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Some New Hermite-Hadamard, Hermite-Hadamard Fejer and Weighted Hardy Type Inequalities Involving $(k-p)$ Riemann-Liouville Fractional Integral Operator

Ankita Chandola[1], *Ritu Agarwal*[2,*] *and Rupakshi Mishra Pandey*[1]

[1]Amity Institute of Applied Sciences, Amity University, Uttar Pradesh, India
[2]Malaviya National Institute of Technology, Jaipur, India

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Abstract: Hermite Hadamard inequality is of immense importance due to its applications in numerical integration and in providing lower and upper limits of the functions mean value. Hardy type inequalities are useful in technical sciences. Various authors have worked for the improvement and generalizations of these inequalities. In this paper, we obtain certain new Hermite-Hadamard, Hermite-Hadamard Fejer and weighted Hardy type inequalities involving $(k-p)$ Riemann-Liouville fractional integral operator using convex and increasing functions. Some inequalities obtained here would provide the extensions of some already known results.

Keywords: Hermite-Hadamard inequality, Hermite-Hadamard Fejer inequality, Weighted Hardy type inequality, $(k-p)$ Riemann-Liouville fractional integral, Convex function.

1 Introduction and Preliminaries

In the recent years, importance of fractional calculus has increased extensively in different fields such as inequality theory, applied mathematics, sciences and engineering. Fractional integrals are used for the description of the various properties of different physical processes like seepage flow in fluid dynamic traffic model and non-linear oscillations of earthquake. Special functions, its generalizations and relation with fractional calculus are extensively investigated by many researchers (see, e.g. [1,2,3,4]).

Fractional calculus provides a powerful tool which has been recently employed to model real life problems. The derivatives and integrals of arbitrary order are used by many researchers and scientists to study various types of problems (see, e.g. [5,6,7,8,9,10,11,12]).

We will now mention some definitions and results useful for our study.

Definition 1.1.(Convex function [13]). A function $g: J \to \mathbb{R}$, where J is an interval in $\mathbb{R}$ is said to be convex if the given inequality holds true for all $x, y \in J$

$$g(\mu x+(1-\mu)y) \leq \mu g(x)+(1-\mu)g(y). \tag{1}$$

When the inequality in equation (1) becomes strict inequality for all different points in J and $\mu \in [0,1]$, then the function g is called strictly convex. Also, g is concave if $(-g)$ is convex.

Definition 1.2.(Hermite-Hadamard inequality [14]). Let $g: J \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $c,d \in J$ with $c<d$, then the inequality

$$g\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d g(x)dx \leq \frac{g(c)+g(d)}{2}, \tag{2}$$

holds and is known as Hermite-Hadamard integral inequality for convex functions.

Fejer introduced the weighted generalization of the Hermite-Hadamard inequality:

Definition 1.3.(Hermite-Hadamard Fejer type inequality [15]). Let $g: [c,d] \to \mathbb{R}$ be a convex function. Then the

* Corresponding author e-mail: ragarwal.maths@mnit.ac.in

inequality given below holds true

$$g\left(\frac{c+d}{2}\right)\int_c^d h(x)dx \le \int_c^d g(x)h(x)dx \le \frac{g(c)+g(d)}{2}\int_c^d h(x)dx, \quad (3)$$

where $h:[c,d]\to\mathbb{R}$ is non-negative, integrable and symmetric to $\frac{c+d}{2}$.

Definition 1.4.(Riemann-Liouville fractional integral [16], [17]). The left and right sided Riemann-Liouville fractional integral are given by

$$\left(J_{c+}^{\alpha}g\right)(x)=\frac{1}{\Gamma(\alpha)}\int_c^x (x-\tau)^{\alpha-1}g(\tau)d\tau, \quad x>c, \quad (4)$$

$$\left(J_{d-}^{\alpha}g\right)(x)=\frac{1}{\Gamma(\alpha)}\int_x^d (\tau-x)^{\alpha-1}g(\tau)d\tau, \quad x<d. \quad (5)$$

Katugampola ([18]) generalized the well known Riemann-Liouville operator, later known as Katugampola fractional integrals.

Definition 1.5.(Katugampola fractional integrals [18, eq-3.3,3.4, pg.4]). The left and right sided Katugampola fractional integrals are given by

$$\left({}^{p}J_{c+}^{\alpha}g\right)(x)=\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_c^x \left(x^{p+1}-\tau^{p+1}\right)^{\alpha-1}\tau^p g(\tau)d\tau, \quad (6)$$

for $x>c$,

$$\left({}^{p}J_{d-}^{\alpha}g\right)(x)=\frac{(p+1)^{1-\alpha}}{\Gamma(\alpha)}\int_x^d \left(\tau^{p+1}-x^{p+1}\right)^{\alpha-1}\tau^p g(\tau)d\tau, \quad (7)$$

for $x<d$.

The Pochhammer k-symbol $(y)_{m,k}$ is defined as (see [19, defn.1, pg-181])

$$(y)_{m,k}=y(y+k)(y+2k)\ldots y+(m-1)k, \quad (8)$$

where $m\in\mathbb{N}_0, k>0$.

The $k-gamma$ function Γ_k is given by (see [19, defn.3, pg-182])

$$\Gamma_k(y)=\lim_{m\to\infty}\frac{m!k^m(mk)^{\frac{y}{k}-1}}{(y)_{m,k}}, \quad (9)$$

where $k>0, y\in\mathbb{C}\setminus k\mathbb{Z}_0^-, k\mathbb{Z}_0^-=[kn:n\in\mathbb{Z}_0^-]$.

When $k=1$, equations (8) and (9) reduces to the Pochhammer symbol $(y)_m$ (see [20, eq.1, pg-22, eq.3, pg-23]) and the gamma function (see [20, eq.8, pg-17]) for $y\neq 0,-1,-2,\ldots$.

$$(y)_m=\begin{cases}\prod_{r=1}^{m}(y+r-1), m\in\mathbb{N}\\ 1, m=0\end{cases}=\frac{\Gamma(y+m)}{\Gamma(y)}$$

and

$$\Gamma(t)=\int_0^\infty e^{-z}z^{t-1}dz, \qquad \Re(t)>0.$$

Definition 1.6.([21, eq.8, pg-91]). The $k-$Riemann-Liouville fractional integral operator ${}_kJ_c^{\alpha}$ of order $\alpha>0$ for a real valued function $g(\tau)$ is defined as

$$\left({}_kJ_c^{\alpha}g\right)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_c^x [x-\tau]^{\frac{\alpha}{k}-1}g(\tau)d\tau, \quad (k>0). \quad (10)$$

