} - \alpha\beta r_1^{n-r} r_2^{n+r} - \alpha\beta r_2^{n-r} \beta^2 r_2^{2n}}{(r_1 - r_2)^2} - \frac{\left(\alpha^2 r_1^{2n} - 2\alpha\beta r_1^n r_2^n + \beta^2 r_2^{2n}\right)}{(r_1 - r_2)^2} \\ &= \frac{\alpha\beta\left(r_1^{n-r} r_2^{n+r} - r_2^{n-r} r_1^{n+r}\right)}{(r_1 - r_2)^2} - \frac{2\alpha\beta r_1^n r_2^n}{(r_1 - r_2)^2} \\ &= \frac{-\alpha\beta}{(r_1 - r_2)^2}(r_1 r_2)^n \left(\frac{r_2^r}{r_1} - \frac{r_1^r}{r_2} - 2\right) \\ &= \frac{-\alpha\beta}{(r_1 - r_2)^2}(-2)^n \left(\frac{r_2^{2r} + r_1^{2r}}{(r_1 r_2)^r} - 2\right) \\ &= -(\alpha\beta)(-2)^{n-r}\left(\frac{r_1^r - r_2^r}{r_1 - r_2}\right)^2 \\ &= -(\alpha\beta)(-2)^{n-r}\left(\widehat{J}_{k,r}\right)^2.\end{aligned}$$

We provide the following theorem.

Theorem 12. *(D′ocagene′s Identity) If $m > n$, then*

$$\widehat{J}_{k,m}\widehat{J}_{k,n+1} - \widehat{J}_{k,m+1}\widehat{J}_{k,n} = \alpha\beta(-2)^n \widehat{J}_{k,m-n},$$

where α and β are given in Theorem 10.

Proof. By (13), we acquire

$$\begin{aligned}\widehat{J}_{k,m}\widehat{J}_{k,n+1} - \widehat{J}_{k,m+1}\widehat{J}_{k,n} &= \left(\frac{\alpha r_1^m - \beta r_2^m}{r_1 - r_2}\right)\left(\frac{\alpha r_1^{n+1} - \beta r_2^{n+1}}{r_1 - r_2}\right) - \left(\frac{\alpha r_1^{m+1} - \beta r_2^{m+1}}{r_1 - r_2}\right)\left(\frac{\alpha r_1^n - \beta r_2^n}{r_1 - r_2}\right) \\ &= \frac{\alpha^2 r_1^{n+m+1} - \alpha\beta r_1^m r_2^{n+1} + \beta^2 r_2^{n+m+1}}{(r_1 - r_2)^2} - \frac{\alpha^2 r_1^{n+m+1} - \alpha\beta r_1^{m+1} r_2^n + \beta^2 r_2^{m+1} r_2^n}{(r_1 - r_2)^2} \\ &= \alpha\beta(r_1 r_2)^n \left[r_2^{m-n} r_1 + r_1^{m-n} r_2 + r_1^{m-n} + r_2^{m-n+1}\right] \\ &= \alpha\beta(-2)^n (r_1 - r_2) \frac{r_1^{m-n} - r_2^{m-n}}{(r_1 - r_2)^2} \\ &= \alpha\beta(-2)^n \frac{r_1^{m-n} - r_2^{m-n}}{r_1 - r_2} \\ &= \alpha\beta(-2)^n \widehat{J}_{k,m-n},\end{aligned}$$

which is the asserted result.

We now state a new extension of the Jacobsthal Lucas sequences as follows.

Definition 2. *The k-Jacobsthal Lucas octinions are defined by*

$$\begin{aligned}\widehat{JL}_{k,n} &= \sum_{s=0}^{7} JL_{k,n+s}e_s \\ &= JL_{k,n}e_0 + JL_{k,n+1}e_1 + JL_{k,n+2}e_2 + \ldots + JL_{k,n+7}e_7,\end{aligned}$$

where $JL_{k,n}$ is defined in (12).

Binet-like formula for k-Jacobsthal Lucas octonions is given below.

Theorem 13. *Binet-like formula for k-Jacobsthal Lucas octonions is*

$$\widehat{JL}_{k,n+s} = \widehat{\widehat{\alpha}}\alpha^n - \widehat{\widehat{\beta}}\beta^n,$$

where $\widehat{\widehat{\alpha}}$ and $\widehat{\widehat{\beta}}$ are given in Theorem 10.

