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Non-Standard Crank-Nicholson Method for Solving the Variable Order Fractional Cable Equation

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Abstract: In this paper, a non-standard Crank-Nicholson finite difference method (NSCN) is presented. NSCN is used to study numerically the variable-order fractional Cable equation, where the variable order fractional derivatives are described in the Riemann-Liouville and the Grünwald-Letnikov sense. The stability analysis of the proposed methods is given by a recently proposed procedure similar to the standard John von Neumann stability analysis. The reliability and efficiency of the proposed approach are demonstrated by some numerical experiments. It is found that NSCN is preferable than the standard Crank-Nicholson finite difference method (SCN).

Keywords: Non-standard finite difference method, Crank-Nicholson method; Variable order fractional Cable equation; Von Neumann stability analysis.

1 Introduction

In modeling neuronal dynamics, the Cable equation is one of the most fundamental equations. Due to its significant deviation from the dynamics of Brownian motion, the anomalous diffusion in biological systems can not be adequately described by the traditional Nernst-Planck equation or its simplification, the Cable equation was introduced for modeling the anomalous diffusion in spiny neuronal dendrites (Henry et al., in [12]). The resulting governing equation, the so-called fractional Cable equation, which is similar to the traditional cable equation except that the order of derivative with respect to the space or time is fractional [15].

In recent years considerable interest in fractional calculus has been stimulated by the applications that finds in numerical analysis and different areas of physics and engineering, possibly including fractal phenomena (see [1]-[3],[16],[18]-[23]). The applications range from control. Variable-order differential equations have been considered in [17],[24]. In this sense, the orders are function in any variable, i.e., space variables, time variable, or any other variables. Samko and Ross [25], first proposed the concept of variable order operator and investigated the properties of variable order integration and differentiation operators of Riemann-Liouville type. Different authors have introduced different definitions of variable order -differential operators, each of these with a specific meaning to suit desired goals, most of these definitions were extension to the fractional calculus definitions as Riemann-Liouville, Grünwald, Caputo and Riesz ([4],[5],[10],[11],[14],[17],[25]) and some not as Coimbra definition [7,8].
The variable order differentials are important tool to study some systems such as the control of nonlinear viscoelasticity oscillator, for more details see ([7,8] and the references sited therein), where the order changes with respect to a parameter or more parameters.

The main aim of this work is to present the efficient numerical method; NSCN which is more accurate than SCN. Then we used NSCN to study numerically the variable order Cable equation, for more details on non-standard finite difference method see [28,31].

The paper is organized as follows: In Section 2, some well-known mathematical preliminaries on variable-order fractional differential equations and non-standard discretization are given. in Section 3, discretization of the variable order fractional Cable equation using NSCN method with shifted Grünwald formula is given. In Section 4, we study the stability of the presented method.

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In Section 5, numerical experiments of a typical variable order fractional cable problem are presented. Finally, in Section 6 we give some conclusions.

2 Preliminaries and Notations

In the following we present some definitions of the variable-order fractional differential operators. Moreover, some fundamental concepts of NSFD for the solution of partial differential equations are presented.

Definition 1*The Riemann-Liouville variable order fractional derivative is defined as [7]:*

$$ {}_0\mathscr{D}_x^{\alpha(x)} f(x,t) = \frac{1}{\Gamma(n-\alpha(x))} \frac{d^n}{dx^n} \int_0^x \frac{f(x,\tau)}{(x-\tau)^{\alpha(x)-n+1}} d\tau, \tag{1} $$

$x > 0$, *where* $n-1 < \alpha(x) < n$.

Definition 2*The Grünwald-Letnikov variable order fractional derivative is defined as:*

$$ {}_0\mathscr{D}_x^{\alpha(x)} f(x) = \lim_{h\to 0} \frac{1}{h^{\alpha(x)}} \sum_{k=0}^{[x/h]} w_k^{(\alpha(x))} f(x-hk), \quad x \geq 0, \tag{2} $$

where $[x/h]$ *means the integer part of* x/h *and* $w_k^{(\alpha(x))}$ *are the normalized Grünwald weights which are defined by* $w_k^{(\alpha(x))} = (-1)^k \binom{\alpha(x)}{k}$.

The Grünwald-Letnikov definition is simply a generalization of the ordinary discretization formula for integer order derivatives. The Riemann-Liouville and the Grünwald-Letnikov approaches coincide under relatively weak conditions; if $f(x)$ is continuous and $f'(x)$ is integrable in the interval $[0,x]$, then for every order $0 < \alpha(x) < 1$, both the Riemann-Liouville and the Grünwald-Letnikov derivatives exist and coincide for any value inside the interval $[0,x]$.

