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Characterizations of Topp–Leone Lomax Distribution based on the Generalized Lower Record Values

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Abstract: In this paper, we derive the exact expressions as well as recurrence relations for single and product moment of generalized lower record values from Topp–Leone Lomax distribution. Further, we characterize the given distribution through conditional expectation, recurrence relations and truncated moment.

Keywords: Generalized lower record values, Topp–Leone Lomax distribution, moment function, recurrence relations and characterization

1 Introduction

A random variale X is said to have a Topp–Leone Lomax (TLLo) distribution [1] if its probability density function (pdf) is of the form

$$f(x) = 2\,\alpha\,\beta\,\lambda\,(1+\lambda\,x)^{-(2\beta+1)}\,[1-(1+\lambda\,x)^{-2\beta}]^{\alpha-1} \quad x>0,\, \alpha>0,\, \beta>0,\, \lambda>0, \tag{1}$$

and the corresponding distribution function (df) is

$$F(x) = [1-(1+\lambda\,x)^{-2\beta}]^{\alpha} \quad x>0,\, \alpha>0,\, \beta>0,\, \lambda>0. \tag{2}$$

The relation between pdf and df can be seen as

$$f(x) = \frac{2\,\alpha\,\beta\,\lambda\,(1+\lambda\,x)^{-(2\beta+1)}}{[1-(1+\lambda\,x)^{-2\beta}]} F(x) \tag{3}$$

simplyfying (3), as

$$\left[2\,\beta\,\lambda\,x + \sum_{a=2}^{2\beta+1} (\lambda\,x)^a \binom{2\beta+1}{a}\right] f(x) = 2\,\alpha\,\beta\,\lambda\,F(x) \tag{4}$$

The shape of the TLLo distribution could either be unimodal or decreasing. The behavior of the failure rate can be used to model real life phenomena with inverted bathtub and decreasing failure rates. This model is more flexible over the Topp–Leone Burr XII, Topp–Leone Flexible Weibull and Lomax distributions as it performs better fit to the the dataset on airbone communication transceivers, for detailed application, one may see [1].

The Lomax ($\alpha = 1$ and $\beta = \beta/2$), exponentiated Pareto ($\lambda = 1$ and $\beta = 1/2$) and standard Pareto ($\alpha = 1, \beta = 1/2$ and $\lambda = 1$) distributions are the special cases of the Topp–Leone Lomax distribution at the different values of the parameyers.

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Let $\{X_n, n \geq 1\}$ be a sequence of independent and identical distributed (iid) continuous random variables with df $F(x)$ and pdf $f(x)$. The j-th order statistic of a sample $X_1, X_2, \ldots, X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k, we define the sequence $\{L_k(n), n \geq 1\}$ of k-th lower record times of $\{X_n, n \geq 1\}$ as follows[2]:

$$L_k(1) = 1$$

$$L_k(n+1) = min\{j > L_k(n) : X_{k:L_k(n)+k-1} > X_{k:j+k-1}\}.$$

The sequence $\{Z_n^{(k)}, n \geq 1\}$ with $Z_n^{(k)} = X_{k:L_k(n)+k-1}$, $n = 1, 2, \ldots,$ is called the sequence of k-th lower record values of $\{X_n, n \geq 1\}$. For convenience, we shall also take $Z_0^{(k)} = 0$. Note that for $k = 1$ we have $Z_n{}^{(1)} = X_{L(n)}$, $n \geq 1$, i.e. the record values of $\{X_n, n \geq 1\}$. Then the pdf of $Z_n^{(k)}$ and the joint pdf of $Z_m^{(k)}$ and $Z_n^{(k)}$ are as follows:

$$f_{Z_n^{(k)}}(x) = \frac{k^n}{(n-1)!}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x), \quad n \geq 1, \tag{5}$$

$$f_{Z_m^{(k)}, Z_n^{(k)}}(x, y) = \frac{k^n}{(m-1)!\,(n-m-1)!}[-\ln F(x)]^{m-1} \frac{f(x)}{F(x)}$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-1}[F(y)]^{k-1} f(y), \quad y < x,\, 1 \leq m < n,\, n \geq 2 \tag{6}$$

respectively [3].

The conditional pdf of $Z_n^{(k)}$ given $Z_m^{(k)} = x$, is

$$f_{Z_n^{(k)}|Z_m^{(k)}}(y|x) = \frac{k^{n-m}}{(n-m-1)!}[\ln F(x) - \ln F(y)]^{n-m-1}\left[\frac{F(y)}{F(x)}\right]^{k-1} \frac{f(y)}{F(x)}, \, y < x. \tag{7}$$

For some recent developments on the k-th lower record values with special reference to those arising from generalized extreme value, Gumble, inverse Pareto, inverse generalized Pareto, inverse Burr, inverse Weibull, generalized inverse weibull, power, uniform, Frechet, Dagum and extended Erlang-truncated exponential distribution, see [4,5,6,7,8]. In this paper we mainly focus on the study of generalized lower record values arising from the TLLo distribution.

2 Relations for Single Moments

To obtain the main result, we require the following lemmas.

