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Odd Generalized Nadarajah-Hagaigai Log-Logistic Model with Modeling

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Abstract: In this article, we intend to introduce and study the odd generalized Nadarajah-Hagaigai log-logistic (OGNH-LL) model as a new extension of the log-logistic (LL) model. Some statistical features of the OGNH-LL model are computed. The four unknown parameters of the OGNH-LL model are estimated by employing the maximum likelihood (ML) technique. One real-world data set is utilized to highlight the importance and applicability of the OGNH-LL model.

Keywords: Statistical Model; Numerical Results; Log-Logistic model; Odd generalized Nadarajah-Hagaigai generated family; Median; Estimation.

1 Motivation and Introduction

The LL model has several applications in various fields, like actuarial science, economics, geophysics, survival analysis, engineering and hydrology [1]. The density function (pdf) and distribution function (cdf) of the LL model are supplied via

$$g(v;\tau,\delta)=\frac{\delta}{\tau^{\delta}}v^{\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-2},v>0,\tau,\delta>0, \quad (1)$$

and

$$G(v;\tau,\delta)=1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1},v>0,\tau,\delta>0, \quad (2)$$

where τ is the scale parameter and δ is the shape parameter.

In the subject field, with the purpose of increasing the flexibility of the LL model in modelling survival time data. Numerous authors proposed developed and generalized new extensions of the LL model such as a transmuted LL model [2], a transmuted generalized LL model [3,4], alpha power LL model [5,6], McDonald LL model [7], an extended LL model [8], a new extension of LL model[9], truncated Cauchy power LL model [10], statistical inference of entropy for LL model using censored samples[11] and Kavya-Manoharan LL model [12]. There are instances in which applying present models to a set of data results in an insufficient fit. Many academics attempt to generalize models in order to deal with this problem and obtain a satisfactory fit for a specific data set. Numerous other authors were inspired in future years to create families of models that included exponentiated power generalized Weibull power series- G [13], Topp-Leone odd Fréchet –G[14], truncated Muth -G [15], Sine Topp-Leone- G [16], Type II general inverse exponential -G [17], Perks- G [18], Kavya-Manoharan family [19], odd inverse power generalized Weibull-G [20], generalized DUS transformation-G in [21], new Muth -G [22], truncated Cauchy power Weibull-G [23], DUS transformation [24], generalized odd Burr III -G [25], odd Dagum [26] and odd Nadarajah-Haghighi-G in [27], for more information about generating a family of distributions see [28,29,30,31,32,33,34,35,36,37,37,39, 40,41,42,43,44].

Recently, [46] introduced the OGNH-generated family of distributions and it has the following cdf and pdf

$$F(v)=1-ee^{-\left(1+\left(\frac{G(v;\varpi)}{1-G(v;\varpi)}\right)^{\lambda}\right)^{\sigma}},v\in\mathbb{R},\lambda,\sigma>0, \quad (3)$$

and

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$$f(v) = \sigma\lambda e g(v;\varpi)\frac{G(v;\varpi)^{\lambda-1}}{(1-G(v;\varpi))^{\lambda+1}} \times \left(1+\left(\frac{G(v;\varpi)}{1-G(v;\varpi)}\right)^{\lambda}\right)^{\sigma-1} e^{-\left(1+\left(\frac{G(v;\varpi)}{1-G(v;\varpi)}\right)^{\lambda}\right)^{\sigma}} \quad (4)$$

Where $g(v;\varpi)$ and $G(v;\varpi)$ consider the pdf and the cdf of baseline model.

The main purpose of this paper is to present a novel extension of the LL model by combining the OGNH family of distributions with the LL model. The rest of this article is organized as follows: We discussed the development of the pdf and cdf, reliability function (rf), the hazard rate function (hrf), the reversed and cumulative hrfs for the OGNH-LL model in Section 2. Section 3 explains the OGNH-LL model's statistical properties. In Section 4, the ML estimation technique is used to estimate the parameters of the OGNH-LL model. Section 5 analyzes the adaptability of the OGNH-LL model using one real data set. Finally, final remarks are offered.