The left and right sided $k-$Riemann-Liouville fractional integral operator are given by

$$\left({}_kJ_{c+}^{\alpha}g\right)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_c^x [x-\tau]^{\frac{\alpha}{k}-1}g(\tau)d\tau, \quad (11)$$

$$\left({}_kJ_{d-}^{\alpha}g\right)(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^d [\tau-x]^{\frac{\alpha}{k}-1}g(\tau)d\tau. \quad (12)$$

Sarikaya ([22]) gave another generalization of Riemann-Liouville fractional integral operator which is known as $(k-p)$ Riemann-Liouville fractional integral:

Definition 1.7.([22, eq-2.1,pg-79]). The $(k-p)$ Riemann-Liouville fractional integral operator ${}_k^pJ_c^{\alpha}$ of order $\alpha>0$ for a real valued function $g(\tau)$ is defined as

$$\left({}_k^pJ_c^{\alpha}g\right)(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_c^x \left[x^{p+1}-\tau^{p+1}\right]^{\frac{\alpha}{k}-1}\tau^p g(\tau)d\tau, \quad (13)$$

where $k>0, p\in\mathbb{R}, p\neq -1$.
The left and right sided $(k-p)$ Riemann-Liouville fractional integral operator are given by

$$\left({}_k^pJ_{c+}^{\alpha}g\right)(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_c^x \left[x^{p+1}-\tau^{p+1}\right]^{\frac{\alpha}{k}-1}\tau^p g(\tau)d\tau, \quad (14)$$

$$\left({}_k^pJ_{d-}^{\alpha}g\right)(x)=\frac{(p+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_x^d \left[\tau^{p+1}-x^{p+1}\right]^{\frac{\alpha}{k}-1}\tau^p g(\tau)d\tau. \quad (15)$$

Special cases-

1.When $p=0$, equations (14) and (15) reduce to equations (11) and (12) respectively, that is, $(k-p)$ Riemann-Liouville fractional integral reduces to $k-$Riemann-Liouville fractional integral.

2. When $k=1$, equations (14) and (15) reduce to equations (6) and (7) respectively, that is, $(k-p)$ Riemann-Liouville fractional integral reduces to Katugampola fractional integral.
3. When $k=1, p=0$, equations (14) and (15) reduce to equations (4) and (5) respectively, that is, $(k-p)$ Riemann-Liouville fractional integral reduces to Riemann-Liouville fractional integral.

Definition 1.8.([23, defn.2]). A space of continuous real valued functions $g(\tau)$ on $[c,d]$, denoted by $L_{q,p}[c,d]$ is given by

$$\left(\int_c^d |g(\tau)|^q \tau^p d\tau\right)^{\frac{1}{q}} < \infty,$$

where $1 \le q < \infty, p \ge 0$. Also, $L_{q,0}[c,d] = L_q[c,d]$.

We will use the following results in our work.

Theorem 1.1. ([24, Thm. 2.1]) Let $(\Xi_1,\Delta_1,\delta_1)$ and $(\Xi_2,\Delta_2,\delta_2)$ be measure spaces with $\sigma - finite$ measures, ω be the weight function on δ_1, m be a non-negative measurable kernel on $\delta_1 \times \delta_2$. Assume that the function, $y \to \omega(y)\frac{m(y,\tau)}{M(y)}$ is integrable on δ_1. Then for each fixed $\tau \in \delta_2$ define υ by

$$\upsilon(\tau) = \int_{\delta_1} \omega(y)\frac{m(y,\tau)}{M(y)} d\delta_1(y) < \infty. \tag{16}$$

If $\Psi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\delta_1} \omega(y)\Psi\left(\left|\frac{h(y)}{M(y)}\right|\right) d\delta_1(y) < \int_{\delta_2} \upsilon(\tau)\Psi(|g(\tau)|)d\delta_2(\tau), \tag{17}$$

holds for all measurable functions $g : \delta_2 \to \mathbb{R}$ and $h(y) = \int_{\delta_2} m(y,\tau)g(\tau)d\delta_2(\tau)$.

We shall assume that $\Theta(g)$ denotes the the class of functions $h : \delta_1 \to \mathbb{R}$ where $h(y) = \int_{\delta_2} m(y,\tau)g(\tau)d\delta_2(\tau)$ and g is a measurable function.

Theorem 1.2. ([25, Thm.1.2, pg-220]) Let $g_i : \delta_2 \to \mathbb{R}$ be measurable functions, $h_i \in \Theta(g_i)$, $(i=1,2)$ where $h_2(y) > 0$ for every $y \in \delta_1$. Let ω be a weight function on δ_1 and m is a non-negative measurable kernel on $\delta_1 \times \delta_2$. Assume that the function $y \to \omega(y)\frac{g_2(\tau)m(y,\tau)}{h_2(y)}$ is integrable on δ_1. Then for each fixed $\tau \in \delta_2$ define φ by

$$\varphi(\tau) = g_2(\tau)\int_{\delta_1} \frac{\omega(y)m(y,\tau)}{h_2(y)} d\delta_1(y) < \infty. \tag{18}$$

If $\Psi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\delta_1} \omega(y)\Psi\left(\left|\frac{h_1(y)}{h_2(y)}\right|\right) d\delta_1(y) < \int_{\delta_2} \varphi(\tau)\Psi\left(\left|\frac{g_1(\tau)}{g_2(\tau)}\right|\right))d\delta_2(\tau), \tag{19}$$

holds.

Theorem 1.3.([26]) Let $(\Xi_1,\Delta_1,\delta_1)$ and $(\Xi_2,\Delta_2,\delta_2)$ be measure spaces with $\sigma - finite$ measures, ω be the weight function on δ_1, m be a non-negative measurable kernel on $\delta_1 \times \delta_2$. Assume that the function $y \to \omega(y)\frac{m(y,\tau)}{M(y)}$ is integrable on δ_1. Then for each fixed $\tau \in \delta_2$ define ζ by

$$\zeta(\tau) = \left(\int_{\delta_1} \omega(y)\left[\frac{m(y,\tau)}{M(y)}\right]^{\frac{s}{r}} d\delta_1(y)\right)^{\frac{r}{s}} < \infty. \tag{20}$$

If $\Psi : (0,\infty) \to \mathbb{R}$ is a non-negative convex function, then the inequality

$$\left(\int_{\delta_1} \omega(y)\left[\Psi(A_k g(y))\right]^{\frac{s}{r}} d\delta_1(y)\right)^{\frac{1}{s}} \le \left(\int_{\delta_2} \zeta(\tau)\Psi(g(\tau))d\delta_2(\tau)\right)^{\frac{1}{r}}, \tag{21}$$

holds for all measurable functions $g : \delta_2 \to \mathbb{R}$.