Proof. By utilizing the binet formula for Jacobsthal Lucas numbers (*cf.* [12,13]), we consider that

$$\begin{aligned}\sum_{s=0}^{7} JL_{k,n+s}e_s = \sum_{s=0}^{7}\left(\alpha^{n+s} - \beta^{n+s}\right)e_s &= \left(\alpha^n - \beta^n\right)e_0 + \left(\alpha^{n+1} - \beta^{n+1}\right)e_1 + \ldots + \left(\alpha^{n+7} - \beta^{n+7}\right)e_7 \\ &= \alpha^n\left(e_0 + \alpha e_1 + \alpha^2 e_2 + \alpha^3 e_3 + \ldots + \alpha^7 e_7\right) - \beta^n\left(e_0 + \beta e_1 + \beta^2 e_2 + \ldots + \beta^7 e_7\right) \\ &= \alpha^n \widehat{\widehat{\alpha}} - \beta^n \widehat{\widehat{\beta}}.\end{aligned}$$

We state the following theorem.

Theorem 14. *(Catalan Identity) We have*

$$\widehat{JL}_{k,n-r}\widehat{JL}_{k,n+r} - \left(\widehat{JL}_{k,n}\right)^2 = -\alpha\beta(-2)^{n-r}\left(\left(\widehat{JL}_{k,r}\right)^2 - 4(-2)^r\right),$$

where α and β are given in Theorem 10.

*Proof.*By Theorem 13, we have

$$\begin{aligned}
\widehat{JL}_{k,n-r}\widehat{JL}_{k,n+r}-\left(\widehat{JL}_{k,n}\right)^2 &= \left(\alpha r_1^{n-r}+\beta r_2^{n-r}\right)\left(\alpha r_1^{n+r}+\beta r_2^{n+r}\right)-\left(\alpha r_1^{n}+\beta r_2^{n}\right)^2\\
&= \alpha^2 r_1^{2n}+\alpha\beta r_1^{n-r}r_2^{n+r}+\alpha\beta r_1^{n+r}r_2^{n-r}\beta^2 r_2^{2n}-\alpha^2 r_1^{2n}-2\alpha\beta r_1^n r_2^n-\beta^2 r_2^{2n}\\
&= \alpha\beta\left(r_1r_2\right)^2\left(\frac{r_1^r}{r_2^r}+\frac{r_2^r}{r_1^r}-2\right)\\
&= \alpha\beta\left(-2\right)^n\left(\frac{r_1^{2r}+r_2^{2r}-2\left(r_1r_2\right)^r}{\left(-2\right)^r}\right)\\
&= -\alpha\beta\left(-2\right)^{n-r}\left(r_1^{2r}+r_2^{2r}-2\left(r_1r_2\right)^r\right)\\
&= -\alpha\beta\left(-2\right)^{n-r}\left(\left(r_1^r+r_2^r\right)^2-4\left(r_1r_2\right)^r\right)\\
&= -\alpha\beta\left(-2\right)^{n-r}\left(\left(\widehat{JL}_{k,r}\right)^2-4\left(-2\right)^r\right).
\end{aligned}$$

One other basic properties of the k-Jacobsthal Lucas octonions is presented below.

Theorem 15.*(D′ocagene′s Identity)The following is valid:*

$$\widehat{JL}_{k,m}\widehat{JL}_{k,n+1}-\widehat{JL}_{k,m+1}\widehat{JL}_{k,n}=\alpha\beta\sqrt{k^2+8}\left(\widehat{JL}_{k,m-n}-2\frac{k+\sqrt{k^2+8}}{2}\right)^{m-n},$$

where α and β are given in Theorem 10.