2 The OGNH-LL Model

The random variable V is believed to have OGNH-LL model signified by OGNH-LL $(\lambda,\sigma,\tau,\delta)$ if the cdf and pdf, rf, hrf, the reversed and cumulative hrfs of V are provided via

$$F(v;\lambda,\sigma,\tau,\delta) = 1-ee^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}, v>0, \lambda,\sigma,\tau,\delta>0, \quad (5)$$

$$f(v;\lambda,\sigma,\tau,\delta) = e\sigma\lambda\frac{\delta}{\tau^{\delta}}v^{\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-1} \times \left(1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1}\right)^{\lambda-1} \times \left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1} e^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}, \quad (6)$$

$$S(v;\lambda,\sigma,\tau,\delta) = ee^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}},$$

$$h(v;\lambda,\sigma,\tau,\delta) = \sigma\lambda\frac{\delta}{\tau^{\delta}}v^{\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-1} \times \left(1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1}\right)^{\lambda-1} \times \left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1},$$

$$\chi(v;\lambda,\sigma,\tau,\delta) = \sigma\lambda\frac{\delta}{\tau^{\delta}}v^{\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-1} \times \left(1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1}\right)^{\lambda-1} \times \left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1} \times \left(1-ee^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}\right)^{-1},$$

and

$$H(v;\lambda,\sigma,\tau,\delta) = \left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma} - 1.$$

Figures 1 and 2 displayed the plots of the pdf and hrf of the OGNH-LL model for numerous numerical values of parameters. The pdf of the OGNH-LL model is decreasing, unimodal, and right skewed. The hrf of the OGNH-LL model can be J-shaped, decreasing, increasing and upside-down shaped.

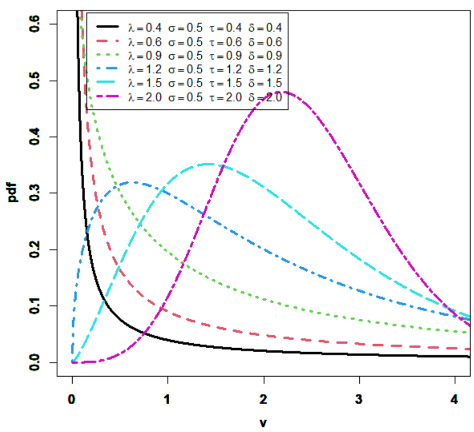

Fig. 1: Plots of the pdf of the OGNH-LL model

3 Statistical Features of The OGNH-LL Model

This section discusses several statistical features of the OGNH-LL model.

3.1 Quantile Function and Median

Assume that $U \sim (0, 1)$ is the uniform model then $V \sim$ OGNH-LL, the quantile function of the OGNH-LL model is provided via

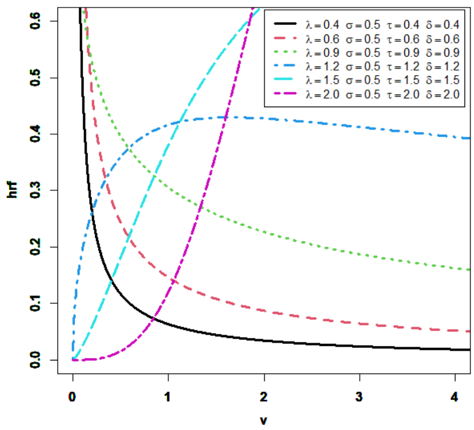

Fig. 2: Plots of the hrf of the OGNH-LL model

$$Q(u)=\tau\left(\left[1-\ln\left(1-u\right)\right]^{\frac{1}{\sigma}}-1\right)^{\frac{1}{\delta\lambda}},$$

and the median is provided via

$$m=\tau\left(\left[1-\ln\left(0.5\right)\right]^{\frac{1}{\sigma}}-1\right)^{\frac{1}{\delta\lambda}}.$$

Table 1 shows some numerical values of Q1, Q2, Q3, Bowley's skewness (BS), and Moor's kurtosis (MK).

From Figure 1, We can note that: when σ and τ increase then the value of BSK and $MKUR$ are decreased.