2.1 Non-standard Discretization

The non-standard finite difference (NSFD) schemes were firstly proposed by Mickens ([28]-[30]), either for ordinary differential equations (ODEs) or partial differential equations (PDEs).
In this part, we would like to introduce several comments related to NSFD schemes. A scheme is called nonstandard if at least one of the following conditions is satisfied:
1- Nonlocal approximation is used.
2- Discretization of derivative is not traditional and use a nonnegative function.
The forward Euler method is one of the simplest discretization schemes. In this method the derivative term $\frac{dy}{dt}$ is replaced by $\frac{y(t+h)-y(t)}{h}$, where h is the step size. However, in the Mickens schemes this term is replaced by $\frac{y(t+h)-y(t)}{\phi(h)}$,where $\phi(h)$ is a continuous function of step size h, and the function $\phi(h)$ satisfies the following conditions:

$$ \phi(h) = h + O(h^2), \quad 0 < \phi(h) < 1, \quad h \to 0. $$

Examples of functions $\phi(h)$ that satisfy these conditions are [27]:
$\phi(h) = h,\ \sinh h,\ e^h - 1,\ \frac{1-e^{-\lambda h}}{\lambda}$, etc., ...
Note that in taking the $\lim h \to 0$ to obtain the derivative, the use of any of these $\phi(h)$ will lead to the usual result for the first derivative

$$ \frac{dy}{dt} = \lim_{h\to 0} \frac{y[t+\phi_1(h)]-y(t)}{\phi_2(h)} = \lim_{h\to 0} \frac{y(t+h)-y(t)}{h}. $$

In addition to this replacement, if there are nonlinear terms in the differential equation [27], these are replaced by

$$ y^2 \to \begin{cases} y_n y_{n+1}, \\ y_{n-1} y_n. \end{cases} $$

In dimensions two and above, nonlinear terms such as $y(t)x(t)$ are either replaced by

$$ yx \to \begin{cases} y_n x_{n+1}, \\ y_{n+1} x_n. \end{cases} $$

One can say that there is no appropriate general method to choose the function $\phi(h)$ or to choose which nonlinear terms are to be replaced, some special techniques may be found in [28] and [29].

3 Discretization of the variable order fractional Cable equation

In this section, we will use NSCN method with shifted Grünwald formula to obtain the discretization finite of the initial-boundary value problem of the variable order fractional Cable equation [9]:

$$ \frac{\partial u(x,t)}{\partial t} = {}_0\mathscr{D}_t^{1-\beta(x,t)} \frac{\partial^2 u(x,t)}{\partial x^2} - \mu\, {}_0\mathscr{D}_t^{1-\alpha(x,t)} u(x,t) + f(x,t), \tag{3} $$

with initial and boundary conditions:

$$ u(x,0) = 0, \quad 0 \leq x \leq L, \tag{4} $$

$$ u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leq t \leq t_{max} \tag{5} $$

where $0 < \beta(x,t), \alpha(x,t) < 1$, $\mu > 0$ is a constant and ${}_0\mathscr{D}_t^{1-\gamma(x,t)}$ is the variable order fractional derivative defined by the Riemann-Liouville operator of order $1-\gamma(x,t)$, where $\gamma(x,t)$ is equal to $\beta(x,t)$ or $\alpha(x,t)$. Let us assume that the coordinates of the mesh points are

$$ x_n = nh, \quad n = 0,1,\dots,N; \quad t_m = m\tau, \quad m = 0,1,\dots,M $$

where $h = L/N$, $\tau = t_{max}/M$, and N, M are the total number of spatial nodes and the number of time steps, respectively. On the grid (x_n, t_m), Eq. (3) can be written as

$$\frac{\partial u(x_n,t_m)}{\partial t} = {}_0\mathscr{D}_t^{1-\beta(x_n,t_m)} \frac{\partial^2 u(x_n,t_m)}{\partial x^2} - \mu \mathscr{D}_t^{1-\alpha(x_n,t_m)} u(x_n,t_m) + f(x_n,t_m).$$

The Grünwald-Letnikov definition is important for our purposes in this paper because it allows us to estimate numerically in a simple and efficient way. From the relationship between the Riemann-Liouville and the Grünwald-Letnikov, then Riemann-Liouville fractional partial derivative of order $1-\gamma_{n,m}$ can be expressed as follows:

$$_0\mathscr{D}_t^{1-\gamma_{n,m}} u(x,t) = \lim_{\tau\to 0} \tau^{\gamma_{n,m}-1} \sum_{l=0}^{[t/\tau]} w_{n,m}^{(l)} u(x, t-l\tau). \tag{6}$$