Theorem 1.4.([26]) Let $h_i \in \Theta(g_i)$, $i=1,2,3$ where $h_2(y) > 0$ for every $y \in \delta_1$, m be a non-negative measurable function on $\delta_1 \times \delta_2$, then ρ is defined by

$$\rho(\tau) = g_2(\tau)\int_{\delta_1} \frac{\omega(y)m(y,\tau)}{h_2(y)} d\delta_1(y) < \infty. \tag{22}$$

If $\Psi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_{\delta_1} \omega(y)\Psi\left(\left|\frac{h_1(y)}{h_2(y)}\right|,\left|\frac{h_3(y)}{h_2(y)}\right|\right) d\delta_1(y) \le \int_{\delta_2} \rho(\tau)\Psi\left(\left|\frac{g_1(\tau)}{g_2(\tau)}\right|,\left|\frac{g_3(\tau)}{g_2(\tau)}\right|\right) d\delta_2(\tau), \tag{23}$$

holds true.

In our next sections we derive some Hermite-Hadamard, Hermite-Hadamard Fejer and weighted Hardy type inequalities using $(k-p)$ Riemann-Liouville fractional integral.

2 Hermite-Hadamard type inequalities

In this section we obtain results of the Hermite-Hadamard inequalities for $(k-p)$ Riemann Liouville fractional integral.

Theorem 2.1.(Generalization of Hermite Hadamard Inequality). Let $\alpha, p > 0$ and $g : [c^p, d^p] \in \mathbb{R}$ be a positive function with $0 \leq c < d$. If g is a convex function on $[c,d]$, then the following inequality hold.

$$g\left(\frac{c^p+d^p}{2}\right) \leq \frac{\alpha \Gamma_k(\alpha) p^{\frac{\alpha}{k}}}{2(d^p-c^p)^{\frac{\alpha}{k}}}\left[{}_k^{p-1}J_{c+}^{\alpha} g(d^p) + {}_k^{p-1}J_{d-}^{\alpha} g(c^p)\right] \leq \frac{g(c^p)+g(d^p)}{2}. \tag{24}$$

Proof. Let $\tau \in [0,1]$. Consider $u, v \in [c,d], c \geq 0$, which are defined by
$u^p = \tau^p c^p + (1-\tau^p)d^p$ and $v^p = (1-\tau^p)c^p + \tau^p d^p$. We first prove

$$g\left(\frac{c^p+d^p}{2}\right) \leq \frac{\alpha \Gamma_k(\alpha) p^{\frac{\alpha}{k}}}{2(d^p-c^p)^{\frac{\alpha}{k}}}\left[{}_k^{p-1}J_{c+}^{\alpha} g(d^p) + {}_k^{p-1}J_{d-}^{\alpha} g(c^p)\right] \tag{25}$$

Since g is convex on $[c,d]$, we have

$$g\left(\frac{u^p+v^p}{2}\right) \leq \frac{g(u^p)+g(v^p)}{2}. \tag{26}$$

Substituting the values of u^p and v^p in equation (26) we get

$$2g\left(\frac{c^p+d^p}{2}\right) \leq g(\tau^p c^p + (1-\tau^p)d^p) + g((1-\tau^p)c^p + \tau^p d^p). \tag{27}$$

Multiplying both sides of equation (27) by $\tau^{\frac{\alpha p}{k}-1}$ and then integrate with respect to τ over $[0,1]$

$$2g\left(\frac{c^p+d^p}{2}\right)\int_0^1 \tau^{\frac{\alpha p}{k}-1} d\tau \leq \int_0^1 \tau^{\frac{\alpha p}{k}-1} g(\tau^p c^p + (1-\tau^p)d^p) d\tau + \int_0^1 \tau^{\frac{\alpha p}{k}-1} g((1-\tau^p)c^p + \tau^p d^p) d\tau.$$

$$\implies 2g\left(\frac{c^p+d^p}{2}\right)\frac{k}{\alpha p} \leq \int_c^d \left(\frac{d^p-u^p}{d^p-c^p}\right)^{\frac{\alpha}{k}-1} g(u^p)\frac{u^{p-1}}{d^p-c^p} du + \int_c^d \left(\frac{v^p-c^p}{d^p-c^p}\right)^{\frac{\alpha}{k}-1} g(v^p)\frac{v^{p-1}}{d^p-c^p} du = \frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}}\left[{}_k^{p-1}J_{c+}^{\alpha} g(d^p) + {}_k^{p-1}J_{d-}^{\alpha} g(c^p)\right].$$

This proves our equation (25).
We now prove

$$\frac{\alpha \Gamma_k(\alpha) p^{\frac{\alpha}{k}}}{2(d^p-c^p)^{\frac{\alpha}{k}}}\left[{}_k^{p-1}J_{c+}^{\alpha} g(d^p) + {}_k^{p-1}J_{d-}^{\alpha} g(c^p)\right] \leq \frac{g(c^p)+g(d^p)}{2}. \tag{28}$$

For a convex function g we have

$$g(\tau^p c^p + (1-\tau^p)d^p) \leq \tau^p g(c^p) + (1-\tau^p) g(d^p), \tag{29}$$

$$g((1-\tau^p)c^p + \tau^p d^p) \leq (1-\tau^p) g(c^p) + \tau^p g(d^p). \tag{30}$$

Adding equations (29) and (30)

$$g(\tau^p c^p + (1-\tau^p)d^p) + g((1-\tau^p)c^p + \tau^p d^p) \leq g(c^p) + g(d^p). \tag{31}$$

Multiplying both sides of equation (31) by $\tau^{\frac{\alpha p}{k}-1}$ and then integrate with respect to τ over $[0,1]$

$$\frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}}\left[{}_k^{p-1}J_{c+}^{\alpha} g(d^p) + {}_k^{p-1}J_{d-}^{\alpha} g(c^p)\right] \leq \frac{k}{\alpha p}\left[g(c^p)+g(d^p)\right].$$

On simplification we get the inequality (28).
From equations (25) and (28) we get our desired result (24). □

Remark 2.1. The above inequality (24) in the Theorem 2.1. is also known as Endpoint Hermite-Hadamard inequality involving $(k-p)$ Riemann-Liouville fractional integral operator because of the use of endpoints c and d.