Table 1: Results of Some numerical values of Q1, Q2, Q3, BS and MK for the OGNH-LL model at $\delta=2$

λ	σ	τ	$Q1$	$Q2$	$Q3$	BSK	$MKUR$
0.8	0.7	2.1	1.248	2.256	3.69	0.174	1.261
	0.9	2.4	1.187	2.08	3.255	0.137	1.225
	1.1	2.7	1.159	1.991	3.034	0.113	1.206
	1.3	3	1.147	1.944	2.91	0.096	1.194
	1.5	3.3	1.144	1.92	2.838	0.083	1.185
	1.7	3.6	1.147	1.911	2.798	0.074	1.18
	1.9	3.9	1.153	1.911	2.776	0.066	1.175
	2.1	4.2	1.162	1.917	2.768	0.06	1.172
	2.3	4.5	1.172	1.926	2.768	0.055	1.169
	2.5	4.8	1.183	1.939	2.775	0.051	1.167
1.5	0.7	2.1	1.591	2.182	2.836	0.051	1.221
	0.9	2.4	1.649	2.223	2.824	0.022	1.21
	1.1	2.7	1.72	2.295	2.873	0.003	1.205
	1.3	3	1.796	2.38	2.952	-0.011	1.202
	1.5	3.3	1.876	2.472	3.045	-0.021	1.201
	1.7	3.6	1.956	2.568	3.147	-0.028	1.2
	1.9	3.9	2.036	2.666	3.253	-0.034	1.199
	2.1	4.2	2.116	2.764	3.362	-0.039	1.199
	2.3	4.5	2.196	2.862	3.473	-0.044	1.198
	2.5	4.8	2.275	2.96	3.583	-0.047	1.198
3	0.7	2.1	1.828	2.141	2.441	-0.021	1.231
	0.9	2.4	1.989	2.31	2.603	-0.045	1.231
	1.1	2.7	2.155	2.489	2.785	-0.061	1.232
	1.3	3	2.321	2.672	2.976	-0.072	1.233
	1.5	3.3	2.488	2.856	3.17	-0.081	1.234
	1.7	3.6	2.654	3.041	3.366	-0.087	1.235
	1.9	3.9	2.818	3.224	3.562	-0.092	1.236
	2.1	4.2	2.981	3.407	3.758	-0.096	1.237
	2.3	4.5	3.144	3.589	3.953	-0.1	1.238
	2.5	4.8	3.304	3.769	4.147	-0.103	1.238

3.2 Moments

Theorem 1: Assume that V be a random variable from the OGNH-LL model then its q^{th} moment is

$$\mu_q'=\sum_{i=0}^{\infty}\sum_{j=0}^{\lambda-1}\sum_{k=0}^{\sigma(i+1)-1}\Omega\, B\left(\frac{q}{\delta}+\lambda k,\, j-\lambda\left(k+1\right)-\frac{q}{\delta}\right). \tag{7}$$

Proof: Assume that V be a random variable having pdf (6). The q^{th} moments of the OGNH-LL model are computed as

$$\begin{aligned}\mu'_q=e\sigma\lambda\frac{\delta}{\tau^{\delta}}\int_0^{\infty}v^{q+\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-1}\\ \times\left(1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1}\right)^{\lambda-1}\\ \times\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1}e^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}dv.\end{aligned} \tag{8}$$

By employing the exponential series $e^{\vartheta v}=\sum_{i=0}^{\infty}\frac{(-1)^i\vartheta^i}{i!}v^i$ to the last term in equation (8), we may rewrite it as below

$$\begin{aligned}\mu'_q=e\sigma\lambda\frac{\delta}{\tau^{\delta}}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\int_0^{\infty}v^{q+\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-1}\\ \times\left(1-\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{-1}\right)^{\lambda-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma(i+1)-1}dv.\end{aligned} \tag{9}$$

By using the binomial theory $(1-v)^{\vartheta-1}=\sum_{j=0}^{\vartheta-1}(-1)^j\binom{\vartheta-1}{j}v^j$ to the last term in equation (9), we have

$$\begin{aligned}\mu'_q=e\sigma\lambda\frac{\delta}{\tau^{\delta}}\sum_{i=0}^{\infty}\sum_{j=0}^{\lambda-1}\frac{(-1)^{i+j}}{i!}\binom{\lambda-1}{j}\int_0^{\infty}v^{q+\delta-1}\\ \times\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-j-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma(i+1)-1}dv.\end{aligned} \tag{10}$$