According to I. Podlubny in [13], the right-hand side of equation (6) is approximated by:

$$\lim_{\tau\to 0} \tau^{\gamma_{n,m}-1} \sum_{l=0}^{[t_m/\tau]} w_{n,m}^{(l)} u(x_n, t_m - l\tau) = \tau^{\gamma_{n,m}-1} \sum_{l=0}^{[t_m/\tau]} w_{n,m}^{(l)} u(x_n, t_m - l\tau) + \mathscr{O}(\tau^p), \tag{7}$$

hence, Eq. (6) yields to

$$_0\mathscr{D}_t^{1-\gamma_{n,m}} u(x_n,t_m) = \tau^{\gamma_{n,m}-1} \sum_{l=0}^{m} w_{n,m}^{(l)} u(x_n, t_{m-l}) + \mathscr{O}(\tau^p). \tag{8}$$

This formula is not unique because there are many different choices for $w_{n,m}^{(l)}$ that lead to approximations of different orders p ([6],[13]). The definition

$$w_{n,m}^{(l)} = (-1)^l \binom{1-\gamma_{n,m}}{l} = (-1)^l \frac{(1-\gamma_{n,m})(-\gamma_{n,m})\cdots(1-\gamma_{n,m}-l+1)}{l!},$$

provides order $p = 1$.
Now, we apply the Mickens discretization scheme to the equation (3) by replacing the step size h by a function of h, $\phi(h)$ and the step size τ by a function of τ, $\psi(\tau)$. The left hand side of Eq. (3) is approximated, i.e.,

$$\frac{\partial u(x_n,t_m)}{\partial t} = \frac{u_n^m - u_n^{m-1}}{\varphi(\tau)}, \tag{9}$$

and the second derivative in the right hand side is approximated with the average of the central difference scheme evaluated at the current and the previous time step

$$\frac{\partial^2 u(x_n,t_m)}{\partial x^2} = \frac{1}{2}\left[\frac{u_{n-1}^m - 2u_n^m + u_{n+1}^m}{\phi(h)^2} + \frac{u_{n-1}^{m-1} - 2u_n^{m-1} + u_{n+1}^{m-1}}{\phi(h)^2}\right]. \tag{10}$$

Also, if $f(x,t)$ has first-order continuous derivative $\frac{\partial f(x,t)}{\partial t}$, then

$$f(x_n,t_m) = f(x_n,t_{m-1}) + \mathscr{O}(\varphi(\tau)). \tag{11}$$

where $\phi(h)$ *and* $\psi(\tau)$ have the properties:

$$\psi(\tau) = \tau + O(\tau^2) \ \textit{and} \ \phi(h) = h^2 + O(h^4).$$

Substituting, equations (8)-(11) into Eq. (3) we obtain:

$$\frac{u_n^m - u_n^{m-1}}{\varphi(\tau)} = \frac{1}{2\phi(h)^2} \mathscr{D}_t^{1-\beta_{n,m}} [u_{n-1}^m - 2u_n^m + u_{n+1}^m + u_{n-1}^{m-1} - 2u_n^{m-1} + u_{n+1}^{m-1}] - \mu \mathscr{D}_t^{1-\alpha_{n,m}} u_n^m + f_n^{m-1} + \mathscr{O}(\varphi(\tau)). \tag{12}$$

Now, using the definition of the Grünwald-Letnikov fractional variable-order given in (8), we get

$$\frac{u_n^m - u_n^{m-1}}{\varphi(\tau)} = \frac{1}{2\phi(h)^2}[\varphi(\tau)^{\beta_{n,m}-1} \sum_{l=0}^{m} w_{n,m}^{(\beta,l)} (u_{n-1}^{m-l} - 2u_n^{m-l} + u_{n+1}^{m-l}) + \varphi(\tau)^{\beta_{n,m}-1} \sum_{l=0}^{m-1} w_{n,m-1}^{(\beta,l)} (u_{n-1}^{m-1-l} - 2u_n^{m-1-l} + u_{n+1}^{m-1-l})] - \mu\varphi(\tau)^{\alpha_{n,m}-1} \sum_{l=0}^{m} w_{n,m}^{(\alpha,l)} u_n^{m-l} + f_n^{m-1},\ n = 1,\dots,N-1;\ m = 1,2,\dots,M, \tag{13}$$

where

$$w_{n,m}^{(\beta,l)} = (-1)^l \binom{1-\beta_{n,m}}{l} = (-1)^l \frac{(1-\beta_{n,m})(-\beta_{n,m})\cdots(1-\beta_{n,m}-l+1)}{l!},$$

$$w_{n,m}^{(\alpha,l)} = (-1)^l \binom{1-\alpha_{n,m}}{l} = (-1)^l \frac{(1-\alpha_{n,m})(-\alpha_{n,m})\cdots(1-\alpha_{n,m}-l+1)}{l!}.$$

