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Bayesian Estimation from Exponentiated Frechet Model using MCMC Approach based on Progressive Type-II Censoring Data

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Abstract: Based on progressively Type-II censored samples, the maximum likelihood (ML) and Bayes estimators for the parameters as well as some lifetime parameters (reliability and hazard functions) of the exponentiated Frchet (EF) distribution are derived. The confidence interval of the parameters are obtained based on an asymptotic distribution of maximum likelihood estimators. Further; we consider delta method and bootstrap method to construct approximate confidence intervals for reliability and hazard functions. The Bayes estimators of the unknown parameters cannot be obtained in closed form. Markov chain Monte Carlo (MCMC) method has been used to compute the approximate Bayes estimates and also to construct the highest posterior density (HPD) credible intervals. The results of Bayes estimators are obtained under both the balanced squared error loss (BSEL) and balanced linear-exponential (BLINEX) loss. A practical example consisting of data represents a relief time of arthritic patients reported by Wu et al. [1] was used for illustration, Finally; some numerical results using simulation study concerning different sample sizes and different progressive censoring schemes were reported.

Keywords: Exponentiated Frchet distribution; Progressively Type-II censored samples; Approximate confidence intervals; Bayesian and non-Bayesian estimations; Gibbs and Metropolis sampler; Bootstrap; Graphical method; Monte Carlo simulation.

1 Introduction

The extrem value distribution is well suited to characterize random variables of large features. Thus it is important for modeling the statistical behavior of materials properties for a variety of engineering applications. It has been used widely in meteorology, hydrology, ocean engineering, pollution studies, strength of materials. It essentially involves three types of extreme value distributions, Types I, II and III. The Fréchet distribution is one of three kinds of general extreme value distribution (the Gumbel (TypeI), Fréchet (Type II) and Weibull (Type III)). The Fréchet distribution has applications ranging from accelerated life testing through to earthquakes, floods, horse racing, rainfall, queues in supermarkets, sea currents, wind speeds, and track race records. Kotz and Nadarajah [2] gave some applications in their book. Exponentiated Fréchet (EF) distribution has been introduced by Nadarajah and Kotz [3] as a generalization of the standard Frèchet distribution. The probability density function (pdf) and cumulative distribution function (cdf) for the two parameters EF distribution (EF(α,θ)), respectively, are

$$f(x)=\theta\alpha x^{-(\alpha+1)}\exp(-x^{-\alpha})(1-\exp(-x^{-\alpha}))^{\theta-1}, x>0, \tag{1}$$

$$F(x)=1-(1-\exp(-x^{-\alpha}))^{\theta},\ x>0,\ \alpha,\theta>0, \tag{2}$$

where α and θ are shape parameters.
The reliability and hazard functions at some time t, are given respectively, by

$$S(t)=\bar{F}(t)=(1-\exp(-t^{-\alpha}))^{\theta}, \tag{3}$$

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$$H(t) = \theta\alpha t^{-(\alpha+1)}\exp(-t^{-\alpha})(1-\exp(-t^{-\alpha}))^{-1}, t > 0. \quad (4)$$

It is common practice in a life testing experiment to terminate the experiment before all the units have failed. The observations obtained in such situation are called censored samples. Type-I and type-II censoring schemes are the two most popular censoring schemes which have been used in practice, see for example, Singh et al. [4] and Kundu [5]. Unfortunately, none of these censoring schemes allows the removal of any experimental units during the experiment. Type-I and type-II progressive censoring schemes allow the removal of experimental units during the experiment. Due to this flexibility progressive censoring scheme has received considerable attention in the applied statistics literature for the last few years. A type-II progressively censored experiment can be briefly described as follows. Consider an experiment, in which n identical units are put on a test and suppose $m < n$ is fixed before the experiment. At the time of the first failure say $X_{1:m:n}$, R_1 surviving units are randomly removed. Similarly, at the time of the second failure, say $X_{2:m:n}$, R_2 surviving units are removed and so on. The test continues until the $m-$th failure say $X_{m:m:n}$ at which time all the remaining $R_m = n-m-R_1-...-R_{m-1}$ units are removed. In this censoring scheme, R_i, and m are pre-fixed. The resulting m ordered values which we denote by $X_{1:m:n}X_{2:m:n},\ldots,X_{m:m:n}$, are referred to as progressively Type-II censored order statistics. As a special case, this scheme includes the conventional Type-II right censoring scheme (when $R_1 = R_2 = ... = R_{m-1} = 0$ and $R_m = n-m$) and complete sampling scheme (when $n = m$ and $R_1 = R_2 = ... = R_m = 0$). For more details about progressive censoring schemes, the readers may refer to Balakrishnan and Aggarwala [6] and Balakrishnan [7]. Some recent studies on progressive Type II censoring have been carried out by many authors including Rastogi and Tripathi [8], Ahmed [9], Huang and [10], and Wu [11].

Although extensive work has been done on the statistical inference of the unknown parameters of different parametric models based on progressively censored observation in the frequentist setup, not that much work has been done in the Bayesian inference, specially for exponentiated Fréchet (EF) distribution.