Remark 2.2. The main result reduces to many known results (see, e.g. [27,28])

1. When $p = 1$, the result reduces to the inequality for $k-$Riemann-Liouville fractional integral:
$$g\left(\frac{c+d}{2}\right) \leq \frac{\alpha \Gamma_k(\alpha)}{2(d-c)^{\frac{\alpha}{k}}}\left[{}_kJ_{c+}^{\alpha} g(d) + {}_kJ_{d-}^{\alpha} g(c)\right] \leq \frac{g(c)+g(d)}{2}.$$
2. When $k = 1$ the result reduces to the inequality for Katumgampola fractional integral.
$$g\left(\frac{c^p+d^p}{2}\right) \leq \frac{\alpha \Gamma(\alpha) p^{\alpha}}{2(d^p-c^p)^{\alpha}}\left[{}^{p-1}J_{c+}^{\alpha} g(d^p) + {}^{p-1}J_{d-}^{\alpha} g(c^p)\right] \leq \frac{g(c^p)+g(d^p)}{2}.$$

3.When $k=1, p=1$ the result reduces to the inequality for Riemann-Liouville fractional integral.

$$g\left(\frac{c+d}{2}\right) \le \frac{\alpha\Gamma(\alpha)p^{\alpha}}{2(d-c)^{\alpha}}\left[J^{\alpha}_{c+}g(d)+J^{\alpha}_{d-}g(c)\right] \le \frac{g(c)+g(d)}{2}.$$

Theorem 2.2. Let $g:[c^p,d^p]\to\mathbb{R}$ be a differentiable mapping where $0\le c<d$. If g' is differentiable on (c^p,d^p) then the inequality given below holds true

$$\left|\left[\frac{g(c^p)+g(d^p)}{2}\right]-A\right| \le \frac{(d^p-c^p)^2}{2(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\left(\frac{\alpha}{k}+\frac{1}{2^{\frac{\alpha}{k}}}\right)sup_{\zeta\in[c^p,d^p]}|g''(\zeta)|, \tag{32}$$

where $A=\frac{\alpha\Gamma_k(\alpha)p^{\frac{\alpha}{k}}}{2(d^p-c^p)^{\frac{\alpha}{k}}}\left[{}^{p-1}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p-1}_{k}J^{\alpha}_{d-}g(c^p)\right]$.

Proof. Consider the equation from the proof of Theorem 2.1.

$$\frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}}\left[{}^{p-1}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p-1}_{k}J^{\alpha}_{d-}g(c^p)\right] = \int_0^1 \tau^{\frac{\alpha p}{k}-1}g(\tau^pc^p+(1-\tau^p)d^p)d\tau + \int_0^1 \tau^{\frac{\alpha p}{k}-1}g((1-\tau^p)c^p+\tau^pd^p)d\tau. \tag{33}$$

Using integration by part on the RHS of equation (33) we get

$$RHS = k\left[\frac{g(c^p)+g(d^p)}{\alpha p}\right]+\frac{k}{\alpha}(d^p-c^p)\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1} \times\left[g'(\tau^pc^p+(1-\tau^p)d^p)-g'((1-\tau^p)c^p+\tau^pd^p)\right]d\tau.$$

Thus equation (33) becomes

$$k\left[\frac{g(c^p)+g(d^p)}{\alpha p}\right] - \frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}}\left[{}^{p-1}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p-1}_{k}J^{\alpha}_{d-}g(c^p)\right] = \frac{k}{\alpha}(d^p-c^p)\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1} \times\left[g'((1-\tau^p)c^p+\tau^pd^p)-g'(\tau^pc^p+(1-\tau^p)d^p)\right]d\tau. \tag{34}$$

Applying mean value theorem for g' on RHS of equation (34) and then taking modulus on both sides of the equation, for $\zeta(\tau)\in(c,d)$ we get

$$\left|k\left[\frac{g(c^p)+g(d^p)}{\alpha p}\right]-\frac{A}{p}\right| \le \frac{k}{\alpha}(d^p-c^p)^2\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1}|2\tau^p-1||g''(\zeta(\tau))|d\tau \le \frac{k}{\alpha}(d^p-c^p)^2 sup_{\zeta\in[c^p,d^p]}|g''(\zeta)| \left[\int_0^{\frac{1}{\sqrt[p]{2}}}\tau^{p(\frac{\alpha}{k}+1)-1}(1-2\tau^p)d\tau+\int_{\frac{1}{\sqrt[p]{2}}}^1\tau^{p(\frac{\alpha}{k}+1)-1}(2\tau^p-1)d\tau\right] = \frac{k}{\alpha}\left[\frac{(d^p-c^p)^2}{p(\frac{\alpha}{k}+1)(\frac{\alpha}{k}+2)}\right]\left(\frac{\alpha}{k}+\frac{1}{2^{\frac{\alpha}{k}}}\right)sup_{\zeta\in[c^p,d^p]}|g''(\zeta)|.$$

where $A=\frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{-\frac{\alpha}{k}}}\left[{}^{p-1}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p-1}_{k}J^{\alpha}_{d-}g(c^p)\right]$.
On simplification we get our desired inequality (32). □

Theorem 2.3. Let $g:[c^p,d^p]\to\mathbb{R}$ be a differentiable mapping where $0\le c<d$. If $|g'|$ is convex on (c^p,d^p) then the inequality given below holds true

$$\left|\left[\frac{g(c^p)+g(d^p)}{2}\right]-\frac{\alpha\Gamma_k(\alpha)p^{\frac{\alpha}{k}}}{2(d^p-c^p)^{\frac{\alpha}{k}}}\left[{}^{p-1}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p-1}_{k}J^{\alpha}_{d-}g(c^p)\right]\right| \le \frac{(d^p-c^p)}{2(\frac{\alpha}{k}+1)}\left[|g'(c^p)|+|g'(d^p)|\right]. \tag{35}$$

Proof. Using equation (34) and taking modulus on both sides

$$\left|k\left[\frac{g(c^p)+g(d^p)}{\alpha p}\right]-\frac{A}{p}\right| \le \frac{k}{\alpha}(d^p-c^p)\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1} \times\left|\left[g'((1-\tau^p)c^p+\tau^pd^p)-g'(\tau^pc^p+(1-\tau^p)d^p)\right]\right|d\tau \le \frac{k}{\alpha}(d^p-c^p)\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1} \times\left[\left|g'((1-\tau^p)c^p+\tau^pd^p)\right|+\left|g'(\tau^pc^p+(1-\tau^p)d^p)\right|\right]d\tau \tag{36}$$

where $A=\frac{1}{(d^p-c^p)^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{p^{-\frac{\alpha}{k}}}\left[{}^{p}_{k}J^{\alpha}_{c+}g(d^p)+{}^{p}_{k}J^{\alpha}_{d-}g(c^p)\right]$.
Using convexity of $|g'|$

$$\left|k\left[\frac{g(c^p)+g(d^p)}{\alpha p}\right]-\frac{A}{p}\right| \le \frac{k}{\alpha}(d^p-c^p)\int_0^1\tau^{p(\frac{\alpha}{k}+1)-1} \left[(1-\tau^p)|g'(c^p)|+\tau^p|g'(d^p)|+\tau^p|g'(c^p)|+(1-\tau^p)|g'(d^p)|\right] = \frac{k}{\alpha}(d^p-c^p)\frac{1}{p(\frac{\alpha}{k}+1)}\left[|g'(c^p)|+|g'(d^p)|\right].$$

On simplification we get our desired inequality (35). □

Remark 2.3. Under certain conditions $p=1$, $k=1$, and $p=k=1$, the above Theorems 2.2. - 2.3. reduce to the results involving $k-$Riemann-Liouville, Katugampola and Riemann-Liouville fractional integral operator respectively.