$$u_n^0 = 0, \quad n = 0,1,2,\dots,N, \tag{14}$$

$$u_0^m = 0, \quad u_N^m = 0, \quad m = 1,2,\dots,M. \tag{15}$$

After doing some algebraic manipulation to equation (13), we obtain,

$$-\frac{\varphi(\tau)^{\beta_{n,m}}}{2\phi(h)^2} w_{n,m}^{(\beta,0)} u_{n+1}^m + [1 + \frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2} w_{n,m}^{(\beta,0)} + \mu\varphi(\tau)^{\alpha_{n,m}} w_{n,m}^{(\alpha,0)}] u_n^m$$

$$-\frac{\varphi(\tau)^{\beta_{n,m}}}{2\phi(h)^2}w_{n,m}^{(\beta,0)}u_{n-1}^{m}= \tag{16}$$
$$\frac{1}{2\phi(h)^2}[\varphi(\tau)^{\beta_{n,m}}w_{n,m}^{(\beta,1)}+\varphi(\tau)^{\beta_{n,m}}w_{n,m-1}^{(\beta,0)}]u_{n+1}^{m-1}+$$
$$[1-\frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2}w_{n,m}^{(\beta,1)}-\frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2}w_{n,m-1}^{(\beta,0)}-\mu\varphi(\tau)^{\alpha_{n,m}}w_{n,m}^{(\alpha,1)}]u_n^{m-1}$$
$$+\frac{1}{2\phi(h)^2}[\varphi(\tau)^{\beta_{n,m}}w_{n,m}^{(\beta,1)}+\varphi(\tau)^{\beta_{n,m}}w_{n,m-1}^{(\beta,0)}]u_{n-1}^{m-1}+$$
$$\sum_{l=0}^{m-2}[\frac{\varphi(\tau)^{\beta_{n,m}}}{2\phi(h)^2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})u_{n+1}^{m-2-l}-$$
$$\frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)}+\mu\varphi(\tau)^{\alpha_{n,m}}w_{n,m}^{(\alpha,l+2)})u_n^{m-2-l}+$$
$$\frac{\varphi(\tau)^{\beta_{n,m}}}{2\phi(h)^2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})u_{n-1}^{m-2-l}]+\varphi(\tau)f_n^{m-1}.$$

Let $\mathbf{U}^m=[u_1^m,u_2^m,...,u_{N-1}^m]$, then Eqs. (14)-(16) can be transferred into a matrix form as follows:

$$\mathbf{A}\mathbf{U}^m=\mathbf{B}\mathbf{U}^{m-1}+\sum_{l=0}^{m-2}\mathbf{C}_l\mathbf{U}^{m-2-l}+\mathbf{f}^{m-1}\equiv\mathbf{b}_m, \tag{17}$$

where $\mathbf{C}_0,\mathbf{C}_1,\cdots,\mathbf{C}_{m-2}$; $\mathbf{A}$; $\mathbf{B}$ are matrices of order $(N-1)\times(N-1)$. The structure of matrix $\mathbf{A}$ is

$$\mathbf{A}=\begin{bmatrix} a_1^{(m)} & \bar{a}_1^{(m)} & & & \\ \bar{a}_2^{(m)} & a_2^{(m)} & \bar{a}_2^{(m)} & & \\ & \ddots & \ddots & \ddots & \\ & & \bar{a}_{N-2}^{(m)} & a_{N-2}^{(m)} & \bar{a}_{N-2}^{(m)} \\ & & & \bar{a}_{N-1}^{(m)} & a_{N-1}^{(m)} \end{bmatrix},$$

where
$a_n^{(m)}=\left(1+\frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2}w_{n,m}^{(\beta,0)}+\mu\varphi(\tau)^{\alpha_{n,m}}w_{n,m}^{(\alpha,0)}\right)$ and $\bar{a}_n^{(m)}=-\frac{\varphi(\tau)^{\beta_{n,m}}}{2\phi(h)^2}w_{n,m}^{(\beta,0)}$,
$n=1,2,\cdots,N-1$. The matrix $\mathbf{A}$ is a strictly diagonally dominant matrix, hence $\mathbf{A}$ is a nonsingular matrix. Thus the linear equation system (17) has a unique solution, and so the numerical method (14)-(16) is uniquely solvable.

4 Stability analysis

In this section, we study the stability analysis of the NSCN scheme (16) for the free force case.

Theorem 1 *The variable order fractional NSCN discretization, using the shifted Grünwald estimates, applied to the variable order fractional Cable Eq. (3) and defined by (16) is unconditional stable for* $0<\beta(x,t)<1$ *and* $0<\alpha(x,t)<1$.