This paper considers the progressive Type-II right censoring scheme, when the lifetime follows two parameters EF distribution. First we provide the maximum likelihood estimators of the unknown parameters. It is observed that the maximum likelihood estimators do not have explicit forms. They can be obtained by solving a non-linear equations. Because the exact distributions of the MLE are not easy derived, we propose to use the asymptotic distributions of the MLE to construct the approximate confidence intervals. We also, propose two bootstrap confidence intervals. The Bayes estimates are obtained under the assumptions of independent gamma priors for the two shape parameters. We use the Gibbs sampling procedure to compute the Bayes estimates and the highest posterior density (HPD) credible intervals. Different methods are compared using Monte Carlo simulations and for illustrative purposes we analyze one real data set.

The rest of the paper is organized as follows: In Section 2, the MLE of the unknown parameter are obtained. Different confidence intervals are presented in Section 3 and 4. Bayesian analysis is provided in Section 5. One real data set has been analyzed in Section 6. In Section 7 we provide a simulation study in order to give an assessment of the performance of the estimation methods. Finally we conclude the paper in Section 8.

2 Maximum likelihood estimation (MLE)

Let $(X_{1:m:n}, X_{2:m:n},\ldots,X_{m:m:n})$, $(1 \le m \le n)$ be a progressively type II censored sample observed from a life test involving n units taken from the EF(α,θ) distribution and $(R_1,R_2,...,R_m)$ being the censoring scheme. Then the joint probability density function of $(X_{1:m:n}, X_{2:m:n},\ldots,X_{m:m:n})$, see Aggarwala and Balakrishnan [12], is

$$f_{X_{1,m,n},\ldots,X_{m,m,n}}(x_{1,m,n},\ldots,x_{m,m,n}) = A\prod_{i=1}^{m} f_{X_{i,m,n}}(x_{i,m,n})\left[1-F_{X_{i,m,n}}(x_{i,m,n})\right]^{R_i}, \quad (5)$$

where $A = n(n-1-R_1)(n-2-R_1-R_2)...(n-\sum_{i=1}^{m-1}(R_i+1))$.

Utilizing Eqs. (1) and (2), the likelihood function of α and θ is given by

$$L(\alpha,\theta|\underline{x}) = A\prod_{i=1}^{m}\alpha\theta x_{i,m,n}^{-(1+\alpha)}\exp(-x_{i,m,n}^{-\alpha})(1-\exp(-x_{i,m,n}^{-\alpha}))^{\theta(1+R_i)-1}, \quad (6)$$

and the corresponding log likelihood function is

$$\ell = \log L(\alpha,\theta|\underline{x}) = \log A + m\log\alpha + m\log\theta + \sum_{i=1}^{m}(\theta(1+R_i)-1)\log(1-\exp(-x_{i,m,n}^{-\alpha}))$$
$$-\sum_{i=1}^{m}\left[x_{i,m,n}^{-\alpha} + (\alpha+1)\log x_{i,m,n}\right]. \quad (7)$$

Consequently, likelihood equations of α and θ are obtained as

$$\frac{\partial \ell}{\partial \alpha}=\frac{m}{\alpha}+\sum_{i=1}^{m}\left(x_{i,m,n}^{-\alpha}-1\right)\log x_{i,m,n}-\sum_{i=1}^{m}\frac{(\theta(1+R_i)-1)x_{i,m,n}^{-\alpha}\exp(-x_{i,m,n}^{-\alpha})\log x_{i,m,n}}{\left[1-\exp(-x_{i,m,n}^{-\alpha})\right]}=0, \tag{8}$$

and

$$\frac{\partial \ell}{\partial \theta}=\frac{m}{\theta}+\sum_{i=1}^{m}(1+R_i)\log(1-\exp(-x_{i,m,n}^{-\alpha}))=0 \tag{9}$$

It follows, from Equation (9), that

$$\hat{\theta}_{ML}=-\frac{m}{\sum_{i=1}^{m}(1+R_i)\log(1-\exp(-x_{i,m,n}^{-\alpha}))}, \tag{10}$$

and $\hat{\alpha}_{ML}$ is the solution of

$$\frac{m}{\alpha}+\sum_{i=1}^{m}\left(x_{i,m,n}^{-\alpha}-1\right)\log x_{i,m,n}-\sum_{i=1}^{m}\frac{\left\{\left(-\frac{m(1+R_i)}{\sum_{i=1}^{m}(1+R_i)\log(1-\exp(-x_{i,m,n}^{-\alpha}))}\right)-1\right\}x_{i,m,n}^{-\alpha}\exp(-x_{i,m,n}^{-\alpha})\log x_{i,m,n}}{\left(1-\exp(-x_{i,m,n}^{-\alpha})\right)}=0. \tag{11}$$

Newton-Raphson iteration is employed to solve (11) to obtain the estimate $\hat{\alpha}_{ML}$, once we obtain $\hat{\alpha}_{ML}$, the maximum likelihood estimators of θ ($\hat{\theta}_{ML}$) can be obtained from (10). The initial values for the parameters are obtain by using graphical techniques, see Balakrishnan and Kateri [13]. We rewrite Equation (11) in the form.

$$\frac{1}{\alpha}=\frac{1}{m}\left(\sum_{i=1}^{m}\left(1-x_{i,m,n}^{-\alpha}\right)\log x_{i,m,n}+\sum_{i=1}^{m}\frac{\left\{\left(-\frac{m(1+R_i)}{\sum_{i=1}^{m}(1+R_i)\log(1-\exp(-x_{i,m,n}^{-\alpha}))}\right)-1\right\}x_{i,m,n}^{-\alpha}\exp(-x_{i,m,n}^{-\alpha})\log x_{i,m,n}}{\left(1-\exp(-x_{i,m,n}^{-\alpha})\right)}\right). \tag{12}$$