*Proof.*For $m>n$, by Theorem 13, we attain

$$\begin{aligned}
\widehat{JL}_{k,m}\widehat{JL}_{k,n+1}-\widehat{JL}_{k,m+1}\widehat{JL}_{k,n} &= \left(\alpha r_1^m+\beta r_2^m\right)\left(\alpha r_1^{n+1}+\beta r_2^{n+1}\right)-\left(\alpha r_1^{m+1}+\beta r_2^{m+1}\right)\left(\alpha r_1^n+\beta r_2^n\right)\\
&= \left(\alpha^2 r_1^{m+n+1}+\alpha\beta r_1^m r_2^{n+1}+\alpha\beta r_2^m r_1^{n+1}+\beta^2 r_2^{m+n+1}\right)\\
&\quad -\left(\alpha^2 r_1^{m+n+1}+\alpha\beta r_1^{m+1}r_2^n+\alpha\beta r_2^{m+1}r_1^n+\beta^2 r_2^{m+n+1}\right)\\
&= \alpha\beta\left(r_1r_2\right)^n\left(r_1^{m-n}r_2+r_1r_2^{m-n}-r_1^{m-n}r_1-r_2r_1^{m-n}\right)\\
&= \alpha\beta\left(-2\right)^n\left[r_1^{m-n}\left(r_2-r_1\right)+r_2^{m-n}\left(r_1-r_2\right)\right]\\
&= \alpha\beta\left(-2\right)^n\left(r_1-r_2\right)\left[r_2^{m-n}-r_1^{m-n}\right]\\
&= \alpha\beta\left(-2\right)^n\sqrt{k^2+8}\left(r_1^{m-n}-r_2^{m-n}-2r_1^{m-n}\right)\\
&= \alpha\beta\left(-2\right)^n\sqrt{k^2+8}\left(\hat{J}_{k,m-n}-2r_1^{m-n}\right)\\
&= \alpha\beta\left(-2\right)^n\sqrt{k^2+8}\left(\widehat{JL}_{k,m-n}-2\frac{k+\sqrt{k^2+8}}{2}\right)^{m-n}.
\end{aligned}$$

We now give some properties for $\widehat{JL}_{k,n}$ as given below.

Theorem 16.*We have*

$$\begin{aligned}
\hat{J}_{k,n+1}+2\hat{J}_{k,n-1} &= \widehat{JL}_{k,n},\\
k\hat{J}_{k,n}+4\hat{J}_{k,n-1} &= \widehat{JL}_{k,n},\\
k\hat{J}_{k,n}+\widehat{JL}_{k,n} &= 2\hat{J}_{k,n+1}.
\end{aligned}$$

*Proof.*From Proposition 1, the proofs of the results in this theorem are, respectively, as follows:

$$\begin{aligned}
\hat{J}_{k,n+1}+2\hat{J}_{k,n-1} &= \sum_{s=0}^{7}\hat{J}_{k,n+1+s}e_s+2\sum_{s=0}^{7}\hat{J}_{k,n-1+s}e_s\\
&= \sum_{s=0}^{7}\left(\hat{J}_{k,n+1+s}+2\hat{J}_{k,n-1+s}\right)e_s\\
&= \sum_{s=0}^{7}\widehat{JL}_{k,n}e_s\\
&= \widehat{JL}_{k,n},
\end{aligned}$$

$$\begin{aligned}
k\hat{J}_{k,n}+4\hat{J}_{k,n-1} &= k\sum_{s=0}^{7}\hat{J}_{k,n+s}e_s+4\sum_{s=0}^{7}\hat{J}_{k,n-1+s}e_s\\
&= \sum_{s=0}^{7}\left(k\hat{J}_{k,n+s}+4\hat{J}_{k,n-1+s}\right)e_s\\
&= \sum_{s=0}^{7}\widehat{JL}_{k,n}\\
&= \widehat{JL}_{k,n}
\end{aligned}$$

and

$$\begin{aligned}
k\hat{J}_{k,n}+\widehat{JL}_{k,n} &= k\sum_{s=0}^{7}\hat{J}_{k,n+s}e_s+\sum_{s=0}^{7}\widehat{JL}_{k,n+s}e_s\\
&= \sum_{s=0}^{7}\left(k\hat{J}_{k,n+s}+\widehat{JL}_{k,n+s}\right)e_s\\
&= \sum_{s=0}^{7}2\hat{J}_{k,n+1+s}e_s\\
&= 2\hat{J}_{k,n+1}.
\end{aligned}$$

4 Perspective

In the present paper, we have considered the Jacobsthal and Jacobsthal Lucas quaternions and octonions. We have investigated diverse new and interesting properties and relations for the Jacobstal and Jacobstal Lucas quaternions and octonions. The binet formulas for the mentioned numbers have been also acquired. Lastly, we have introduced the k-Jacobstal octonions and k-Jacobstal Lucas octonions and we then derived diverse properties covering Catalan identities, D′ocagene′s identities and binet-like formulas.

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