By using the binomial theory $(1+v)^{\vartheta-1}=\sum_{k=0}^{\vartheta-1}\binom{\vartheta-1}{k}v^k$ to the last term in equation (10), we have

$$\mu'_q=e\sigma\lambda\frac{\delta}{\tau^\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\lambda-1}\frac{(-1)^{i+j}}{i!}\binom{\lambda-1}{j}\int_0^\infty v^{q+\delta-1}\times\left(1+\left(\frac{v}{\tau}\right)^{\delta}\right)^{\lambda-j-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma(i+1)-1}dv.$$

where $\Xi_{i,j,k}=e\sigma\lambda\delta\frac{(-1)^{i+j}}{\tau^{\delta(\lambda k+1)}i!}\binom{\lambda-1}{j}\binom{\sigma(i+1)-1}{k}$.

Suppose that $x=\left(\frac{v}{\tau}\right)^{\delta}$, then

$$\mu'_q=\sum_{i=0}^{\infty}\sum_{j=0}^{\lambda-1}\sum_{k=0}^{\sigma(i+1)-1}\Omega\int_0^\infty x^{\frac{q}{\delta}+\lambda k}(1+x)^{\lambda-j-1}dv,$$

where $\Omega=\Xi_{i,j,k}\frac{\tau^{1-\delta}}{\delta}$. Then,

$$\mu'_q=\sum_{i=0}^{\infty}\sum_{j=0}^{\lambda-1}\sum_{k=0}^{\sigma(i+1)-1}\Omega\,B\left(\frac{q}{\delta}+\lambda k,\,j-\lambda(k+1)-\frac{q}{\delta}\right).$$

The moment generating function of V is provided via

$$M_V(t)=\sum_{q=0}^{\infty}\frac{t^q}{q!}\mu'_q=\sum_{q,i=0}^{\infty}\sum_{j=0}^{\lambda-1}\sum_{k=0}^{\sigma(i+1)-1}\frac{\Omega t^q}{q!}B\left(\frac{q}{\delta}+\lambda k,\,j-\lambda(k+1)-\frac{q}{\delta}\right).$$

Table 2 shows the numerical values of the first four moments also the numerical values of variance (var), coefficient of skewness (CS), coefficient of kurtosis (CK) and CV for the OGNH-LL model.

From Figure 2, We can note that: when σ and τ increase then the value of var, CS, CK and CV are decreased.

3.3 Order Statistics

Let $v_1,v_2,\ldots,v_n$ be random sample from the OGNH-LL model having order statistics $v_{(1)},v_{(2)},\ldots,v_{(n)}$. The pdf of $v_{(k)}$ of order statistics is provided as below

$$f_{V_{(k)}}(v)=\frac{n!}{(k-1)!(n-k)!}f(v;\lambda,\sigma,\tau,\delta)\times(F(v;\lambda,\sigma,\tau,\delta))^{k-1}(1-F(v;\lambda,\sigma,\tau,\delta))^{n-k},$$

The pdf of $v_{(k)}$ can be calculated as

$$f_{V_{(k)}}(v)=\frac{n!e^{n-k+1}\sigma\lambda\delta\Lambda}{\tau^\delta(k-1)!(n-k)!}v^{\delta-1}\,\Pi^{\lambda-1}\left(1-\Pi^{-1}\right)^{\lambda-1}\times\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1}(1-e\Lambda)^{k-1}\Lambda^{n-k},$$

where $\Pi=1+\left(\frac{v}{\tau}\right)^{\delta}$ and $\Lambda=e^{-\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma}}$ The pdf of the lowest and biggest order statistics, in particular, may be computed as

$$f_{V_{(1)}}(v)=\frac{ne^n\sigma\lambda\delta\Lambda^n}{\tau^\delta\Pi^{1-\lambda}\left(1-\Pi^{-1}\right)^{1-\lambda}}v^{\delta-1}\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1},$$

and

$$f_{V_{(n)}}(v)=\frac{ne\sigma\lambda\delta\Lambda}{\tau^\delta}v^{\delta-1}\,\Pi^{\lambda-1}\left(1-\Pi^{-1}\right)^{\lambda-1}\times\left(1+\left(\frac{v}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1}(1-e\Lambda)^{n-1},$$

respectively.