We denote the right-hand side of Equation (12) by $H_1(\alpha,x)$, and we can show that, for a given sample x_i, $i=1,2,\ldots,m$ $H_1(\alpha,x)$ is a monotone increasing function of with a finite and positive limit as $\alpha\to\infty$. Since $\frac{1}{\alpha}$ is strictly decreasing with a right limit ∞ at 0, it would then follow that the plots of $\frac{1}{\alpha}$ and $H_1(\alpha,x)$ would intersect exactly once at the value of α, and the resulting value can be used as a starting value for the Newton–Raphson iterative method. Then, using invariance property of maximum likelihood estimation, the MLEs of the reliability and hazard functions are obtained, respectively, from (3) and (4) after replacing α and θ by their MLEs $\hat{\alpha}_{ML}$ and $\hat{\theta}_{ML}$ as

$$\hat{S}_{ML}(t)=(1-\exp(-t^{-\hat{\alpha}_{ML}}))^{\hat{\theta}_{ML}}, \tag{13}$$

and

$$\hat{H}_{ML}(t)=\hat{\alpha}_{ML}\hat{\theta}_{ML}t^{-(\hat{\alpha}_{ML}+1)}\exp(-t^{-\hat{\alpha}_{ML}})(1-\exp(-t^{-\hat{\alpha}_{ML}}))^{-1}. \tag{14}$$

3 Variances and covariances of the MLE

We obtain approximate confidence intervals (CI) of the parameters α and θ based on the asymptotic distribution of the maximum likelihood estimator of the parameters. The asymptotic variances and covariances of the MLE for parameters α and θ are given by the elements of the inverse of the Fisher information matrix. The observed information matrix of α and θ, denoted by $\mathbf{I}^{-1}$ is

$$\mathbf{I}^{-1}=\begin{bmatrix}-\frac{\partial^2 L}{\partial \alpha^2} & -\frac{\partial^2 L}{\partial\alpha\partial\theta}\\ -\frac{\partial^2 L}{\partial\theta\partial\alpha} & -\frac{\partial^2 L}{\partial\theta^2}\end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\theta})}=\begin{bmatrix}var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\theta})\\ cov(\hat{\alpha},\hat{\theta}) & var(\hat{\theta})\end{bmatrix}, \tag{15}$$

where, the second partial derivatives of the log-likelihood function are

$$\frac{\partial^2 \ell}{\partial \theta^2} = -\frac{m}{\theta^2}, \tag{16}$$

$$\frac{\partial^2 L}{\partial \theta \partial \alpha} = \frac{\partial^2 L}{\partial \alpha \partial \theta} = \sum_{i=1}^{m} \frac{\exp(-x_{i,m,n}^{-\alpha}) \log(\frac{1}{(x_{i,m,n})})(1+R_i) x_{i,m,n}^{-\alpha}}{1-\exp(-x_{i,m,n}^{-\alpha})}, \tag{17}$$

and

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{m}{\alpha^2} - \sum_{i=1}^{m} x_{i,m,n}^{-\alpha} \log^2 x_{i,m,n} - \sum_{i=1}^{m} \frac{(\theta(1+R_i)-1)\, u(x_{i,m,n},\alpha)}{\left[1-\exp(-x_{i,m,n}^{-\alpha})\right]^2}, \tag{18}$$

where

$$u(x_{i,m,n},\alpha) = x_{i,m,n}^{-\alpha} \exp(-x_{i,m,n}^{-\alpha}) \log^2 x_{i,m,n} \left[\left(x_{i,m,n}^{-\alpha}-1\right)\left[1-\exp(-x_{i,m,n}^{-\alpha})\right] + x_{i,m,n}^{-\alpha}\right]$$

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for parameters. Therefore, $(1-\gamma)100\%$ confidence intervals for parameters θ and α become

$$\hat{\theta} \mp Z_{\gamma/2}\sqrt{var(\hat{\theta})} \quad \text{and} \quad \hat{\alpha} \mp Z_{\gamma/2}\sqrt{var(\hat{\alpha})}, \tag{19}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.
Furthermore; to construct the asymptotic confidence interval of the reliability and hazard functions, we need to find $Var(\hat{S}(t))$ and $Var(\hat{H}(t))$. In order to find the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$, we use the delta method see Greene [14]. The delta method is a general approach for computing confidence intervals for functions of maximum likelihood estimates. It takes a function that is too complex for analytically computing the variance, creates a linear approximation of that function and then computes the variance of the simpler linear function that can be used for large sample inference, for details, see Greene [14]. Let

$$\acute{G}_1 = \left(\frac{\partial S(t)}{\partial \alpha}, \frac{\partial S(t)}{\partial \theta}\right) \quad \text{and} \quad \acute{G}_2 = \left(\frac{\partial H(t)}{\partial \alpha}, \frac{\partial H(t)}{\partial \theta}\right), \tag{20}$$

where

$$\frac{\partial S(t)}{\partial \alpha} = -\theta(1-\exp(-t^{-\alpha}))^{\theta-1}\exp(-t^{-\alpha})t^{-\alpha}\log(t), \frac{\partial S(t)}{\partial \theta} = (1-\exp(-t^{-\alpha}))^{\theta}\log(1-\exp(-t^{-\alpha})), \tag{21}$$

and

$$\frac{\partial H(t)}{\partial \alpha} = \frac{t^{-(1+2\alpha)}\theta[t^{\alpha}-(t^{\alpha}-1)\alpha\log(t)]}{\exp(-t^{-\alpha})-1} + \frac{\exp(-2t^{-\alpha})t^{-(1+2\alpha)}\theta\alpha\log(t)}{(1-\exp(-t^{-\alpha}))^2}, \frac{\partial H(t)}{\partial \theta} = \frac{\alpha t^{-\alpha-1}\exp(-t^{-\alpha})}{1-\exp(-t^{-\alpha})}. \tag{22}$$