Lemma 2.1. For the TLLo distribution as given in (1). If a is non negative finite integer and $r < 2\beta$, then we have

$$\Phi_j(a) = \frac{\alpha}{\lambda^j} \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} B\left(\alpha(a+1), 1 - \frac{r}{2\beta}\right), \tag{8}$$

consequently,

$$\Phi_0(a) = \frac{1}{a+1}, \tag{9}$$

where

$$\Phi_j(a) = \int_0^\infty x^j \, [F(x)]^a \, f(x) \, dx \tag{10}$$

and $B(a, b)$ is the complete beta function.

Proof. From (10), we have

$$\Phi_j(a) = \int_0^\infty x^j \, [F(x)]^a \, f(x) \, dx, \tag{11}$$

using (3) and (2) in (11), we get

$$\Phi_j(a) = 2\alpha\beta\lambda \int_0^\infty x^j \, (1+\lambda x)^{-(2\beta+1)} \, [1 - (1+\lambda x)^{-2\beta}]^{\alpha(a+1)-1} dx. \tag{12}$$

Making the substitution $t = [1-(1+\lambda x)^{-2\beta}]$ in (12), we find that

$$\Phi_j(a) = \frac{\alpha}{\lambda^j} \int_0^1 [(1-t)^{\frac{-1}{2\beta}} - 1]^j \, t^{\alpha(a+1)-1} dt. \tag{13}$$

Using binomial expansion in (13), we get

$$\Phi_j(a) = \frac{\alpha}{\lambda^j} \sum_{r=0}^{j} (-1)^{j-r} \binom{j}{r} \int_0^1 (1-t)^{\frac{-r}{2\beta}} \, t^{\alpha(a+1)-1} dt. \tag{14}$$

Let $r < 2\beta$ and using the beta function in (14), hence the result given in (8).

To prove (9), put $j = 0$ in (8).

Theorem 2.1. For the TLLo distribution (1), $n > 1$, is

$$E(Z_n^{(k)})^j = \frac{k^n}{(n-1)\,!} \sum_{p=0}^{\infty} \sum_{q=0}^{p+n-1} (-1)^q \, a_p(n-1) \binom{p+n-1}{q} \Phi_j(q+k-1). \tag{15}$$

Proof. We have

$$E(Z_n^{(k)})^j = \frac{k^n}{(n-1)!} \int_0^\infty x^j \, [-\ln F(x)]^{n-1} [F(x)]^{k-1} f(x) dx. \tag{16}$$

By [9], note that

$$[-\ln(1-t)]^j = \left(\sum_{p=0}^{\infty} \frac{t^p}{p\,!} \right)^j = \sum_{p=0}^{\infty} a_p(j) \, t^{j+p}, \qquad |t| < 1 \tag{17}$$

where $a_p(j)$ is the coefficient of t^{j+p} in the expression.

$$E(Z_n^{(k)})^j = \frac{k^n}{(n-1)!} \sum_{p=0}^{\infty} a_p(n-1) \int_0^\infty x^j \, [1-F(x)]^{p+n-1} [F(x)]^{k-1} f(x) dx. \tag{18}$$

using the binomial expression on $[1-F(x)]^{p+n-1}$, we get

$$E(Z_n^{(k)})^j = \frac{k^n}{(n-1)!} \sum_{p=0}^{\infty} a_p(n-1) \sum_{q=0}^{p+n-1} (-1)^q \binom{p+n-1}{q} \times \int_0^\infty x^j \, [F(x)]^{q+k-1} f(x) dx. \tag{19}$$

using lemma 2.1 in (19) and the resulting expression, which gives the yield given in (15).

Identity 2.1. For $1 \le n \le p$

$$\sum_{p=0}^{\infty} \sum_{q=0}^{p+n-1} (-1)^q \binom{p+n-1}{q} \frac{a_p(n-1)}{(q+k)} = \frac{(n-1)\,!}{k^n} \tag{20}$$

Proof. (20) can be proved by setting $j = 0$ in (15).

Remark 2.1. Setting $k = 1$ in (15), we get the single moments of upper records from the TLLo distribution as

$$E(X_{L_n})^j = \frac{1}{(n-1)\,!} \sum_{p=0}^{\infty} \sum_{q=0}^{p+n-1} (-1)^q \, a_p(n-1) \binom{p+n-1}{q} \Phi_j(q).$$

The following theorem gives the recurrence relations for single moments of generalized record values from df (2).