Proof. For simplicity, let us assume that

$$\phi_{n,m}=\frac{\varphi(\tau)^{\beta_{n,m}}}{\phi(h)^2},\quad \textit{and}\quad \psi_{n,m}=\mu\varphi(\tau)^{\alpha_{n,m}},$$

then we can rewrite Eq. (16)

$$-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}u_{n+1}^m+(1+\phi_{n,m}w_{n,m}^{(\beta,0)}+\psi_{n,m}w_{n,m}^{(\alpha,0)})u_n^m-$$
$$\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}u_{n-1}^m=$$
$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})u_{n+1}^{m-1}+$$
$$(1-\phi_{n,m}w_{n,m}^{(\beta,1)}-\phi_{n,m}w_{n,m-1}^{(\beta,0)}-\psi_{n,m}w_{n,m}^{(\alpha,1)})u_n^{m-1}+$$
$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})u_{n-1}^{m-1}+$$
$$\sum_{l=0}^{m-2}\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})u_{n+1}^{m-2-l}-$$
$$(\phi_{n,m}\left(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)}\right)+\psi_{n,m}w_{n,m}^{(\alpha,l+2)})u_n^{m-2-l}+$$
$$\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})u_{n-1}^{m-2-l}].$$

Assume that $u_n^m=\xi_m e^{iqnh}$ (as assumed in the von Neumann stability procedure), then we get

$$-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}\xi_m e^{iq(n+1)h}+$$
$$(1+\phi_{n,m}w_{n,m}^{(\beta,0)}+\psi_{n,m}w_{n,m}^{(\alpha,0)})\xi_m e^{iqnh}-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}\xi_m e^{iq(n-1)h}-$$
$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})\xi_{m-1}e^{iq(n+1)h}-$$
$$(1-\phi_{n,m}w_{n,m}^{(\beta,1)}-\phi_{n,m}w_{n,m-1}^{(\beta,0)}-\psi_{n,m}w_{n,m}^{(\alpha,1)})\xi_{m-1}e^{iqnh}-$$
$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})\xi_{m-1}e^{iq(n-1)h}-$$
$$\sum_{l=0}^{m-2}[\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})\xi_{m-2-l}e^{iq(n+1)h}-$$
$$(\phi_{n,m}\left(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)}\right)+\psi_{n,m}w_{n,m}^{(\alpha,l+2)})\xi_{m-2-l}e^{iqnh}+$$
$$\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})\xi_{m-2-l}e^{iq(n-1)h}]=0$$

Divided by e^{iqnh}

$$-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}\xi_m e^{iqnh}+(1+\phi_{n,m}w_{n,m}^{(\beta,0)}+\psi_{n,m}w_{n,m}^{(\alpha,0)})\xi_m$$

$$-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}\xi_m e^{-iqh}-\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})\xi_{m-1}e^{iqh}-$$

$$(1-\phi_{n,m}w_{n,m}^{(\beta,1)}-\phi_{n,m}w_{n,m-1}^{(\beta,0)}-\psi_{n,m}w_{n,m}^{(\alpha,1)})\xi_{m-1}-$$

$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})\xi_{m-1}e^{-iqh}-$$

$$\sum_{l=0}^{m-2}[\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})\xi_{m-2-l}e^{iqh}-$$

$$(\phi_{n,m}\left(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)}\right)+\psi_{n,m}w_{n,m}^{(\alpha,l+2)})\xi_{m-2-l}+$$

$$\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})\xi_{m-2-l}e^{-iqh}]=0$$

Now, using $e^{i\theta}=\cos\theta+i\sin\theta$, we have

$$[-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}(\cos(qh)+i\sin(qh))+(1+\phi_{n,m}w_{n,m}^{(\beta,0)}+\psi_{n,m}w_{n,m}^{(\alpha,0)})$$

$$-\frac{\phi_{n,m}}{2}w_{n,m}^{(\beta,0)}(\cos(qh)-i\sin(qh))]\xi_m-$$

$$[\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(\cos(qh)+i\sin(qh))+$$

$$(1-\phi_{n,m}w_{n,m}^{(\beta,1)}-\phi_{n,m}w_{n,m-1}^{(\beta,0)}-\psi_{n,m}w_{n,m}^{(\alpha,1)})+$$

$$\frac{1}{2}(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(\cos(qh)-i\sin(qh))]\xi_{m-1}-$$

$$\sum_{l=0}^{m-2}[\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(\cos(qh)+i\sin(qh))-$$

$$(\phi_{n,m}\left(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)}\right)+\psi_{n,m}w_{n,m}^{(\alpha,l+2)})+$$

$$\frac{\phi_{n,m}}{2}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(\cos(qh)-i\sin(qh))]\xi_{m-2-l}=0.$$