Then the approximate estimates of $Var(\hat{S})$and $Var(\hat{H})$ are given, respectively, by

$$\widehat{Var}(\hat{S}) \simeq \left[\acute{G}_1 \mathbf{I}^{-1} G_1\right]_{(\alpha,\beta)=(\hat{\alpha}_{ML},\hat{\beta}_{ML})} \text{ and } \widehat{Var}(\hat{H}) \simeq \left[\acute{G}_2 \mathbf{I}^{-1} G_2\right]_{(\alpha,\beta)=(\hat{\alpha}_{ML},\hat{\beta}_{ML})}. \tag{23}$$

$$\text{Thus, } \frac{\left(\hat{S}(t)-S(t)\right)}{\sqrt{\widehat{Var}(\hat{S})}} \sim N(0,1) \text{ and } \frac{\left(\hat{H}(t)-H(t)\right)}{\sqrt{\widehat{Var}(\hat{H})}} \sim N(0,1)$$

asymptotically. Theses results yields the approximate confidence intervals for $S(t)$ and $H(t)$ as

$$\hat{S}(t) \mp Z_{\frac{\gamma}{2}}\sqrt{\widehat{Var}(\hat{S})} \text{ and } \hat{H}(t) \mp Z_{\frac{\gamma}{2}}\sqrt{\widehat{Var}(\hat{H})}. \tag{24}$$

4 Bootstrap confidence intervals

It is evident that the confidence intervals based on the asymptotic results do not perform very well for small sample size. So, we propose two confidence intervals based on the parametric bootstrap methods: (i) percentile bootstrap method (we call it Boot-p) based on the idea of Efron [15] and (ii) bootstrap-t method (we refer to it as Boot-t) based on the idea of Hall [16]. We illustrate briefly how to estimate C.I.'s of α, θ, $S(t)$ and $H(t)$, using both methods.

(i) Boot-p method

1. Generate a bootstrap sample of size m, $\{x_1^*,...,x_m^*\}$ from EF$(\hat{\alpha},\hat{\theta})$ by using $\{x_1,...,x_m\}$. Based on $\{x_1^*,...,x_m^*\}$, compute the bootstrap estimate of α, θ, $S(t)$ and $H(t)$, say $\hat{\alpha}^*$, $\hat{\theta}^*$, $\hat{S}^*(t)$ and $\hat{H}^*(t)$, using (10-14)
2. Repeat step1 B times, to obtain $\hat{\alpha}^{*(i)}$, $\hat{\theta}^{*(i)}$, $\hat{S}^{*(i)}(t)$ and $\hat{H}^{*(i)}(t)$, $i=1,2,...,B$.
3. For $i=1,2,...,B$, arrange $\hat{\alpha}^{*(i)}$, $\hat{\theta}^{*(i)}$, $\hat{S}^{*(i)}(t)$ and $\hat{H}^{*(i)}(t)$, in ascending order and obtain $\hat{\alpha}^{*[i]}$, $\hat{\theta}^{*[i]}$, $\hat{S}^{*[i]}(t)$ and $\hat{H}^{*[i]}(t)$.

The approximate $100(1-\gamma)\%$ confidence interval of $g=(\alpha,\theta,S(t),H(t))$, is given by

$$\left(\hat{g}_{Bp}^{*[B\,\gamma/2]},\hat{g}_{Bp}^{*[B\,(1-\gamma/2)]}\right), \tag{25}$$

(ii) Boot-t method

1. Same as in Boot-p method, first generate bootstrap sample $\{x_1^*,...,x_m^*\}$.
2. Based on $\{x_1^*,...,x_m^*\}$ compute the bootstrap estimate of α, θ, $S(t)$ and $H(t)$ using 10- 14, say $\hat{\alpha}^*$, $\hat{\theta}^*$, $\hat{S}^*(t)$ and $\hat{H}^*(t)$ and following statistics

$$T_1^*=\frac{\sqrt{m}(\widehat{g}^*-\widehat{g})}{\sqrt{Var(\widehat{g}^*)}},\quad \text{where } g=\alpha,\theta,S(t),H(t). \tag{26}$$

3. Repeat step 2, B times.
4. Let $K_i(x)=P(T_i^*\leq x)$ be the cumulative distribution function of T_i^* , $i=1,2,3,4$. For a given x define

$$\widehat{g}_{Bt}(x)=\widehat{g}+m^{-1/2}\sqrt{Var(\widehat{g})}K_i^{-1}(x), i=1,2,3,4. \tag{27}$$