Theorem 2.2. For the distribution given in (1), fix a positive integer $k \geq 1$, for $n \geq 1$, $n \geq k$ and $j = 0, 1, \ldots$

$$\left(1+\frac{j}{\alpha k}\right)E\left(Z_n^{(k)}\right)^j = E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{2\,\alpha\,\beta\,\lambda\,k}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E\left(Z_n^{(k)}\right)^{j+a-1} \tag{21}$$

Proof. From (5) for $n \geq 1$ and $j = 0, 1, \ldots$, we have

$$E\left(Z_n^{(k)}\right)^j = \frac{k^n}{(n-1)!}\int_0^\infty x^j\,[-\ln F(x)]^{n-1}[F(x)]^{k-1}f(x)dx. \tag{22}$$

Integrating (19) with respect to x, we get

$$E\left(Z_n^{(k)}\right)^j = \frac{k^{n-1}}{(n-2)!}\int_0^\infty x^j\,[-\ln F(x)]^{n-2}[F(x)]^{k-1}f(x)dx$$

$$-\frac{j\,k^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}\,[-\ln F(x)]^{n-1}[F(x)]^{k}dx. \tag{23}$$

From (4), in the second term of (23) and simplifying

$$E\left(Z_n^{(k)}\right)^j = \frac{k^{n-1}}{(n-2)!}\int_0^\infty x^j\,[-\ln F(x)]^{n-2}[F(x)]^{k-1}f(x)dx - \frac{j\,k^{n-1}}{(n-1)!}$$

$$\times\int_0^\infty x^{j-1}\left\{\frac{x}{\alpha}+\frac{1}{2\alpha\beta\lambda}\sum_{a=2}^{2\beta+1}(\lambda x)^a\binom{2\beta+1}{a}\right\}[-\ln F(x)]^{n-1}[F(x)]^{k-1}f(x)dx$$

$$= E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{\alpha k}E\left(Z_n^{(k)}\right)^j - \frac{j}{2\alpha\,\beta\,\lambda\,k}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E\left(Z_n^{(k)}\right)^{j+a-1} \tag{24}$$

arranging the (24), which gives the result (21).

Remark 2.1.

i) At $k = 1$ in (21), the recurrence relation for the single moments of lower records from the TLLo distribution as,

$$\left(1+\frac{j}{\alpha}\right)E\left(X_n\right)^j = E\left(X_{n-1}\right)^j - \frac{j}{2\,\alpha\,\beta\,\lambda}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E\left(X_n\right)^{j+a-1}$$

ii) Setting $\alpha = 1$ and $\beta = \frac{\beta}{2}$ in (21) we get the recurrence relations of generalized lower records from Lomax distribution as

$$\left(1+\frac{j}{k}\right)E\left(Z_n^{(k)}\right)^j = E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{\beta\,\lambda\,k}\sum_{a=2}^{\beta+1}\lambda^a\binom{\beta+1}{a}E\left(Z_n^{(k)}\right)^{j+a-1}$$

iii) Setting $\beta = \frac{1}{2}$ and $\lambda = 1$ in (21) we get the recurrence relations of generalized lower records from Exponentiated Pareto distribution as

$$\left(1+\frac{j}{\alpha k}\right)E\left(Z_n^{(k)}\right)^j = E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{\alpha k}E\left(Z_n^{(k)}\right)^{j+1}$$

iv) Setting $\alpha = 1$, $\beta = \frac{1}{2}$ and $\lambda = 1$ in (21) we get the recurrence relations of generalized lower records from Pareto distribution as

$$\left(1+\frac{j}{k}\right)E\left(Z_n^{(k)}\right)^j = E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{k}E\left(Z_n^{(k)}\right)^{j+1}$$

3 Relations for Product Moments

In this section, we derived the exact moment and recurrence relations for product moments of generalized lower record values. To obtain the main result, we require the following lemmas.

Lemma 3.1. For the TLLo distribution as given in (1). If a and b are non negative finite integers, then we have

$$\Phi_{i,j}(a,b) = \frac{\alpha}{\lambda^{i+j}(b+1)} \sum_{p=0}^{j} \sum_{q=0}^{i} (-1)^{i+j-p-q} \binom{j}{p} \binom{i}{q} B\left(\alpha(a+b+2), 1-\frac{q}{2\beta}\right)$$

$$\times {}_3F_2\left[\alpha(b+1), \frac{p}{2\beta}, \alpha(a+b+2); \alpha(b+1)+1, \alpha(a+b+2)+1-\frac{q}{2\beta}; 1\right], \tag{25}$$

where

$$\Phi_{i,j}(a,b) = \int_0^\infty \int_0^x x^i y^j [F(x)]^a [F(y)]^b f(x) f(y) dy dx \tag{26}$$

and

$${}_pF_q[a_1,\ldots,a_p; b_1,\ldots,b_q; x] = \sum_{r=0}^{\infty} \left[\prod_{j=1}^{p} \frac{\Gamma(a_j+r)}{\Gamma(a_j)}\right] \left[\prod_{j=1}^{q} \frac{\Gamma(b_j)}{\Gamma(b_j+r)}\right] \frac{x^r}{r!},$$

for $p = q+1$ and $\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j > 0$. [See [10]].