Under some simplifications, we get

$$[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))]\xi_m-$$

$$[1-\psi_{n,m}w_{n,m}^{(\alpha,1)}-(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(2\sin^2(qh/2))]\xi_{m-1}$$

$$-\sum_{l=0}^{m-2}[\phi_{n,m}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})$$

$$(-2\sin^2(qh/2))+\psi_{n,m}w_{n,m}^{(\alpha,l+2)}]\xi_{m-2-l}=0.$$

Inserting the expression $\xi_{m-1}=\eta^{-1}\xi_m$, in the above equation

$$[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))]\xi_m$$

$$-[1-\psi_{n,m}w_{n,m}^{(\alpha,1)}-(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})$$

$$(2\sin^2(qh/2))]\eta^{-1}\xi_m-$$

$$\sum_{l=0}^{m-2}[\phi_{n,m}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(-2\sin^2(qh/2))+\psi_{n,m}w_{n,m}^{(\alpha,l+2)}]\eta^{-2-l}\xi_m=0,$$

divided by ξ_m to obtain the following formula of η:

$$\eta=\frac{-\left[1-\psi_{n,m}w_{n,m}^{(\alpha,1)}-(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(2\sin^2(qh/2))\right]}{\left[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))\right]}+$$
$$\frac{\sum_{l=0}^{m-2}\left[\phi_{n,m}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(-2\sin^2(qh/2))+\psi_{n,m}w_{n,m}^{(\alpha,l+2)}\right]\eta^{-1-l}}{\left[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))\right]}.$$

The mode will be stable as long as $|\eta|\leq 1$, so

$$-1\leq\frac{-\left[1-\psi_{n,m}w_{n,m}^{(\alpha,1)}-(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(2\sin^2(qh/2))\right]}{\left[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))\right]}+$$
$$\frac{\sum_{l=0}^{m-2}\left[\phi_{n,m}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(-2\sin^2(qh/2))+\psi_{n,m}w_{n,m}^{(\alpha,l+2)}\right]\eta^{-1-l}}{\left[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))\right]}\leq 1.$$

By considering, $\eta=-1$, and since

$$\left[1+\psi_{n,m}w_{n,m}^{(\alpha,0)}+\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))\right]>0,$$

then,

$$-(1-\psi_{n,m}w_{n,m}^{(\alpha,0)})-\phi_{n,m}w_{n,m}^{(\beta,0)}(2\sin^2(qh/2))+$$

$$(1-\psi_{n,m}w_{n,m}^{(\alpha,1)})-(\phi_{n,m}w_{n,m}^{(\beta,1)}+\phi_{n,m}w_{n,m-1}^{(\beta,0)})(2\sin^2(qh/2))-$$

$$\sum_{l=0}^{m-2}\phi_{n,m}(w_{n,m}^{(\beta,l+2)}+w_{n,m-1}^{(\beta,l+1)})(2\sin^2(qh/2))(-1)^{-l}+$$

$$\sum_{l=0}^{m-2}\psi_{n,m}w_{n,m}^{(\alpha,l+2)}(-1)^{-l}\leq 0,$$

$$-w_{n,m}^{(\beta,0)} - w_{n,m}^{(\beta,1)} - w_{n,m-1}^{(\beta,0)} - \sum_{l=0}^{m-2}(w_{n,m}^{(\beta,l+2)} + w_{n,m-1}^{(\beta,l+1)})(-1)^{-l} \leq \frac{w_{n,m}^{(\alpha,0)} + w_{n,m}^{(\alpha,1)} + \sum_{l=0}^{m-2} w_{n,m}^{(\alpha,l+2)}(-1)^{-l}}{\frac{2\phi_{n,m}}{\psi_{n,m}}\sin^2(qh/2)},$$

$$\text{let } S_{m,\alpha} = \sum_{l=0}^{m-2} w_{n,m}^{(\alpha,l+2)}(-1)^{-l},$$

$$\text{and } S_{m,\beta} = \sum_{l=0}^{m-2}(w_{n,m}^{(\beta,l+2)} + w_{n,m-1}^{(\beta,l+1)})(-1)^{-l},$$

$$-w_{n,m}^{(\beta,0)} - w_{n,m}^{(\beta,1)} - w_{n,m-1}^{(\beta,0)} - S_{m,\beta} \leq \frac{w_{n,m}^{(\alpha,0)} + w_{n,m}^{(\alpha,1)} + S_{m,\alpha}}{\frac{2\phi_{n,m}}{\psi_{n,m}}\sin^2(qh/2)}.$$

Since $w_{n,m}^{(\alpha,0)} = w_{n,m}^{(\beta,0)} = 1$, $w_{n,m}^{(\alpha,1)} + S_{m,\alpha} > -1$, $w_{n,m}^{(\beta,1)} + S_{m,\beta} > -1$and $\frac{2\phi_{n,m}}{\psi_{n,m}}\sin^2(qh/2)$ is always positive, we can conclude that the scheme is unconditional stable.