4 Method of Maximum Likelihood Estimation

The ML estimators for the four unknown parameters in the OGNH-LL model are obtained employing complete samples. Let $v_1,v_2,\ldots,v_n$ be observed values from the OGNH-LL model with vector of parameters $\Theta=(v;\lambda,\sigma,\tau,\delta)^T$ The total log-likelihood function for the vector of parameters Θ may be phrased as

$$\begin{aligned}\ln L(\Theta)&=n+n\ln\sigma+n\ln\lambda+n\ln\delta-n\delta\ln\tau\\&+(\delta-1)\sum_{i=1}^{n}\ln v_i+(\lambda-1)\sum_{i=1}^{n}\ln\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)\\&+(\lambda-1)\sum_{i=1}^{n}\ln\left(1-\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-1}\right)\\&+(\sigma-1)\sum_{i=1}^{n}\ln\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)-\sum_{i=1}^{n}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)^{\sigma}.\end{aligned}$$

The elements of the score function $U(\Theta)=(U_\lambda,U_\sigma,U_\tau,U_\delta)$ are provided via

$$\begin{aligned}\frac{\ln L(\Theta)}{\partial\lambda}&=\frac{n}{\lambda}+\sum_{i=1}^{n}\ln\left(1-\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-1}\right)\\&+\sum_{i=1}^{n}\ln\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)+\delta(\sigma-1)\sum_{i=1}^{n}\frac{\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\ln\left(\frac{v_i}{\tau}\right)}{1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}}\\&-\sigma\delta\sum_{i=1}^{n}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1}\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\ln\left(\frac{v_i}{\tau}\right),\end{aligned}$$

Table 2: Results of some moments measures for the OGNH-LL model at $\delta = 2$

λ	σ	τ	μ_1'	μ_2'	μ_3'	μ_4'	var	CS	CK	CV
0.8	0.7	2.1	0.277	0.23	0.321	0.629	0.153	2.868	15.359	1.411
	0.9	2.4	0.266	0.153	0.129	0.139	0.082	1.877	7.701	1.077
	1.1	2.7	0.275	0.135	0.089	0.072	0.059	1.34	5.027	0.885
	1.3	3	0.292	0.134	0.078	0.054	0.049	0.991	3.804	0.757
	1.5	3.3	0.311	0.139	0.076	0.048	0.043	0.739	3.164	0.665
	1.7	3.6	0.331	0.148	0.079	0.047	0.039	0.545	2.805	0.595
	1.9	3.9	0.351	0.159	0.084	0.049	0.036	0.389	2.599	0.54
	2.1	4.2	0.371	0.171	0.09	0.052	0.034	0.259	2.485	0.495
	2.3	4.5	0.39	0.184	0.097	0.056	0.032	0.149	2.429	0.457
	2.5	4.8	0.408	0.197	0.105	0.061	0.03	0.054	2.41	0.426
1.5	0.7	2.1	0.443	0.318	0.296	0.33	0.121	1.134	4.329	0.787
	0.9	2.4	0.464	0.299	0.233	0.206	0.084	0.675	3.086	0.623
	1.1	2.7	0.492	0.307	0.222	0.177	0.065	0.378	2.635	0.52
	1.3	3	0.52	0.325	0.228	0.173	0.054	0.164	2.477	0.448
	1.5	3.3	0.547	0.346	0.24	0.179	0.047	-0.0011	2.452	0.395
	1.7	3.6	0.573	0.369	0.257	0.19	0.041	-0.134	2.494	0.353
	1.9	3.9	0.596	0.392	0.276	0.204	0.036	-0.244	2.571	0.32
	2.1	4.2	0.618	0.415	0.295	0.219	0.033	-0.338	2.667	0.293
	2.3	4.5	0.639	0.437	0.315	0.236	0.03	-0.418	2.773	0.27
	2.5	4.8	0.657	0.459	0.335	0.253	0.027	-0.489	2.883	0.25
3	0.7	2.1	0.668	0.531	0.473	0.457	0.085	0.184	2.54	0.436
	0.9	2.4	0.705	0.557	0.476	0.431	0.06	-0.1	2.531	0.347
	1.1	2.7	0.738	0.59	0.5	0.442	0.046	-0.299	2.676	0.289
	1.3	3	0.767	0.624	0.531	0.468	0.036	-0.448	2.869	0.248
	1.5	3.3	0.792	0.657	0.564	0.498	0.03	-0.565	3.073	0.218
	1.7	3.6	0.814	0.687	0.597	0.531	0.025	-0.66	3.273	0.194
	1.9	3.9	0.833	0.716	0.629	0.565	0.021	-0.739	3.463	0.175
	2.1	4.2	0.851	0.742	0.66	0.598	0.018	-0.806	3.642	0.159
	2.3	4.5	0.866	0.766	0.69	0.63	0.016	-0.863	3.809	0.146
	2.5	4.8	0.88	0.789	0.718	0.661	0.014	-0.913	3.963	0.135