3 Hermite-Hadamard Fejer type inequalities

In this section we generalize Hermite-Hadamard Fejer type inequalities using $(k-p)$ Riemann-Liouville fractional integrals. Here we define $G(x)=g(x)+g(c+d-x)$ where $g:[c,d]\to\mathbb{R}$ is the given convex function. Hence $G(x)$ is also convex. We also consider certain properties of $G(x)$

1. $G(x)$ is symmetric to $\frac{c+d}{2}$.
2. $G(c)=G(d)=g(c)+g(d)$.
3. $G\left(\frac{c+d}{2}\right)=2g\left(\frac{c+d}{2}\right)$.

Theorem 3.1. (Generalization of Hermite-Hadamard Fejer type inequality). Let $g:[c,d]\to\mathbb{R}$ be a convex function with $c<d$ and $g\in L[c,d]$. Then $G(x)$ is also convex and $G\in L[c,d]$. If $h:[c,d]\to\mathbb{R}$ is non-negative and integrable, then the inequality given below holds true

$$G\left(\frac{c+d}{2}\right)\left[{}_k^{p-1}J_{c+}^{\alpha}h(d)+{}_k^{p-1}J_{d-}^{\alpha}h(c)\right] \leq\left[{}_k^{p-1}J_{c+}^{\alpha}(hG)(d)+{}_k^{p-1}J_{d-}^{\alpha}(hG)(c)\right] \leq\frac{G(c)+G(d)}{2}\left[{}_k^{p-1}J_{c+}^{\alpha}h(d)+{}_k^{p-1}J_{d-}^{\alpha}h(c)\right]. \tag{37}$$

Proof. Let $\tau\in[0,1]$. Consider $u,v\in[c,d],c\geq 0$ which are defined by $u=\tau c+(1-\tau)d$ and $v=(1-\tau)c+\tau d$. We first prove

$$G\left(\frac{c+d}{2}\right)\left[{}_k^{p-1}J_{c+}^{\alpha}h(d)+{}_k^{p-1}J_{d-}^{\alpha}h(c)\right] \leq\left[{}_k^{p-1}J_{c+}^{\alpha}(hG)(d)+{}_k^{p-1}J_{d-}^{\alpha}(hG)(c)\right]. \tag{38}$$

Since g is convex on $[c,d]$, we have

$$g\left(\frac{u+v}{2}\right)\leq\frac{g(u)+g(v)}{2}.$$

$$\implies 2g\left(\frac{c+d}{2}\right)\leq g(\tau c+(1-\tau)d)+g((1-\tau)c+\tau d).$$

$$\implies G\left(\frac{c+d}{2}\right)\leq G((1-\tau)c+\tau d). \tag{39}$$

Multiplying both sides of equation (39) by

$$\frac{((1-\tau)c+\tau d)^{p-1}}{\left(d^p-[(1-\tau)c+\tau d]^p\right)^{1-\frac{\alpha}{k}}}h\left((1-\tau)c+\tau d\right). \tag{40}$$

and integrating with respect to τ over $[0,1]$, we get

$$G\left(\frac{c+d}{2}\right){}_k^{p-1}J_{c+}^{\alpha}h(d)\leq{}_k^{p-1}J_{c+}^{\alpha}(hG)(d). \tag{41}$$

Similarly we get

$$G\left(\frac{c+d}{2}\right){}_k^{p-1}J_{d-}^{\alpha}h(c)\leq{}_k^{p-1}J_{d-}^{\alpha}(hG)(c). \tag{42}$$

Adding equations (41) and (42) we get the inequality (38). We now prove

$$\left[{}_k^{p-1}J_{c+}^{\alpha}(hG)(d)+{}_k^{p-1}J_{d-}^{\alpha}(hG)(c)\right] \tag{43}$$
$$\leq\frac{G(c)+G(d)}{2}\left[{}_k^{p-1}J_{c+}^{\alpha}h(d)+{}_k^{p-1}J_{d-}^{\alpha}h(c)\right]. \tag{44}$$

Since g is convex, then for all $\tau\in[0,1]$ we have

$$g(\tau c+(1-\tau)d)+g((1-\tau)c+\tau d)\leq g(c)+g(d).$$

$$\implies G((1-\tau)c+\tau d)\leq\frac{G(c)+G(d)}{2}. \tag{45}$$

Multiplying both sides of equation (45) by (40) and integrating with respect to τ over $[0,1]$, we have,

$${}_k^{p-1}J_{c+}^{\alpha}(hG)(d)\leq\left(\frac{G(c)+G(d)}{2}\right){}_k^{p-1}J_{c+}^{\alpha}h(d). \tag{46}$$

Similarly we get

$${}_k^{p-1}J_{d-}^{\alpha}(hG)(c)\leq\left(\frac{G(c)+G(d)}{2}\right){}_k^{p-1}J_{d-}^{\alpha}h(c). \tag{47}$$

Adding equations (46) and (47) we get the inequality (43).
From equations (38) and (43) we get the desired inequality (37). □

Remark 3.1. The above inequality (37) in the Theorem 3.1. is also known as Endpoint Hermite-Hadamard Fejer type inequality involving $(k-p)$ Riemann-Liouville fractional integral operator because of the use of endpoints c and d.

Lemma 3.1. Let $g:[c,d]\to\mathbb{R}$ be a differentiable mapping on (c,d) with $0\leq c<d$ and $g'\in L[c,d]$. Then $G(x)$ is also differentiable and $G'\in L[c,d]$. If $h:[c,d]\to\mathbb{R}$ is integrable, then the equality given below holds true

$$\frac{G(c)+G(d)}{2}\left[{}_k^{p-1}J_{c+}^{\alpha}h(d)+{}_k^{p-1}J_{d-}^{\alpha}h(c)\right]-\left[{}_k^{p-1}J_{c+}^{\alpha}(hG)(d)+{}_k^{p-1}J_{d-}^{\alpha}(hG)(c)\right]=\frac{p^{1-\frac{\alpha}{k}}}{2k\Gamma_k(\alpha)}\times\int_c^d\left[\int_c^{\tau}H(x)h(x)dx-\int_{\tau}^{d}H(x)h(x)dx\right]G'(\tau)d\tau, \tag{48}$$

where $\alpha>0,p>0$ and

$$H(x)=\frac{x^{p-1}}{(d^p-x^p)^{1-\frac{\alpha}{k}}}+\frac{x^{p-1}}{(x^p-c^p)^{1-\frac{\alpha}{k}}}. \tag{49}$$