5 Numerical experiments

In this section, we present numerical example to illustrate the efficiency and the validation of the proposed numerical method when it applied to solve numerically the variable order fractional Cable equation.

Example 1*Consider the following initial-boundary problem of the variable-order fractional nonlinear Cable equation.*

$$u_t(x,t) = {}_0\mathscr{D}_t^{1-\beta(x,t)} u_{xx}(x,t) - {}_0\mathscr{D}_t^{1-\alpha(x,t)} u(x,t) + f(x,t),$$

on a finite domain $0 < x < 1$, *with* $0 \leq t \leq t_{max}$.
The source term is given by:

$$f(x,t) = 2\left(t + \frac{\pi^2 t^{\beta(x,t)+1}}{\Gamma(2+\beta(x,t))} + \frac{t^{\alpha(x,t)+1}}{\Gamma(2+\alpha(x,t))}\right)\sin(\pi x),$$

the initial and boundary conditions are:

$$u(x,0) = 0; \quad u(0,t) = u(1,t) = 0,$$

let

$$\psi(\tau) = \sinh(\tau) \text{ and } \phi(h) = e^h - 1.$$

and the exact solution is:

$$u(x,t) = t^2\sin(\pi x).$$

Table 1, shows the maximum error "`Error`" of proposed method between the exact solution and the numerical solution at $t_M = 1$, using different values of $\beta(x,t)$ and $\alpha(x,t)$.

In Tables 1 and 2 , a comparison between the NSCN and the standard Crank-Nicholson (SCN) solutions, where the accuracy of the NSCN is better than the SCN. Figs. 1 and 3 show the behavior of the exact solutions and the numerical solutions of the proposed method with different values of $\alpha(x,t), \beta(x,t)$. Moreover, in Figs. 2 and 4 show 3D-drawing of the absolute error "Error" of the numerical solutions with different values of $\alpha(x,t), \beta(x,t)$. From the results displayed in the Table 1 and in all the figures, it is obvious that the proposed method is an efficient and able to give numerical solutions coincide closely with the exact solutions.

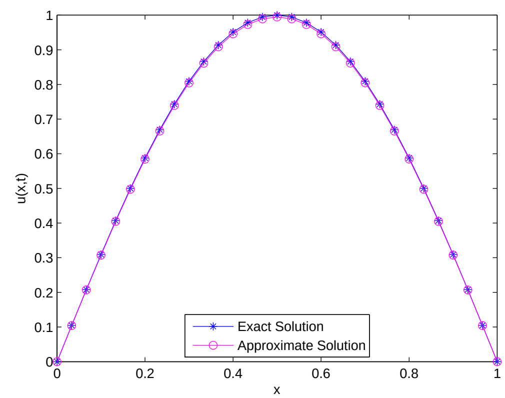

Fig. 1: The behavior of the numerical and exact solutions at $\beta(x,t) = 5^{xt-3}$, $\alpha(x,t) = \frac{11-\cos^2(xt)}{11}$.

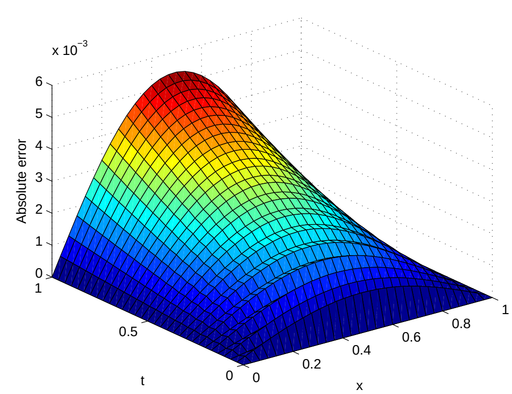

Fig. 2: The absolute error of the numerical solution at $\beta(x,t) = 5^{xt-3}$, $\alpha(x,t) = \frac{11-\cos^2(xt)}{11}$.

Table 1: The maximum error "Error" of the numerical solutions using different values of $\beta(x,t)$ and $\alpha(x,t)$.