Proof. From (26), we have

$$\Phi_{i,j}(a,b) = \int_0^\infty x^i [F(x)]^a f(x) I(x) dx \tag{27}$$

where

$$I(x) = \int_0^x y^j [F(y)]^b f(y) dy. \tag{28}$$

In view of lemma 2.1, we get

$$I(x) = \frac{\alpha}{\lambda^i} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} B_{[1-(1+\lambda x)^{-2\beta}]}\left(\alpha(b+1), 1-\frac{p}{2\beta}\right). \tag{29}$$

Thus

$$\Phi_{i,j}(a,b) = \frac{\alpha}{\lambda^i} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} \int_0^\infty x^i [F(x)]^a f(x) B_{[1-(1+\lambda x)^{-2\beta}]}\left(\alpha(b+1), 1-\frac{p}{2\beta}\right) dx$$

$$= \frac{\alpha}{\lambda^i} 2\alpha\beta\lambda \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} \int_0^\infty x^i (1+\lambda x)^{-(2\beta+1)} [1-(1+\lambda x)^{-2\beta}]^{\alpha(a+1)-1}$$

$$\times B_{[1-(1+\lambda x)^{-2\beta}]}\left(\alpha(b+1), 1-\frac{p}{2\beta}\right) dx \tag{30}$$

where

$$B_x(p,q) = \int_0^x u^{p-1} (1-u)^{q-1} du. \tag{31}$$

We know that (Mathai and Saxena, 1973),

$$B_x(p,q) = p^{-1} x^p \, {}_2F_1(p, 1-q; p+1; x) \tag{32}$$

and

$$\int_0^1 u^{a-1} (1-u)^{b-1} \, {}_2F_1(c,d;e;x) du = B(a,b) \, {}_3F_2(c,d,a;e,a+b;1) \tag{33}$$

Substituting these results in (30), we get

$$\Phi_{i,j}(a,b) = \frac{2\alpha\beta\lambda}{\lambda^i(b+1)} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} \int_0^\infty x^i (1+\lambda x)^{-(2\beta+1)} [1-(1+\lambda x)^{-2\beta}]^{\alpha(a+b+2)-1}$$

$$\times {}_2F_1\left(\alpha(b+1), \frac{p}{2\beta}; \alpha(b+1)+1; [1-(1+\lambda x)^{-2\beta}]\right) dx.$$

Making the substitution $t=[1-(1+\lambda x)^{-2\beta}]$, we find that

$$\Phi_{i,j}(a,b) = \frac{\alpha}{\lambda^{i+j}(b+1)} \sum_{p=0}^{j} \sum_{q=0}^{i} (-1)^{i+j-p-q} \binom{j}{p} \binom{i}{q} \int_0^1 (1-t)^{-\frac{q}{2\beta}} t^{\alpha(a+b+2)-1}$$
$$\times {}_2F_1\left(\alpha(b+1), \frac{p}{2\beta}; \alpha(b+1)+1; t\right) dt.$$

$$= \frac{\alpha}{\lambda^{i+j}(b+1)} \sum_{p=0}^{j} \sum_{q=0}^{i} (-1)^{i+j-p-q} \binom{j}{p} \binom{i}{q} B\left(\alpha(a+b+2), 1-\frac{q}{2\beta}\right)$$

$$\times {}_3F_2\left[\alpha(b+1), \frac{p}{2\beta}, \alpha(a+b+2); \alpha(b+1)+1, \alpha(a+b+2)+1-\frac{q}{2\beta}; 1\right],$$

Lemma 3.2. For the TLLo distribution as given in (1). If a and b are non negative finite integers, then we have

$$\Phi_{i,0}(a,b) = \frac{1}{(b+1)} \Phi_i(a+b+1) \tag{34}$$

where $\Phi_i(a)$ is defined in (10),
and

$$\Phi_{0,0}(a,b) = \frac{1}{(b+1)} \Phi_0(a+b+1)$$
$$= \frac{1}{(b+1)(a+b+2)} \tag{35}$$

Proof. From (26), set $j=0$ we have

$$\Phi_{i,0}(a,b) = \int_0^\infty \int_0^x x^i [F(x)]^a [F(y)]^b f(x) f(y) dy dx$$
$$= \int_0^\infty x^i [F(x)]^a f(x) \left[\int_0^x [F(y)]^b f(y) dy\right] dx$$
$$= \int_0^\infty x^i [F(x)]^a f(x) \left[2\alpha\beta\lambda \int_0^x (1+\lambda y)^{-(2\beta+1)} [1-(1+\lambda y)^{-2\beta}]^{\alpha(b+1)-1} dy\right] dx$$

Making the substitution $t=[1-(1+\lambda y)^{-2\beta}]$, we find that

$$\Phi_{i,0}(a,b) = \int_0^\infty x^i [F(x)]^a f(x) \left[\frac{[1-(1+\lambda y)^{-2\beta}]^{\alpha(b+1)}}{(b+1)}\right] dx.$$

In view of (2)

$$\Phi_{i,0}(a,b) = \frac{1}{(b+1)} \int_0^\infty x^i [F(x)]^{a+b+1} f(x) dx.$$

In view of (10), we get the result given in (34).
(35) can be proved by setting $i=0$ in (34).