$$\frac{\ln L(\Theta)}{\partial\sigma} = \frac{n}{\sigma} + \sum_{i=1}^{n} \ln\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right) - \sum_{i=1}^{n}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)^{\sigma} \ln\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right),$$

$$\frac{\ln L(\Theta)}{\partial\tau} = -\frac{n\delta}{\tau} - \delta(\lambda-1)\sum_{i=1}^{n}\frac{{v_i}^{\delta}\tau^{-\delta-1}}{1+\left(\frac{v_i}{\tau}\right)^{\delta}} - \delta(\lambda-1)\sum_{i=1}^{n}\frac{{v_i}^{\delta}\tau^{-\delta-1}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-2}}{1-\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-1}} - \lambda\delta(\sigma-1)\sum_{i=1}^{n}\frac{{v_i}^{\lambda\delta}\tau^{-\lambda\delta-1}}{1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}} + \sigma\lambda\delta\sum_{i=1}^{n}{v_i}^{\lambda\delta}\tau^{-\lambda\delta-1}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1},$$

and

$$\frac{\ln L(\Theta)}{\partial\delta} = \frac{n}{\delta} - n\ln\tau + \sum_{i=1}^{n}\ln v_i + (\lambda-1)\sum_{i=1}^{n}\frac{\left(\frac{v_i}{\tau}\right)^{\delta}\ln\left(\frac{v_i}{\tau}\right)}{1+\left(\frac{v_i}{\tau}\right)^{\delta}} + (\lambda-1)\sum_{i=1}^{n}\frac{\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-2}\left(\frac{v_i}{\tau}\right)^{\delta}\ln\left(\frac{v_i}{\tau}\right)}{1-\left(1+\left(\frac{v_i}{\tau}\right)^{\delta}\right)^{-1}} + \lambda(\sigma-1)\sum_{i=1}^{n}\frac{\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\ln\left(\frac{v_i}{\tau}\right)}{1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}} - \sigma\lambda\sum_{i=1}^{n}\left(1+\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\right)^{\sigma-1}\left(\frac{v_i}{\tau}\right)^{\delta\lambda}\ln\left(\frac{v_i}{\tau}\right).$$

The system of non-linear equations $U(\Theta) = 0$ are solved numerically, we investigate the ML estimators of Θ .