The approximate $100(1-\gamma)\%$ confidence interval of $g=\alpha,\ \theta, S(t)$ and $H(t)$ are given by

$$\left[\widehat{g}_{Bt}(\frac{\gamma}{2}),\widehat{g}_{Bt}(1-\frac{\gamma}{2})\right]. \tag{28}$$

5 Bayes Estimation

This section presents Bayes estimates of the parameters θ, α, $S(t)$ and $H(t)$. It is assumed here that the parameters θ and α are independent and follow the gamma prior distributions. we consider these prior because, it is flexible in nature and mathematical ease. We can obtain non-informative prior for gamma prior by taking the value of hyper-parameters are equal to zero. Therefore, the prior density functions of θ and α becomes

$$\pi_1(\theta|a_1,b_1)=\frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta},\ \theta>0, \tag{29}$$

and

$$\pi_2(\alpha|a_2,b_2)=\frac{b_2^{a2}}{\Gamma(a_2)}\alpha^{a_2-1}e^{-b_2\alpha},\ \alpha>0. \tag{30}$$

Here, a_1, b_1, a_2 and b_2 are chosen to reflect prior knowledge about α and θ.
Based on (29) and (30) the joint posterior density of θ and α given the data is

$$\begin{aligned} q(\theta,\alpha|\underline{x}) &= \frac{\ell(\underline{x}|\theta,\alpha)\times\pi_1(\theta|a_1,b_1)\times\pi_2(\alpha|a_2,b_2)}{\int_0^\infty\int_0^\infty \ell(\underline{x}|\theta,\alpha)\times\pi_1\,(\theta|a_1,b_1)\times\pi_2(\alpha|a_2,b_2)d\theta d\alpha}, \\ &= k\theta^{m+a_1-1}\alpha^{m+a_2-1}e^{[-b_1\theta-b_2\alpha]}\prod_{i=1}^{m}x_i^{-(1+\alpha)}e^{[-x_i^{-\alpha}]}\left(1-e^{[-x_i^{-\alpha}]}\right)^{\theta(1+R_i)-1}, \end{aligned} \tag{31}$$

where

$$k^{-1}=\int_0^\infty\int_0^\infty \theta^{m+a_1-1}\alpha^{m+a_2-1}e^{[-b_1\theta-b_2\alpha]}\prod_{i=1}^{m}x_i^{-(1+\alpha)}e^{[-x_i^{-\alpha}]}\left(1-e^{[-x_i^{-\alpha}]}\right)^{\theta(1+R_i)-1}d\theta d\alpha. \tag{32}$$

In order to make the statistical Bayesian inferences more practical and applicable, we often need to choose an asymmetric loss function. A number of asymmetric loss functions proposed for use, one of the most popular is the LINEX loss function. This loss function was introduced by Varian [17] and several others; among of them Basu and Ebrahimi [18], Soliman et al.[19] and Abd Ellah [20]. Recently, A more generalized loss function called the balanced loss function (see

Jozani et al. [21]).
Under the balanced squared error loss (BSEL) function (see Ahmed [9]) the Bayes estimate of a function $g \equiv g(\theta,\alpha) = \alpha$, θ, $S(t)$ or $H(t)$, is given by

$$\widehat{g}_{BS} = \omega \widehat{g}_{ML} + (1-\omega)\int_0^\infty \int_0^\infty g q(\theta,\alpha|\underline{x}) d\theta d\alpha, \tag{33}$$

Also, based on the balanced linear-exponential (BLINEX) loss function (see Ahmed [9]), the Bayes estimate of the function g is given by

$$\widehat{g}_{BL} = \frac{-1}{c}\log\left[\omega e^{-c\widehat{g}_{ML}} + (1-\omega)\int_0^\infty \int_0^\infty e^{-cg} q(\theta,\alpha|\underline{x}) d\theta d\alpha\right], \tag{34}$$

where $\widehat{g}_{ML}$ is the MLE of g and $q(\theta,\alpha|\underline{x})$ is as given by (30). For more details of balanced loss function see Soliman et al. [22].
Unfortunately, (33) and (34) can not be obtained in simple closed form for general $g = g(\theta,\alpha)$. Therefore, we propose the use of MCMC approximation for obtaining the Bayes estimator of g.

5.1 Bayesian estimation using MCMC

In this area we consider the MCMC method to generate samples from the posterior distributions and then compute the Bayes estimates of θ and α under progressively Type-II censored EF distribution. A wide variety of MCMC schemes are available and it can be difficult to choose among them. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis-within-Gibbs samplers; see, for example, Smith and Roberts [23], Gilks et al. [24], Robert and Casella [25], Soliman et al. [22] and Recently, Mahmoud et al. [26]. We propose using the Gibbs sampling procedure to generate a sample from the posterior density function $l(\theta,\alpha|\underline{x})$ and in turn compute the Bayes estimates and also construct the corresponding credible intervals based on the generated posterior sample. In order to use the method of MCMC for estimating the parameters of EF distribution, namely, θ and α. From (31) the posterior can be obtained up to proportionality by multiplying the likelihood with the prior and this can be written as

$$\begin{aligned} h_{\theta,\alpha}(\theta,\alpha|\underline{x}) \propto\ & \theta^{m+a_1-1}\alpha^{m+a_2-1}\exp[-b_1\theta - b_2\alpha] \\ & \times \prod_{i=1}^{m} x_i^{-(1+\alpha)}\exp[-x_i^{-\alpha}]\left(1-\exp[-x_i^{-\alpha}]\right)^{\theta(1+R_i)-1}. \end{aligned} \tag{35}$$