Theorem 3.1. For the TLLo distribution (1), for $1 \le m \le n-1$ and $i,j=0,1,\ldots,$

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = \frac{k^n}{(m-1)!(n-m-1)!} \sum_{p=0}^{n-m-1} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{s=0}^{p+q+m-1} \sum_{t=0}^{n+r-m-p-1}$$

$$\times(-1)^{p+s+t}\binom{n-m-1}{p}\binom{p+q+m-1}{s}\binom{n+r-m-p-1}{t}$$

$$\times a_q(p+m-1)\,a_r(n-m-p-1)\,\Phi_{i,j}(s-1,t+k-1). \tag{36}$$

Proof. From (6), we have

$$E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty\int_0^x x^i y^j\,[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}$$

$$\times[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)\,dy\,dx. \tag{37}$$

$$=\frac{k^n}{(m-1)!\,(n-m-1)!}\sum_{p=0}^{n-m-1}(-1)^p\binom{n-m-1}{p}\int_0^\infty\int_0^x x^i y^j$$

$$\times[-\ln F(x)]^{p+m-1}\frac{f(x)}{F(x)}[-\ln F(y)]^{n-m-p-1}[F(y)]^{k-1}f(y)\,dy\,dx. \tag{38}$$

Now using the (17), we get

$$E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=\frac{k^n}{(m-1)!\,(n-m-1)!}\sum_{p=0}^{n-m-1}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^p\binom{n-m-1}{p}$$

$$\times a_q(p+m-1)\,a_r(n-m-p-1)\int_0^\infty\int_0^x x^i y^j[1-F(x)]^{p+q+m-1}$$

$$\times\frac{f(x)}{F(x)}[1-F(y)]^{n+r-m-p-1}[F(y)]^{k-1}f(y)\,dy\,dx. \tag{39}$$

Again using the binomial expansion

$$E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=\frac{k^n}{(m-1)!\,(n-m-1)!}\sum_{p=0}^{n-m-1}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{p+q+m-1}\sum_{t=0}^{n+r-m-p-1}$$

$$\times(-1)^{p+s+t}\binom{n-m-1}{p}\binom{p+q+m-1}{s}\binom{n+r-m-p-1}{t}$$

$$\times a_q(p+m-1)\,a_r(n-m-p-1)\int_0^\infty\int_0^x x^i y^j[F(x)]^{s-1}f(x)\,[F(y)]^{t+k-1}f(y)\,dy\,dx. \tag{40}$$

Using the lemma 3.1 in (40), we get the yield given in (36).

Theorem 3.2. For $1\le m\le n-2$ and $i,j=0,1,\dots,$

$$\left(1+\frac{j}{\alpha k}\right)E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=E[(Z_m^{(k)})^i(Z_{n-1}^{(k)})^j]-\frac{j}{2\,\alpha\,\beta\,\lambda\,k}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E[(Z_m^{(k)})^i(Z_n^{(k)})^{j+a-1}] \tag{41}$$

for $m\ge 1$ and $i,j=0,1,\dots,$

$$\left(1+\frac{j}{\alpha k}\right)E[(Z_m^{(k)})^i(Z_{m+1}^{(k)})^j]=E[(Z_m^{(k)})^{i+j}]-\frac{j}{2\,\alpha\,\beta\,\lambda\,k}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E[(Z_m^{(k)})^i(Z_{m+1}^{(k)})^{j+a-1}] \tag{42}$$

Proof. From (6), we have

$$E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=\frac{k^n}{(m-1)!\,(n-m-1)!}\int_0^\infty x^i\,[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}I(x)\,dx \tag{43}$$

where

$$I(x)=\int_0^x y^j\,[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^{k-1}f(y)\,dy. \tag{44}$$

Integrating $I(x)$ by parts taking $'F(y)]^{k-1}f(y)'$ for integeration and the rest of integrand for differentiation,

$$I(x)=\frac{n-m-1}{k}\int_0^x y^j\,[\ln F(x)-\ln F(y)]^{n-m-2}[F(y)]^{k-1}f(y)\,dy$$

$$-\frac{j}{k}\int_0^x y^{j-1}\,[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^k\,dy. \tag{45}$$

From (4), in the second term of (45) and simplifying

$$I(x)=\frac{n-m-1}{k}\int_0^x y^j\,[\ln F(x)-\ln F(y)]^{n-m-2}[F(y)]^{k-1}f(y)\,dy$$

$$-\frac{j}{k}\int_0^x y^{j-1}\left\{\frac{y}{\alpha}+\frac{1}{2\alpha\beta\lambda}\sum_{a=2}^{2\beta+1}(\lambda x)^a\binom{2\beta+1}{a}\right\}[\ln F(x)-\ln F(y)]^{n-m-1}[F(y)]^k\,dy. \tag{46}$$

Putting the value of $I(x)$ in (43), we get

$$E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=E[(Z_m^{(k)})^i(Z_{n-1}^{(k)})^j]-\frac{j}{\alpha k}E[(Z_m^{(k)})^i(Z_n^{(k)})^j]-\frac{j}{2\alpha\beta\lambda k}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E[(Z_m^{(k)})^i(Z_n^{(k)})^{j+a-1}] \tag{47}$$

arranging the terms, we get the expression given in (41). Now putting $n=m+1$ in (41) and noting that $E[(Z_m^{(k)})^i(Z_m^{(k)})^j]=E[(Z_m^{(k)})^{i+j}]$, the recurrence relation given in (42) can be easily established.