5 Modelling to Real Dataset

In this section, we have provided application using real data set to illustrate the importance and potentiality of the

OGNH-LL distribution. The goodness-of-fit statistics for these distributions and other competitive distributions are compared and the MLEs of their parameters are provided. The data set representing the fracture toughness of Alumina (Al2O3) [46], as follows: 5.2 , 5.2 , 5.15 , 5.15 , 5.1 , 5.1 , 4 , 4 , 3.95 , 3.95 , 5.01 , 5 , 5 , 5 , 5 , 5 , 4.9 , 4.9 , 4.9 , 4.9 , 4.9 , 4.85 , 4.75 , 6.81 , 6.4 , 6.25 , 6.05 , 5.9 , 5.9 , 5.85 , 5.8 , 5.75 , 5.73 , 5.5 , 5.5 , 5.45 , 5.4 , 5.35 , 5.3 , 5.3 , 5.3 , 5.3 , 5.25 , 5.25 , 5.2 , 4.7 , 4.7 , 4.7 , 4.65 , 4.6 , 4.6 , 4.6 , 4.56 , 4.55 , 4.5 , 4.5 , 4.5 , 4.5 , 4.5 , 4.5 , 4.5 , 4.5 , 4.4 , 4.38 , 4.35 , 4.3 , 4.3 , 4.3 , 4.3 , 4.3 , 4.3 , 4.25 , 4.25 , 4.2 , 4.2 , 4.2 , 4.15 , 4.13 , 4.1 , 4.1 , 4.1 , 3.3 , 3.28 , 3.2 , 3.2 , 4.1 , 4.1 , 4.05 , 4 , 4 , 4 , 3.9 , 3.9 , 3.8 , 3.8 , 3.75 , 3.73 , 3.71 , 3.7 , 3.6 , 3.51 , 3.5 , 3.4 , 3.13 , 3.12 , 3 , 2.77 , 2.7 , 2.68 , 2.66 , 2.61 , 2.5 , 2.36 , 2.13 , 2.1 , 2.08 , 2.04 , 1.91 , 1.68. These real data sets are utilized to assess the goodness of fit of the OGNH-LL distribution. The suggested model is compared with exponentiated generalized linear exponential (EGL-Ex) [47], alpha power transformed Weibull (APT-W) [48], alpha power exponential (AP-Ex) [49] and exponential Lomax (Ex-Lo) [50] distributions. We obtain the MLEs and standard errors (SEs) of the model parameters. To compare the distribution models, we consider criteria like the Akaike information criterion (AK-IC), the correct AK-IC (C-AK-IC), Bayesian-IC (B-IC), Hannan-Quinn-IC (H-Q-IC), Kolmogorov–Smirnov (Ko-Sm) test, Cramer-von Mises (C-VM) and Anderson-Darling (A-D) test. The wider distribution, on the other hand, refers to lower AK-IC, C-AK-IC, B-IC, H-Q-IC, Ko-Sm, C-VM and A-D. The MLEs of the competitive models and their SEs and values of AK-IC, C-AK-IC, B-IC, H-Q-IC, C-VM, A-D and Ko-Sm for the proposed data set is presented in Tables 3- 4. We find that the OGNH-LL distribution with four parameters provides a better fit than other models. It has the smallest values of AK-IC, C-AK-IC, B-IC, H-Q-IC, C-VM, A-D and Ko-Sm among those distributions considered here. Moreover, the plots of empirical cdf, empirical pdf, and PP plots of our competitive model for proposed data set is displayed in Figures 3- 5, respectively.

6 Concluding Remarks

In this article, we offer the odd generalized Nadarajah-Hagaigai log-logistic (OGNH-LL) model. The OGNH-LL model is very flexible because its pdf of the OGNH-LL model is decreasing, unimodal, and right-skewed. Moreover, the hrf of the OGNH-LL model can be J-shaped, decreasing, increasing and upside-down shaped. Various statistical aspects of the OGNH-LL model were determined, including quantile function, median, moments, mean, variance, skewness, kurtosis, generating function, and order statistics. The four unknown parameters of the OGNH-LL model were estimated employing ML methodology of estimation. By

Table 3: Results of MLEs, and SEs for the data set

Dist.	*MLE* and *SE*			
	λ	σ	τ	δ
OGNH-LL	10.53306	1.206615	4.987816	0.4474
	(67.3666)	(0.5753)	(0.7364)	(2.8614)
EGL-Ex	0.0064	3.0452	2.3923	1.0535
	(0.1245)	(0.5231)	(0.5983)	(0.4834)
APT-W	2.329	0.0329	3.9312	
	(0.7308)	(0.0789)	(0.1318)	
AP-Ex	21.442	0.4295		
	(6.4880)	(3.0274)		
Ex-Lo	14.7402	9.852	0.0039	
	(3.1329)	(2.2892)	(0.0059)	