The posterior is obviously complicated and no closed form inferences appear possible. We, therefore, propose to consider MCMC methods, namely the Gibbs sampler, to simulate samples from the posterior so that sample-based inferences can be easily drawn. From (35), the marginal posterior density of θ is proportional to

$$\begin{aligned} h_{\alpha}(\alpha|\theta,\underline{x}) \propto\ & \alpha^{m+a_2-1}\exp[-b_2\alpha]\prod_{i=1}^{m} x_i^{-(1+\alpha)}\exp[-x_i^{-\alpha}] \\ & \times \left(1-\exp[-x_i^{-\alpha}]\right)^{\theta(1+R_i)-1}. \end{aligned} \tag{36}$$

Similarly, the full posterior conditional distribution for α as the following

$$h_{\theta}(\theta|\alpha,\underline{x}) \propto \theta^{m+a_1-1}\exp\left[-\theta\left(b_1 - \sum_{i=1}^{m}(R_i+1)\log(1-\exp[-x_i^{-\alpha}])\right)\right]. \tag{37}$$

It can be seen that Equation (37) is a gamma density with shape parameter $(m+a_1)$ and scale parameter $\left(b_1 - \sum_{i=1}^{m}(R_i+1)\log(1-\exp[-x_i^{-\alpha}])\right)$ and, therefore, samples of θ can be easily generated using any gamma generating routine. However, in our case, the conditional posterior distribution of α equation (36) cannot be reduced analytically to well known distributions and therefore it is not possible to sample directly by standard methods, but the plot of it show that it is similar to normal distribution. So to generate random numbers from this distribution, we use the Metropolis-Hastings method with normal proposal distribution.
The following steps illustrate the process of the Metropolis–Hastings algorithm within the Gibbs sampling to simulate the posterior samples:

Step1:Start with an $(\alpha^{(0)},\theta^{(0)})$
Step2:Set $i = 1$.

Table 1: Relief time (in hours) for 50 arthritic patients

0.70	0.84	0.58	0.50	0.55	0.82	0.59	0.71	0.72	0.61
0.62	0.49	0.54	0.36	0.36	0.71	0.35	0.64	0.84	0.55
0.59	0.29	0.75	0.46	0.46	0.60	0.60	0.36	0.52	0.68
0.80	0.55	0.84	0.34	0.34	0.70	0.49	0.56	0.71	0.61
0.57	0.73	0.75	0.44	0.44	0.81	0.80	0.87	0.29	0.50

Step3:Generate $\theta^{(i)}$ from $Gamma(m+a_1, b_1+\sum_{i=1}^{m}(R_i+1)\log(1-\exp[-x_i^{-\alpha}]))$.
Step4:Using Metropolis-Hastings, generate $\alpha^{(i)}$ from h_α with the N($\alpha^{(i-1)}$,Var(α)) proposal distribution.
Step5:Compute $S^{(i)} = (1-\exp(-t^{-\alpha^{(i)}}))^{\theta^{(i)}}$ and $H^{(i)} = \theta^{(i)}\alpha^{(i)}t^{-(\alpha^{(i)}+1)}\exp(-t^{-\alpha^{(i)}})(1-\exp(-t^{-\alpha^{(i)}}))^{-1}$.
Step6:Set $i=i+1$.
Step7:Repeat steps $3-6$ N times.

Note that in step 4, we used Metropolis-Hastings algorithm with N($\alpha^{(i-1)}$,$Var(\alpha)$) proposal distribution as follows:

a-Generate a proposal α^* from $N(\alpha^{(i-1)}$,Var(α)).
b-Evaluate the acceptance probabilities $\rho_\alpha = \min\left[1, \frac{h_\alpha(\alpha^*|\theta^{(j)},\mathrm{x})}{h_\alpha(\alpha^{(j-1)}|\theta^{(j)},\mathrm{x})}\right]$.
c-Generate a u from a Uniform(0,1) distribution.
d-If $u \leq \rho_\alpha$, accept the proposal and set $\alpha^{(j)} = \alpha^*$, else set $\alpha^{(j)} = \alpha^{(j-1)}$.

In order to guarantee the convergence and to remove the affection of the selection of initial value, the first M simulated variates are discarded. Then the selected sample are $\alpha^{(i)}$ and $\theta^{(i)}$, $i = M+1,...,N$, for sufficiently large N, forms an approximate posterior sample which can be used to develop the Bayesian inference.
step 8:Obtain the Bayes estimates of θ, α, $S(t)$ and $H(t)$ as follows:
An approximate Bayes estimate under MCMC of g under BSEL function is

$$\widehat{g}_{BS} = \omega\widehat{g}(\theta,\alpha)_{ML} + \frac{(1-\omega)}{N-M}\sum_{i=M+1}^{N} g(\theta^{(i)},\alpha^{(i)}). \tag{38}$$

Also; the approximate Byes estimate under MCMC of the g under BLINEX loss is then given by

$$\widehat{g}_{BL} = \frac{-1}{c}\log\left(\omega e^{-c\widehat{g}(\theta,\alpha)_{ML}} + \frac{(1-\omega)}{N-M}\sum_{i=M+1}^{N} e^{-cg(\theta^{(i)},\alpha^{(i)})}\right) \tag{39}$$