Proof. Consider

$$
\begin{aligned}
I &= \int_c^d \left[\int_c^\tau H(x)h(x)dx - \int_\tau^d H(x)h(x)dx\right] G'(\tau)d\tau \\
&= \int_c^d \int_c^\tau H(x)h(x)dxG'(\tau)d\tau - \int_c^d \int_\tau^d H(x)h(x)dxG'(\tau)d\tau \\
&= I_1 + I_2.
\end{aligned}
$$

Using integration by parts

$$
I_1 = \left[\int_c^\tau H(x)h(x)dxG(\tau)\right]_c^d - \int_c^d G(\tau)H(\tau)H(\tau)d\tau.
$$

$$
\begin{aligned}
\Longrightarrow I_1 =& \frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}} \left[{}_k^{p-1}J_{c+}^{\alpha} h(d) + {}_k^{p-1}J_{d-}^{\alpha} h(c)\right] G(d) \\
&- \frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}} \left[{}_k^{p-1}J_{c+}^{\alpha} (hG)(d) + {}_k^{p-1}J_{d-}^{\alpha} (hG)(c)\right].
\end{aligned} \tag{50}
$$

Similarly

$$
\begin{aligned}
I_2 =& \frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}} \left[{}_k^{p-1}J_{c+}^{\alpha} h(d) + {}_k^{p-1}J_{d-}^{\alpha} h(c)\right] G(c) \\
&- \frac{k\Gamma_k(\alpha)}{p^{1-\frac{\alpha}{k}}} \left[{}_k^{p-1}J_{c+}^{\alpha} (hG)(d) + {}_k^{p-1}J_{d-}^{\alpha} (hG)(c)\right].
\end{aligned} \tag{51}
$$

Adding equations (50) and (51), and multiplying the resultant on both sides by $\frac{p^{1-\frac{\alpha}{k}}}{2k\Gamma_k(\alpha)}$ we get our desired result (48). □

Theorem 3.2. Let $g : [c,d] \to \mathbb{R}$ be a differentiable mapping on (c,d) with $0 \le c < d$ and $g' \in L[c,d]$. Then $G(x)$ is also differentiable and $G' \in L[c,d]$. If $|g'|$ is convex on [c,d] and $h : [c,d] \to \mathbb{R}$ is continuous then the inequality given below holds true

$$
\begin{aligned}
&\frac{G(c)+G(d)}{2} \left[{}_k^{p-1}J_{c+}^{\alpha} h(d) + {}_k^{p-1}J_{d-}^{\alpha} h(c)\right] \\
&- \left[{}_k^{p-1}J_{c+}^{\alpha} (hG)(d) + {}_k^{p-1}J_{d-}^{\alpha} (hG)(c)\right] \\
&\le \frac{||h||_\infty (d-c)}{p^{\frac{\alpha}{k}} \alpha \Gamma_k(\alpha)} \left[|g'(c)| + |g'(d)|\right] \int_0^1 |M(t)|dt,
\end{aligned} \tag{52}
$$

where $\alpha > 0, p > 0, ||h||_\infty = sup_{\tau\in[c,d]} |h(x)|$ and

$$
M(t) = ([(1-\tau)c + \tau d]^p - c^p)^{\frac{\alpha}{k}} - (d^p - [(1-\tau)c + \tau d]^p)^{\frac{\alpha}{k}}. \tag{53}
$$

Proof. From Lemma 3.1. we have

$$
\begin{aligned}
&\frac{G(c)+G(d)}{2} \left[{}_k^{p-1}J_{c+}^{\alpha} h(d) + {}_k^{p-1}J_{d-}^{\alpha} h(c)\right] \\
&- \left[{}_k^{p-1}J_{c+}^{\alpha} (hG)(d) + {}_k^{p-1}J_{d-}^{\alpha} (hG)(c)\right] \\
&= \frac{p^{1-\frac{\alpha}{k}}}{2k\Gamma_k(\alpha)} \\
&\times \int_c^d \left[\int_c^\tau H(x)h(x)dx - \int_\tau^d H(x)h(x)dx\right] G'(\tau)d\tau \\
&\le \frac{p^{1-\frac{\alpha}{k}} sup_{\tau\in[c,d]} |h(x)|}{2k\Gamma_k(\alpha)} \\
&\times \int_c^d \left[\int_c^\tau H(x)dx - \int_\tau^d H(x)dx\right] |G'(\tau)|d\tau.
\end{aligned} \tag{54}
$$

Consider $|G'(\tau)|$ and using the fact that $|g'|$ is convex, we have

$$
\begin{aligned}
|G'(\tau)| &= |g'(\tau) - g'(c+d-\tau)| \\
&\le |g'(\tau)| + g|'(c+d-\tau)| \\
&= \left|g'\left(\frac{d-\tau}{d-c}c + \frac{\tau-c}{d-c}d\right)\right| \\
&+ \left|g'\left(\frac{\tau-c}{d-c}c + \frac{d-\tau}{d-c}d\right)\right| \\
&\le \frac{d-\tau}{d-c}|g'(c)| + \frac{\tau-c}{d-c}|g'(d)| \\
&+ \frac{\tau-c}{d-c}|g'(c)| + \frac{d-\tau}{d-c}|g'(d)| \\
&= |g'(c)| + |g'(d)|.
\end{aligned} \tag{55}
$$

Now consider

$$
\begin{aligned}
&\int_c^\tau H(x)dx - \int_\tau^d H(x)dx \\
&= \int_c^\tau \left[\frac{x^{p-1}}{(d^p - x^p)^{1-\frac{\alpha}{k}}} + \frac{x^{p-1}}{(x^p - c^p)^{1-\frac{\alpha}{k}}}\right] dx \\
&- \int_\tau^d \left[\frac{x^{p-1}}{(d^p - x^p)^{1-\frac{\alpha}{k}}} + \frac{x^{p-1}}{(x^p - c^p)^{1-\frac{\alpha}{k}}}\right] dx \\
&= \frac{2k}{\alpha p}\left[(\tau^p - c^p)^{\frac{\alpha}{k}} - (d^p - \tau^p)^{\frac{\alpha}{k}}\right].
\end{aligned} \tag{56}
$$