Remark 3.2.

i) At $k=1$ in (41), the recurrence relation for the product moments of lower records from the TLLo distribution as,

$$\left(1+\frac{j}{\alpha}\right)E[(X_m)^i(X_n)^j]=E[(X_m)^i(X_{n-1})^j]-\frac{j}{2\alpha\beta\lambda}\sum_{a=2}^{2\beta+1}\lambda^a\binom{2\beta+1}{a}E[(X_m)^i(X_n)^{j+a-1}]$$

ii) Setting $\alpha=1$ and $\beta=\frac{\beta}{2}$ in (41) we get the recurrence relations of generalized lower records from Lomax distribution as

$$\left(1+\frac{j}{k}\right)E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=E[(Z_m^{(k)})^i(Z_{n-1}^{(k)})^j]-\frac{j}{\beta\lambda k}\sum_{a=2}^{\beta+1}\lambda^a\binom{\beta+1}{a}E[(Z_m^{(k)})^i(Z_n^{(k)})^{j+a-1}]$$

iii) Setting $\beta=\frac{1}{2}$ and $\lambda=1$ in (41) we get the recurrence relations of generalized lower records from Exponentiated Pareto distribution as

$$\left(1+\frac{j}{\alpha k}\right)E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=E[(Z_m^{(k)})^i(Z_{n-1}^{(k)})^j]-\frac{j}{\alpha k}E[(Z_m^{(k)})^i(Z_n^{(k)})^{j+1}]$$

iv) Setting $\alpha=1$, $\beta=\frac{1}{2}$ and $\lambda=1$ in (41) we get the recurrence relations of generalized lower records from Pareto distribution as

$$\left(1+\frac{j}{k}\right)E[(Z_m^{(k)})^i(Z_n^{(k)})^j]=E[(Z_m^{(k)})^i(Z_{n-1}^{(k)})^j]-\frac{j}{k}E[(Z_m^{(k)})^i(Z_n^{(k)})^{j+1}]$$

v) When $i=0$, in (41) we get the recurrence relations for single moment as given in (2.14) of generalized lower records from the Topp–Leone Lomax distribution.

4 Characterization

This section contains the characterizations of TLLo distribution, we start with the following result of Lin [11].

PROPOSITION. Let n_0 be any fixed non-negative integer, $-\infty < a < b < \infty$ and $g(x) \geq 0$ an absolutely continuous function with $g^{'}(x) \neq 0$ i.e. on (a, b). Then the sequence of functions $\{(g(x))^n e^{-g(x)}, n \geq n_0\}$ is complete in L(a, b) iff $g(x)$ is strictly monotone on (a, b).

Theorem 4.1. Fix a positive integer $k \geq 1$ and let j be a non negative integers. A necessary and sufficient condition for a random variable X to be distributed with pdf given by (1) is that

$$\left(1+\frac{j}{\alpha k}\right) E\left(Z_n^{(k)}\right)^j = E\left(Z_{n-1}^{(k)}\right)^j - \frac{j}{2\alpha\beta\lambda k}\sum_{a=2}^{2\beta+1}\lambda^a \binom{2\beta+1}{a} E\left(Z_n^{(k)}\right)^{j+a-1} \tag{48}$$

for $n = 1, 2, \ldots$ and $n \geq k$.

Proof. The necessary part follows from (21). On the other hand if the recuerrence relations (48) is satisfied, then on rearranging the terms in (48)

$$E\left(Z_n^{(k)}\right)^j - E\left(Z_{n-1}^{(k)}\right)^j = -\frac{j}{\alpha k} E\left(Z_n^{(k)}\right)^j - \frac{j}{2\alpha\beta\lambda k}\sum_{a=2}^{2\beta+1}\lambda^a \binom{2\beta+1}{a} E\left(Z_n^{(k)}\right)^{j+a-1} \tag{49}$$

using the lemma by Bieniek and Szynal [5]

$$-\frac{jk^{n-1}}{(n-1)!}\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^k dx = -\frac{jk^n}{(n-1)!\,\alpha k}\Big\{\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}$$

$$[F(x)]^{k-1} f(x)\,dx + \int_0^\infty \frac{j}{2\beta\lambda}\sum_{a=2}^{2\beta+1}\lambda^a \binom{2\beta+1}{a} x^{j+a-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x) dx\Big\}.$$

$$\int_0^\infty x^{j-1}[-\ln F(x)]^{n-1}[F(x)]^{k-1} f(x)\Big\{\frac{F(x)}{f(x)} - \frac{x}{\alpha} - \frac{1}{2\alpha\beta\lambda}\sum_{a=2}^{2\beta+1}(\lambda x)^a \binom{2\beta+1}{a}\Big\} dx = 0 \tag{50}$$

It now follow from the above proposition

$$\Big[2\beta\lambda x + \sum_{a=2}^{2\beta+1}(\lambda x)^a \binom{2\beta+1}{a}\Big] f(x) = 2\alpha\beta\lambda F(x)$$

which proves that $f(x)$ has the form as given in (4).