Substituting, from (37) and (38) by $g(\theta,\alpha) = \alpha$, θ, $S(t)$ and $H(t)$, the the approximate Byes estimate under MCMC of α, θ, $S(t)$ and $H(t)$ under both $bSEL$ and $bLINEX$ loss functions can be obtained.
Step9:To compute the credible intervals of g, as $g_{(1)} < ... < g_{(N)}$,. Then the $100(1-\gamma)\%$ symmetric credible intervals of g become

$$(g_{(N\,\gamma/2)},\;\; g_{(N(1-\gamma/2))}). \tag{40}$$

Substituting, from (40) by $g(\theta,\alpha) = \alpha$, θ, $S(t)$ and $H(t)$, the $100(1-\gamma)\%$ symmetric credible intervals of α, θ, $S(t)$ and $H(t)$ can be obtained.

6 Application to real life data

In this section, we consider a real-life data set and illustrate the methods proposed in the previous sections. The data set is given by Wingo [27] and used recently by Wu et al. [1]. and it represents a clinical trial describe a relief time (in hours) for 50 arthritic patients. The data are given in Table 1.

Before progressing further, we have examined the goodness-of-fit of the previous data to EF distribution graphically. We have computed the Kolmogorov-Smirnov (KS) distance between the empirical and the fitted distribution functions. it is 0.11 and the associated p-value is 0.53. Since the p-value is quite high, we cannot reject the null hypothesis that the data is coming from the EF distribution. We plot both the empirical and the estimated survival functions in Fig.1 and we found that the EF fits the data very well.

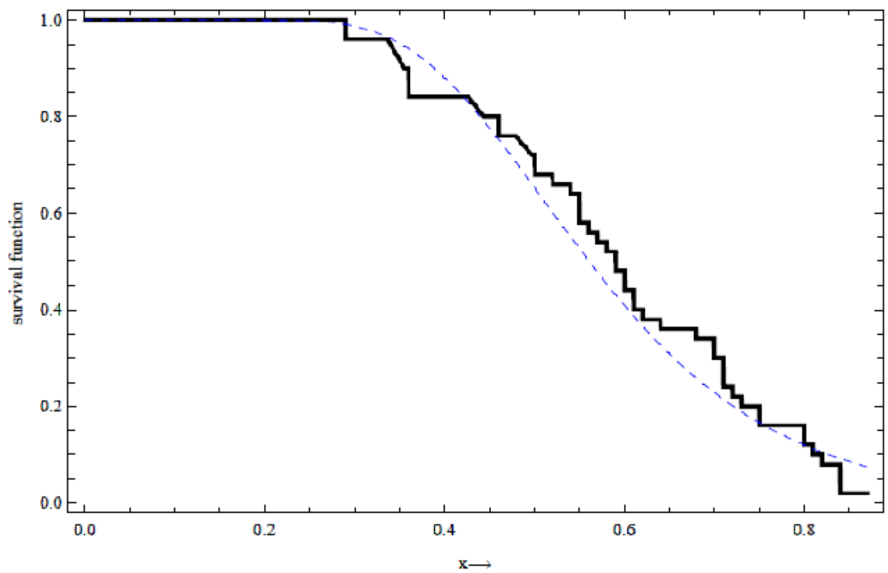

Fig. 1: Empirical survival function (bold line) and the fitted survival function (dotted lines) for Wu et al. [1].

Now, we consider the case when the data are progressively type-II censored with the sample size $n = 50$ and $m = 45$ with the censoring scheme ($R_1 = 5$, $R_i = 0, i = 2, ...m$), we obtained the following data:

0.29	0.29	0.34	0.35	0.36	0.36	0.36	0.44	0.46	0.46	0.49	0.49	0.50	0.52	0.54
0.55	0.55	0.56	0.58	0.59	0.59	0.60	0.60	0.61	0.61	0.62	0.64	0.68	0.70	0.70
0.71	0.71	0.71	0.72	0.73	0.75	0.75	0.80	0.80	0.81	0.82	0.84	0.84	0.84	0.87

By using the graphical method introduced by Balakrishnan and Kateri [13], the initial values of the parameters θ and α have been obtained to be $\theta = 7.138$ and $\alpha = 1.543$. Using thus initial values, the MLEs of α and θ, are computed to be $\hat{\theta}_{ML} = 7.1899$ and $\hat{\alpha}_{ML} = 1.5525$. With $t = 0.33$, from (3) and (4) the MLEs of $S(t)$ and $H(t)$ become $\hat{S}(t) = 0.9734$ and $\hat{H}(t) = 0.7082$.

Based on the large sample inference property and assume the regularity conditions are satisfied, the 95% confidence interval (CIs) for the parameters θ and α are obtained. Furthermore, we used delta method to obtain the 95% CIs for the $S(t)$ and $H(t)$. We also obtain the 95% Boot-p and Boot-t confidence intervals, the results are displayed in Table (2).

Table 2: 95% confidence and credible intervals of α, θ, $S(t = 0.33)$ and $H(t = 0.33)$.