		$N=50, M=10$	$N=100, M=15$	$N=100, M=20$	$N=100, M=30$
$\beta(x,t)$	$\alpha(x,t)$	Error	Error	Error	Error
$\frac{9-x^2+t^3}{120}$	$\frac{15+\sin^5(xt)}{16}$	1.04e-2	8.90e-3	3.95e-3	6.39e-4
$\frac{10+(xt)^4-(xt)^5}{130}$	$\frac{16+(xt)^5}{17}$	1.23e-2	1.02e-2	5.08e-3	1.97e-4
$\frac{8+x^2-t^4}{100}$	$\frac{12-(xt)^2}{12}$	1.51e-2	1.23e-2	6.72e-3	1.39e-4
$\frac{10+\cos^2(xt)}{120}$	$\frac{12-\sin^3(xt)}{12}$	1.40e-2	1.14e-2	5.98e-3	8.11e-5
$\frac{5+(xt)^3}{50}$	$\frac{9-\sin^4(xt)}{9}$	1.38e-2	1.12e-2	5.79e-3	6.50e-4
$\frac{9-(xt)^2+\sin^3(xt)}{60}$	$\frac{15-(xt)^4}{15}$	1.95e-2	1.50e-2	8.74e-3	2.70e-3
5^{xt-3}	$\frac{11-\cos^2(xt)}{11}$	4.64e-3	5.20e-3	1.08e-4	2.61e-5

Table 2: The maximum error "Error" of the SCN method using different values of $\beta(x,t)$ and $\alpha(x,t)$.

		$N=50, M=10$	$N=100, M=15$	$N=100, M=20$	$N=100, M=30$
$\beta(x,t)$	$\alpha(x,t)$	Error	Error	Error	Error
$\frac{9-x^2+t^3}{120}$	$\frac{15+\sin^5(xt)}{16}$	2.70e-2	1.73e-2	1.24e-2	7.97e-3
$\frac{10+(xt)^4-(xt)^5}{130}$	$\frac{16+(xt)^5}{17}$	2.89e-2	1.87e-2	1.36e-2	8.80e-3
$\frac{8+x^2-t^4}{100}$	$\frac{12-(xt)^2}{12}$	3.16e-2	2.07e-2	1.52e-2	9.99e-3
$\frac{10+\cos^2(xt)}{120}$	$\frac{12-\sin^3(xt)}{12}$	3.05e-2	1.98e-2	1.44e-2	9.41e-3
$\frac{5+(xt)^3}{50}$	$\frac{9-\sin^4(xt)}{9}$	3.04e-2	1.96e-2	1.42e-2	9.22e-3
$\frac{9-(xt)^2+\sin^3(xt)}{60}$	$\frac{15-(xt)^4}{15}$	3.58e-2	2.33e-2	1.71e-2	1.12e-2
5^{xt-3}	$\frac{11-\cos^2(xt)}{11}$	2.15e-2	1.37e-2	9.68e-3	6.09e-2

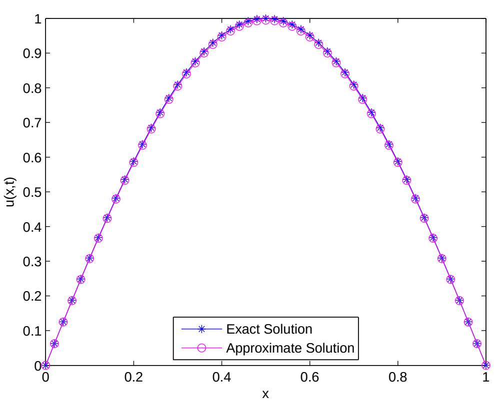

Fig. 3: The behavior of the numerical and exact solutions at $\beta(x,t) = \frac{10+\cos^2(xt)}{120}$, $\alpha(x,t) = \frac{12-\sin^3(xt)}{12}$.

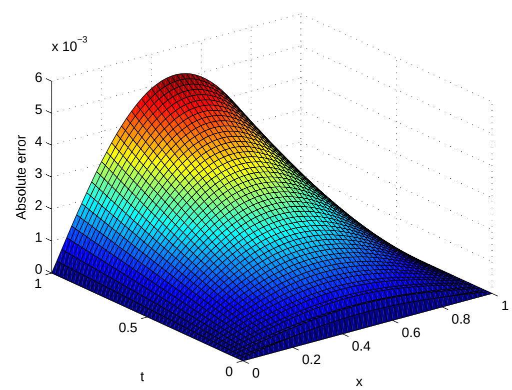

Fig. 4: The absolute error of the numerical solution at $\beta(x,t) = \frac{10+\cos^2(xt)}{120}$, $\alpha(x,t) = \frac{12-\sin^3(xt)}{12}$.

6 Conclusions

In this paper, NSCN method with shifted Grünwald estimate applied for solving variable order fractional Cable equation. Special attention is given to study the stability of the method. The obtained numerical results are presented and compared with the exact and the SCN solutions. The comparison between NSCN and SCN show that NSCN more accurate than SCN. From this comparison, we can conclude that the numerical solutions are in good agreement with the exact solutions. All computations in this paper are performed using Matlab programming.

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