Theorem 4.2. For a positive integer k, i and j be a non-negative integer, a necessary and sufficient condition for a random variable X to be distributed with pdf given by (1), is that

$$\left(1+\frac{j}{\alpha k}\right) E[(Z_m^{(k)})^i (Z_n^{(k)})^j] = E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] - \frac{j}{2\alpha\beta\lambda k}\sum_{a=2}^{2\beta+1}\lambda^a \binom{2\beta+1}{a} E[(Z_m^{(k)})^i (Z_n^{(k)})^{j+a-1}] \tag{51}$$

Proof. The necessary part can be seen in view of Theorem 3.2. To prove sufficiency part, we have

$$E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - E[(Z_m^{(k)})^i (Z_{n-1}^{(k)})^j] = -\frac{j}{\alpha k} E[(Z_m^{(k)})^i (Z_n^{(k)})^j] - \frac{j}{2\alpha\beta\lambda k}\sum_{a=2}^{2\beta+1}\lambda^a \binom{2\beta+1}{a} E[(Z_m^{(k)})^i (Z_n^{(k)})^{j+a-1}] \tag{52}$$

using the lemma by Singh and Khan [8]

$$-\frac{jk^{n-1}}{(m-1)!(n-m-1)!}\int_0^\infty\int_0^x x^i y^{j-1}[-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}[\ln F(x) - \ln F(y)]^{n-m-1}$$

$$\times [F(y)]^k dy\,dx = -\frac{jk^{n-1}}{(m-1)!(n-m-1)!\,\alpha}\Big\{\int_0^\infty\int_0^x x^i y^j [-\ln F(x)]^{m-1}\frac{f(x)}{F(x)}$$

$$\times [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, dy\, dx - \frac{1}{2\beta\lambda} \int_0^\infty \int_0^x \sum_{a=2}^{2\beta+1} \lambda^a \binom{2\beta+1}{a}$$

$$\times x^i y^{j+a-1} [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)\, dy\, dx \Big\}. \tag{53}$$

This gives

$$\int_0^\infty \int_0^x x^i y^{j-1} [-\ln F(x)]^{m-1} \frac{f(x)}{F(x)} [\ln F(x) - \ln F(y)]^{n-m-1} [F(y)]^{k-1} f(y)$$

$$\times \left\{ \frac{F(y)}{f(y)} - \frac{y}{\alpha} - \frac{1}{2\alpha\beta\lambda} \sum_{a=2}^{2\beta+1} (\lambda y)^a \binom{2\beta+1}{a} \right\} dy\, dx = 0. \tag{54}$$

It now follow from the above proposition, we get

$$\left[2\beta\lambda x + \sum_{a=2}^{2\beta+1} (\lambda x)^a \binom{2\beta+1}{a} \right] f(x) = 2\alpha\beta\lambda F(x)$$

which is a form of relation given in (4).

Theorem 4.3. Let X be a non-negative random variable having an absolutely continuous df $F(x)$ and pdf $f(x)$ over the support $(0,\infty)$ and let $h(x)$ be a continuous and differentiable function of x, then for two consecutive value of m and $m+1$, then

$$E[F(Z_n^{(k)})|(Z_l^{(k)}) = x] = g_{n|l}(x) = [1-(1+\lambda x)^{-2\beta}]^\alpha \left(\frac{k}{k+1}\right)^{n-l}, \tag{55}$$

$$l = m, m+1, m \geq k$$

if and only if

$$F(x) = [1-(1+\lambda x)^{-2\beta}]^\alpha \quad x > 0,\, \alpha > 0,\, \beta > 0,\, \lambda > 0.$$

Proof. From (7), we have

$$E[F(Z_n^{(k)})|(Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!} \int_0^x [1-(1+\lambda y)^{-2\beta}]^\alpha [\ln F(x) - \ln F(y)]^{n-m-1} \left[\frac{F(y)}{F(x)}\right]^{k-1} \frac{f(y)}{F(x)} dy. \tag{56}$$

By setting $u = \frac{F(y)}{F(x)}$, from (2) in (56), we have

$$E[F(Z_n^{(k)})|(Z_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!} [1-(1+\lambda x)^{-2\beta}]^\alpha \int_0^1 u^k [-\ln u]^{n-m-1} du.$$

We have [12]

$$\int_0^1 x^{\nu-1} [-\ln x]^{\mu-1} dx = \frac{\Gamma\mu}{\nu^\mu}. \tag{57}$$

Which gives the result given in (55).

To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!} \int_0^x [1-(1+\lambda y)^{-2\beta}]^\alpha [\ln F(x) - \ln F(y)]^{n-m-1} [F(x)]^{k-1} f(y) dy = [F(x)]^k g_{n|m}(x), \tag{58}$$

where

$$g_{n|m}(x) = [1-(1+\lambda x)^{-2\beta}]^\alpha \left(\frac{k}{k+1}\right)^{n-m}. \tag{59}$$

Differentiating (58) both sides with respect to x, we get

$$\frac{k^{n-m} f(x)}{(n-m-2)!\, F(x)} \int_0^x [1-(1+\lambda y)^{-2\beta}]^\alpha [\ln F(x) - \ln F(y)]^{n-m-2} [F(x)]^{k-1} f(y) dy$$

$$= g^{'}_{n|m}(x)\,[F(x)]^k + k\,g_{n|m}(x)[F(x)]^{k-1} f(x)$$

or

$$k\,g_{n|m+1}(x)[F(x)]^{k-1} f(x) = g^{'}_{n|m}(x)\,[F(x)]^k + k\,g_{n|m}(x)[F(x)]^{k-1} f(x).$$