Using equations (55) and (56) in (54) we get

$$\frac{G(c)+G(d)}{2}\left[{}_{k}^{p-1}J_{c+}^{\alpha}h(d)+{}_{k}^{p-1}J_{d-}^{\alpha}h(c)\right]$$
$$-\left[{}_{k}^{p-1}J_{c+}^{\alpha}(hG)(d)+{}_{k}^{p-1}J_{d-}^{\alpha}(hG)(c)\right]$$
$$\leq \frac{p^{1-\frac{\alpha}{k}}||h||_{\infty}}{2k\Gamma_k(\alpha)}\int_c^d\left[\frac{2k}{\alpha p}\left[(\tau^p-c^p)^{\frac{\alpha}{k}}-(d^p-\tau^p)^{\frac{\alpha}{k}}\right]\right]d\tau$$
$$\times\left(|g'(c)|+|g'(d)|\right)$$
$$=\frac{||h||_{\infty}}{p^{\frac{\alpha}{k}}\alpha\Gamma_k(\alpha)}\int_c^d\left[(\tau^p-c^p)^{\frac{\alpha}{k}}-(d^p-\tau^p)^{\frac{\alpha}{k}}\right]d\tau$$
$$\times\left(|g'(c)|+|g'(d)|\right)$$
$$=\frac{||h||_{\infty}(d-c)}{p^{\frac{\alpha}{k}}\alpha\Gamma_k(\alpha)}$$
$$\int_0^1\left([((1-\tau)c+\tau d)^p-c^p]^{\frac{\alpha}{k}}-[d^p-((1-\tau)c+\tau d)^p]^{\frac{\alpha}{k}}\right)$$
$$d\tau\times\left(|g'(c)|+|g'(d)|\right).$$

Hence we get our desired result (52). □

Remark 3.2. Under certain conditions $p=1$, $k=1$, and $p=k=1$, the above Theorems 3.1. - 3.2. reduce to the results involving $k-$Riemann-Liouville, Katugampola and Riemann-Liouville fractional integral operator respectively.

4 Weighted Hardy type inequalities

In this section we obtain certain Weighted Hardy type inequalities for $(k-p)$ Riemann-Liouville fractional integral.

Theorem 4.1. Let $g\in L[c,d], \alpha\geq 0$ and $p\neq -1$. Let ω be a weight function on (c,d). Assume that the function $y\to \omega(y)\frac{\alpha(p+1)(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}$ is integrable on (c,d). Then for each $\tau\in(c,d)$ define

$$\upsilon(\tau)=\alpha(p+1)\int_{\tau}^{d}\omega(y)\frac{(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\tau^p dy<\infty. \tag{57}$$

If $\Psi:(0,\infty)\to\mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_c^d\omega(y)\Psi$$
$$\times\left(\left|\frac{\alpha(p+1)}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\int_c^y(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p g(\tau)d\tau\right|\right)dy$$
$$\leq\int_c^d\upsilon(\tau)\Psi(|g(\tau)|)\,d\tau, \tag{58}$$

holds for all measurable functions $g:(c,d)\to\mathbb{R}$.

Proof. Using Theorem 1.1. with $\delta_1=\delta_2=(c,d), d\delta_1(y)=dy, d\delta_2(y)=d\tau$

$$m(y,\tau)=\begin{cases}\frac{(p+1)^{1-\frac{\alpha}{k}}(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{k\Gamma_k(\alpha)}, c\leq\tau\leq y\\ 0, y<\tau\leq d\end{cases} \tag{59}$$

$$M(y)$$
$$=\frac{1}{k\Gamma_k(\alpha)}\int_c^d(p+1)^{1-\frac{\alpha}{k}}(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p d\tau$$
$$=\frac{1}{\alpha\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}}(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}, \tag{60}$$

and

$$A_k g(y)=\frac{\alpha(p+1)}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\int_c^y(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p g(\tau)d\tau, \tag{61}$$

we get our desired inequality (58). □

Corollary 4.1. If in particular, we take the weight function to be $\omega(y)=y^p(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}$, $\upsilon(\tau)$ is calculated as $\upsilon(\tau)=k\tau^p(d^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}}$, then the inequality (58) becomes

$$\int_c^d y^p(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}$$
$$\times\Psi\left(\left|\frac{\alpha(p+1)}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\frac{k\Gamma_k(\alpha)}{(p+1)^{1-\frac{\alpha}{k}}}{}_{k}^{p}J_c^{\alpha}g(y)\right|\right)dy$$
$$\leq\int_c^d\upsilon(\tau)\Psi(|g(\tau)|)\,d\tau. \tag{62}$$

If $s>1$ and function $\Psi:(0,\infty)\to\mathbb{R}$ is defined by $\Psi(y)=y^s$, then we can write inequality (62) as

$$\left[\alpha k\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}\right]^s$$
$$\times\int_c^d y^p(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}(1-s)}\left|{}_{k}^{p}J_c^{\alpha}g(y)\right|^s dy$$
$$\leq\int_c^d k\tau^p(d^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}}|g(\tau)|^s d\tau. \tag{63}$$

Since $y\in(c,d)$ and $\frac{\alpha}{k}(1-s)<0$, LHS of equation (63) can be written as

$$\left[\alpha k\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}\right]^s$$
$$\times\int_c^d y^p(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}(1-s)}\left|{}_{k}^{p}J_c^{\alpha}g(y)\right|^s dy$$
$$\geq\left[\alpha k\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}\right]^s c^p(d^{p+1}-c^{p+1})^{\frac{\alpha}{k}(1-s)} \tag{64}$$
$$\times\int_c^d\left|{}_{k}^{p}J_c^{\alpha}g(y)\right|^s dy,$$

and RHS of equation (63) can be written as

$$\int_c^d k\tau^p (d^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}} |g(\tau)|^s d\tau \leq kd^p (d^{p+1}-c^{p+1})^{\frac{\alpha}{k}} \int_c^d |g(\tau)|^s d\tau. \quad (65)$$

Using inequalities (64) and (65) in (63) we get

$$\int_c^d |{}^p_k J_c^\alpha g(y)|^s dy \leq \frac{d^p}{c^p}\left[\frac{(d^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}}\right]^s \int_c^d |g(\tau)|^s d\tau.$$

Taking $\frac{1}{s}$ in power on both sides, we have

$$||{}^p_k J_c^\alpha g(y)||_s \leq \mathfrak{Q}||g||_s, \quad (66)$$

where $\mathfrak{Q} = \left(\frac{d}{c}\right)^{\frac{p}{s}} \frac{(d^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}{\alpha\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}}$.

Theorem 4.2. Let ω be a weight function on (c,d). ${}^p_k J_c^\alpha g$ is the $(k-p)$ Riemann-Liouville fractional integral and ${}^p_k J_c^\alpha g_2(y) > 0$. Assume that the function $y \to \omega(y)\frac{g_2(\tau)(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{{}^p_k J_c^\alpha g_2(y)}$ is integrable on (c,d). Define φ on (c,d) by

$$\varphi(\tau) = \frac{(p+1)^{1-\frac{\alpha}{k}} g_2(\tau)}{k\Gamma_k(\alpha)} \int_\tau^d \omega(y)\frac{(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{{}^p_k J_c^\alpha g_2(y)} dy < \infty. \quad (67)$$

If $\Psi : (0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_c^d \omega(y)\Psi\left(\left|\frac{{}^p_k J_c^\alpha g_1(y)}{{}^p_k J_c^\alpha g_2(y)}\right|\right) dy \leq \int_c^d \varphi(\tau)\Psi\left(\left|\frac{g_1(\tau)}{g_2(\tau)}\right|\right) d\tau, \quad (68)$$

holds for all measurable functions $g_i \in L_1(c,d), i = 1,2$.