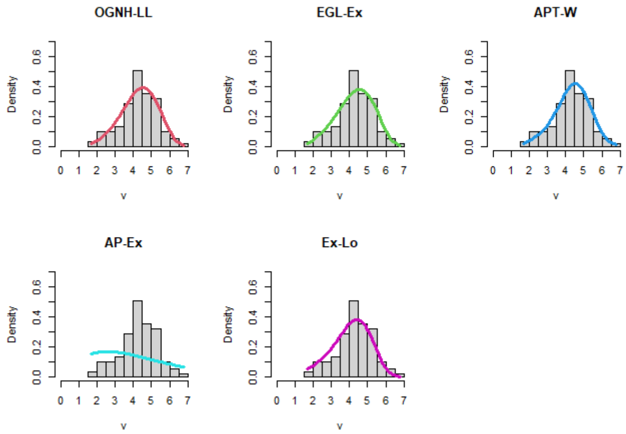

Fig. 3: Estimated pdf plots of competitive model for the data set

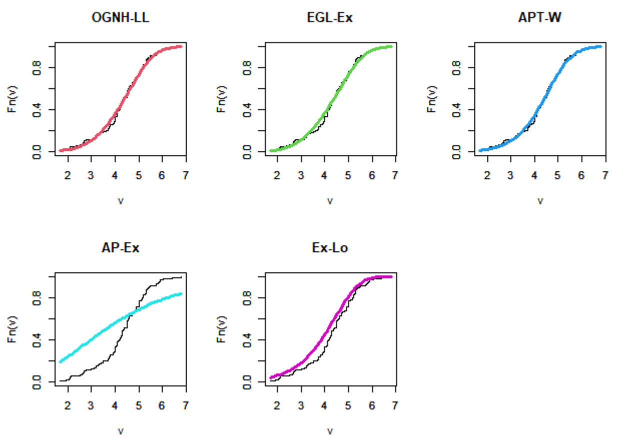

Fig. 4: Estimated cdf plots of competitive model for the data set

Table 4: Results of AK-IC, C-AK-IC, B-IC, H-Q-IC, C-VM, A-D and Ko-Sm for the data set

Dist.	LL	AK-IC	C-AK-IC	B-IC	H-Q-IC	KO-SM	A-D	C-VM
OGNH-LL	168.618	345.237	345.588	354.353	349.751	0.071	0.525	0.083
EGL-Ex	168.840	345.680	346.030	354.790	350.190	0.087	0.565	0.084
APT-W	170.600	347.210	347.420	355.550	350.600	0.144	0.573	0.087
AP-Ex	245.590	495.180	495.280	500.740	497.440	0.310	2.041	0.334
Ex-Lo	170.820	347.650	347.860	355.990	351.030	0.093	0.591	0.149

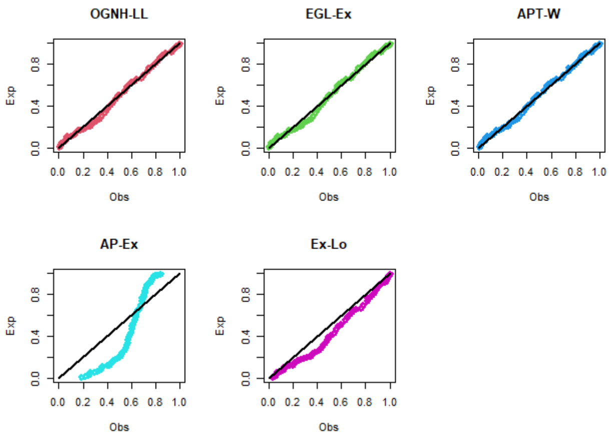

Fig. 5: The PP plots of the fitted model for the data set

using one real-world dataset, we proved the OGNH-LL model's flexibility. For future works, many authors can use the OGNH-LL model to estimate its parameters under different censored schemes and under different ranked set sampling.

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