Therefore,

$$\frac{f(x)}{F(x)} = \frac{g^{'}_{n|m}(x)}{k[g_{n|m+1}(x) - g_{n|m}(x)]}$$

where

$$g^{'}_{n|m}(x) = 2\alpha\beta\lambda\,(1+\lambda x)^{-(2\beta+1)}\,[1-(1+\lambda x)^{-2\beta}]^{\alpha-1}\left(\frac{k}{k+1}\right)^{n-m}$$

$$g_{n|m+1}(x) - g_{n|m}(x) = [1-(1+\lambda x)^{-2\beta}]^{\alpha}\frac{1}{k}\left(\frac{k}{k+1}\right)^{n-m}$$

then

$$\frac{f(x)}{F(x)} = \frac{2\alpha\beta\lambda\,(1+\lambda x)^{-(2\beta+1)}\,[1-(1+\lambda x)^{-2\beta}]^{\alpha-1}}{[1-(1+\lambda x)^{-2\beta}]^{\alpha}}$$

which implying that $F(x) = [1-(1+\lambda x)^{-2\beta}]^{\alpha}$, the sufficiency part is proved.

Theorem 4.3. Suppose an absolutely continuous (with respect to Lebesque measure) random variable X has the df $F(x)$ and the pdf $f(x)$ for $0 < x < \infty$, such that $f'(x)$ and $E(X|X \le x)$ exist for all x, then

$$E(X|X \le x) = g(x)\eta(x), \tag{60}$$

where

$$\eta(x) = \frac{f(x)}{F(x)}$$

and

$$g(x) = \frac{x[1-(1+\lambda x)^{-2\beta}]}{2\alpha\beta\lambda\,(1+\lambda x)^{-(2\beta+1)}} + \frac{1}{f(x)}\int_0^x [1-(1+\lambda u)^{-2\beta}]^{\alpha} du$$

if and only if

$$f(x) = 2\alpha\beta\lambda\,(1+\lambda x)^{-(2\beta+1)}\,[1-(1+\lambda x)^{-2\beta}]^{\alpha-1}, \quad x>0, \alpha>0, \beta>0, \lambda>0,$$

Proof. From (1), we have

$$E(X|X \le x) = \frac{2\alpha\beta\lambda}{F(x)}\int_0^x u\,(1+\lambda u)^{-(2\beta+1)}\,[1-(1+\lambda u)^{-2\beta}]^{\alpha-1} du. \tag{61}$$

Integrating (61) by parts treating '$2\alpha\beta\lambda\,(1+\lambda u)^{-(2\beta+1)}\,[1-(1+\lambda u)^{-2\beta}]^{\alpha-1}$' for integration and rest of the integrand for differentiation, we get

$$E(X|X \le x) = \frac{1}{F(x)}\left[x[1-(1+\lambda x)^{-2\beta}]^{\alpha} + \int_0^x [1-(1+\lambda u)^{-2\beta}]^{\alpha} du\right]. \tag{62}$$

After multiplying and dividing by $f(x)$ in (62), we have the result given in (60).

To prove the sufficient part, we have from (60)

$$\int_0^x u f(u) dt = g(x) f(x). \tag{63}$$

Differentiating (63) on both the sides with respect to x, we find that

$$x f(x) = g'(x) f(x) + g(x) f'(x).$$

Therefore,

$$\frac{f'(x)}{f(x)} = \frac{x - g'(x)}{g(x)}$$

$$= \frac{\lambda(-2\beta-1)}{(1+\lambda x)} + \frac{2\beta\lambda(\alpha-1)(1+\lambda x)^{-2\beta-1}}{[1-(1+\lambda x)^{-2\beta}]}, \tag{64}$$

where

$$g'(x) = x - g(x)\left[\frac{\lambda(-2\beta-1)}{(1+\lambda x)} + \frac{2\beta\lambda(\alpha-1)(1+\lambda x)^{-2\beta-1}}{[1-(1+\lambda x)^{-2\beta}]}\right].$$

Integrating both the sides in (64) with respect to x, we get

$$f(x) = C\,(1+\lambda x)^{-(2\beta+1)}[1-(1+\lambda x)^{-2\beta}]^{\alpha-1}.$$

Now, using the condition $\int_0^\infty f(x)dx = 1$, we obtains

$$f(x) = 2\alpha\beta\lambda\,(1+\lambda x)^{-(2\beta+1)}[1-(1+\lambda x)^{-2\beta}]^{\alpha-1} \quad x>0, \alpha>0, \beta>0, \lambda>0.$$

5 Conclusion

In this study, we demonstrate the explicit expression as well as recurrence relation for the moments of k-th lower record values from Topp–Leone Lomax distribution. The recurrence relations can be used to reduce the amount of direct computation and moments of any order can be calculated easily. To verify the designed models which is required in probability distribution, we used the results of the characterization. At the different values of parameters, we reduced some well known results. We can explore our study for dual generalized order statistics which contains several models of order random variates.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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