Proof. Using Theorem 1.2. with $\delta_1 = \delta_2 = (c,d)$, $d\delta_1(y) = dy, d\delta_2(y) = d\tau$, we get our desired inequality (67). □

Theorem 4.3. Let $0 < r \leq s < \infty$ and $g \in L_{1,p}[c,d]$. ${}^p_k J_c^\alpha g$ is the $(k-p)$ Riemann-Liouville fractional integral of order $\alpha > 0$ and $p \neq -1$. Let ω be a weight function and assume that $y \to \omega(y)\left(\frac{(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\right)^{\frac{s}{r}}$ is integrable on (c,d). For each fixed $\tau \in (c,d)$ define

$$\zeta(\tau) = \frac{\alpha}{k}(p+1)\left[\int_\tau^d \omega(y)\left(\frac{(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\right)^{\frac{s}{r}} dy\right]^{\frac{s}{r}} < \infty. \quad (69)$$

If $\Psi : (0,\infty) \to \mathbb{R}$ is a non-negative convex function, then the inequality

$$\left[\int_c^d \omega(y)\left(\Psi\left(\frac{k\alpha\Gamma_k(\alpha)(p+1)^{\frac{\alpha}{k}}\,{}^p_k J_c^\alpha g(y)}{(y^{p+1}-c^{p+1})^{\frac{\alpha}{k}}}\right)\right)^{\frac{s}{r}} dy\right]^{\frac{1}{s}} \leq \left[\int_c^d \zeta(\tau)\Psi(g(\tau))d\tau\right]^{\frac{1}{r}}, \quad (70)$$

holds for all measurable functions $g : [c,d] \to \mathbb{R}$.

Proof. Using Theorem 1.3. with $\delta_1 = \delta_2 = (c,d)$, $d\delta_1(y) = dy, d\delta_2(y) = d\tau, m(y,\tau), M(y)$ and $A_k g(y)$ are given by (59), (60), (61) we get our desired inequality (70). □

Theorem 4.4. Let $g \in L_{1,p}[c,d]$, ${}^p_k J_c^\alpha g$ is the $(k-p)$ Riemann Liouville fractional integral of order $\alpha > 0$ and $p \neq -1$ and ${}^p_k J_c^\alpha g_2(y) > 0$ for every $y \in (c,d)$. Let ω be a weight function on (c,d). Then

$$\rho(\tau) = \frac{(p+1)^{1-\frac{\alpha}{k}} g_2(\tau)}{k\Gamma_k(\alpha)} \int_\tau^d \omega(y)\frac{(y^{p+1}-\tau^{p+1})^{\frac{\alpha}{k}-1}\tau^p}{{}^p_k J_c^\alpha g_2(y)} dy < \infty. \quad (71)$$

If $\Psi : (0,\infty)\times(0,\infty) \to \mathbb{R}$ is a convex and increasing function, then the inequality

$$\int_c^d \omega(y)\Psi\left(\left|\frac{{}^p_k J_c^\alpha g_1(y)}{{}^p_k J_c^\alpha g_2(y)}\right|, \left|\frac{{}^p_k J_c^\alpha g_3(y)}{{}^p_k J_c^\alpha g_2(y)}\right|\right) dy \quad (72)$$

$$\leq \int_c^d \rho(\tau)\Psi\left(\left|\frac{g_1(\tau)}{g_2(\tau)}\right|, \left|\frac{g_3(\tau)}{g_2(\tau)}\right|\right) d\tau, \quad (73)$$

holds for all measurable functions $g_i \in L_1(c,d), i = 1,2,3$.

Proof. Using Theorem 1.4. with $\delta_1 = \delta_2 = (c,d)$, $d\delta_1(y) = dy, d\delta_2(y) = d\tau$ we get our desired inequality (72). □

Remark 4.1. Under certain conditions $p = 0$, $k = 1$, and $p = 0, k = 1$, the above Theorems 4.1. - 4.4. reduce to the results involving k-Riemann-Liouville, Katugampola and Riemann-Liouville fractional integral operator respectively.

5 Conclusion

In our course of study, we have obtained various Hermite-Hadamard, Hermite-Hadamard Fejer, weighted Hardy type inequalities using $(k-p)$ Riemann-Liouville fractional integral operator. Under certain special conditions these inequalities reduce to some known inequalities involving the Riemann-Liouville, k-Riemann-Liouville and Katugampola fractional integral (see, e.g. [29,30]). Various authors have worked on the inequalities for different types of functions and fractional operators. Rashid et al. [31] discussed Hermite-Hadamard and Ostrowski type inequalities for n-polynomials, s-type convex functions by employing k-fractional integral operators and studied the quadrature rules that are helpful in fractal theory, optimization and machine learning. Hermite-Hadamard inequalities for the differentiable exponentially convex and exponential quasi-convex functions, which were applied to numerical analysis and statistics, have also been discussed in the recent years [32]. Inequalities for the p-th order differentiation useful in the Banach Spaces are also obtained [33]. Further, Grüss type inequalities for the generalized k-fractional integral operator are discussed and applied in the real world mathematical problems [34,35]. As future scope, these results can be extended using $(k-p)$ Riemann-Liouville fractional integral operator and applications can be found in the fractal theory, machine learning, numerical analysis, statistics and various others.

Conflict of Interest The authors declare that they have no conflict of interest

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Ankita Chandola is a Research Scholar in Department of Mathematics, Amity Institute of Applied Sciences, Amity University Uttar Pradesh India. Her research interests are Special functions, integral transforms and Fractional Calculus. She has published research articles in reputed international journals of mathematical and engineering sciences.

Ritu Agarwal is Assistant Professor of Mathematics at Malaviya National Institute of Technology Jaipur Rajasthan India. She has done her PhD from Department of Mathematics, University of Rajasthan Jaipur India. Her Research interests are Bicomplex analysis, Geometric function theory and Fractional calculus. She has published many research papers and books in reputed international and national journals. She is a part of many membership programs such as International Association of Engineers, Indian Mathematical Society, Ramanujan Mathematical Society, Indian Science Congress Association among many others. She is a reviewer in many national and international journals. She has chaired and organized many national, international level workshops and conferences. She has delivered talks in various national, international level workshops and conferences. She has also worked on various granted research projects, supervised many PhD's.

Rupakshi Mishra Pandey is Associate Professor of Mathematics at Amity Institute of Applied Sciences, Amity University Uttar Pradesh India. She has done her PhD from Department of Mathematics, University of Rajasthan Jaipur India. Her Research interests are Special functions, Integral transforms and Fractional calculus. She has published many research papers in reputed international and national journals. She is a member of many Mathematical societies of international and national levels. She has chaired and organized many national, international level workshops and conferences.