	α			θ	
	Interval	Length		Interval	Length
MLE	(1.3480, 1.7569)	0.4089	MLE	(4.8123, 9.5675)	4.7552
Boot-p	(1.3558, 1.8977)	0.5418	Boot-p	(6.1068, 9.4177)	3.3109
Boot-t	(1.3326, 1.8307)	0.4981	Boot-t	(5.8461, 8.8354)	2.9893
MCMC	(1.3359, 1.7373)	0.4014	MCMC	(5.0739,9.7938)	4.7199
	$S(t)$			$H(t)$	
MLE	(0.9433, 1.0036)	0.0603	MLE	(0.1323, 1.2841)	1.1518
Boot-p	(0.9219, 0.9975)	0.0756	Boot-p	(0.1181, 1.4776)	1.3594
Boot-t	(0.9291, 0.9956)	0.0666	Boot-t	(0.1847, 1.3100)	1.1253
MCMC	(0.9251, 0.9917)	0.0666	MCMC	(0.2745,1.3812)	1.1067

Under assumption that these data are from EF distribution. We run the Gibbs sampler with MH algorithm to generate a Markov chain with $11,000$ observations. Discarding the first 1000 values as 'burn-in' and taking every tenth variate as iid observations, starting from 1001. This is done to minimize the auto correlation among the generated deviates. The convergence is monitored using trace plots. We used the non-informative priors for α and θ, that is, when the hyperparameters are 0 ($a_1 = a_2 = b_1 = b_2 = 0$). The marginal posterior density estimates of the parameters, reliability and

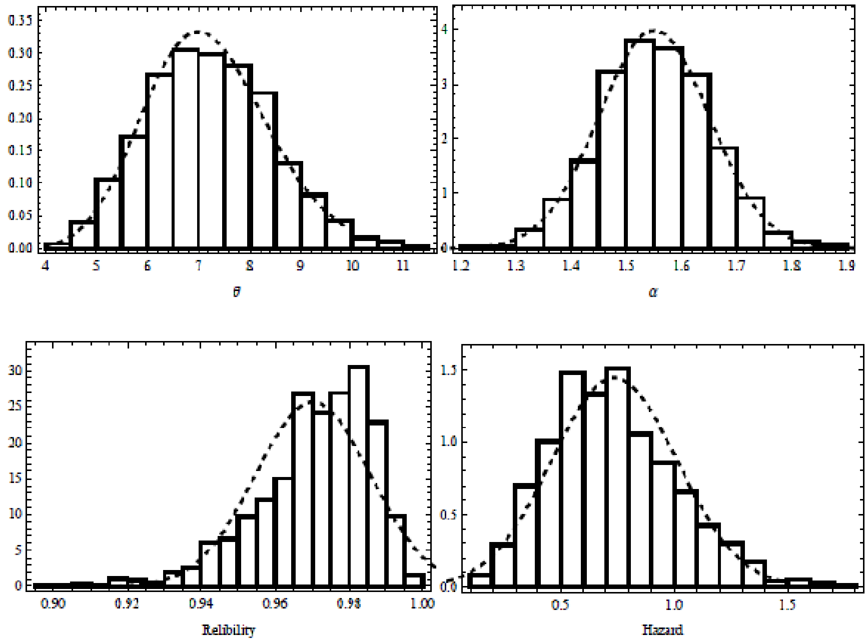

Fig. 2: Histogram and kernel density estimates of α, θ, $S(t)$ and $H(t)$ from Example 1.

hazard functions and their histograms based on samples of size 10,000 are shown in Fig. 2 using the Gaussian kernel. It is evident from the estimates that all the marginal distributions are almost symmetrical. We can take the posterior mean as the best estimate for symmetric distribution and the posterior mode for a skewed distribution. A traceplot is a plot of the iteration number against the value of the draw of the parameter at each iteration. Fig. 3 display 10,000 chain values for the two parameters θ and α as well as reliability $S(t)$ and hazard $H(t)$ functions, with their sample mean and 95% credible intervals. The plots syndicate a good mixing performance. Interval estimates of the unknown concentrations are easily obtained from the percentiles of the posterior distributions. Table 2 lists the 95% probability intervals for the parameters, reliability and hazard functions. The result of Bayes MCMC estimates relative to both balanced squared error loss function (BSEL) and balanced LINEX loss function (BLINEX) with different values of the shape parameter (c) of LINEX loss function and various values of ω for the parameters θ and α as well as reliability $S(t=0.33)$ and hazard $H(t=0.33)$ are displayed in Table 3.

7 Simulation study and comparisons

In previous sections we proposed several estimators for unknown parameters θ, α, $S(t)$ and $H(t)$. In this section, the performance of all estimators is evaluated and compared in terms of their MSE values. Bayes estimators are evaluated under the prior assumptions that θ, α follow Gamma(a_1,b_1) and Gamma(a_2,b_2) distributions, respectively. Approximate expressions for all Bayes estimators are obtained using the MCMC method in Section 5. Applying the algorithm of Balakrishnan and Sandhu [28]. to generate a progressive Type II censored sample from the EF distribution as following steps:

(1).For given values of the prior parameters a_1,b_1,a_2,b_2 we generated random values for θ and α from the gamma distribution whose density functions given by Equations (29) and (30).

(2).Using the results for θ and α from step (1), we generated a progressive Type II censored sample of size m from the EF distribution based on algorithm of Balakrishnan and Sandhu [28], using the inverse cdf, $x_i=\left\{\ln\left[1-(1-U_i)^{-\frac{1}{\theta}}\right]\right\}^{-\frac{1}{\alpha}}, i=1,2,...,m.$

(3).Compute the estimates as the following:

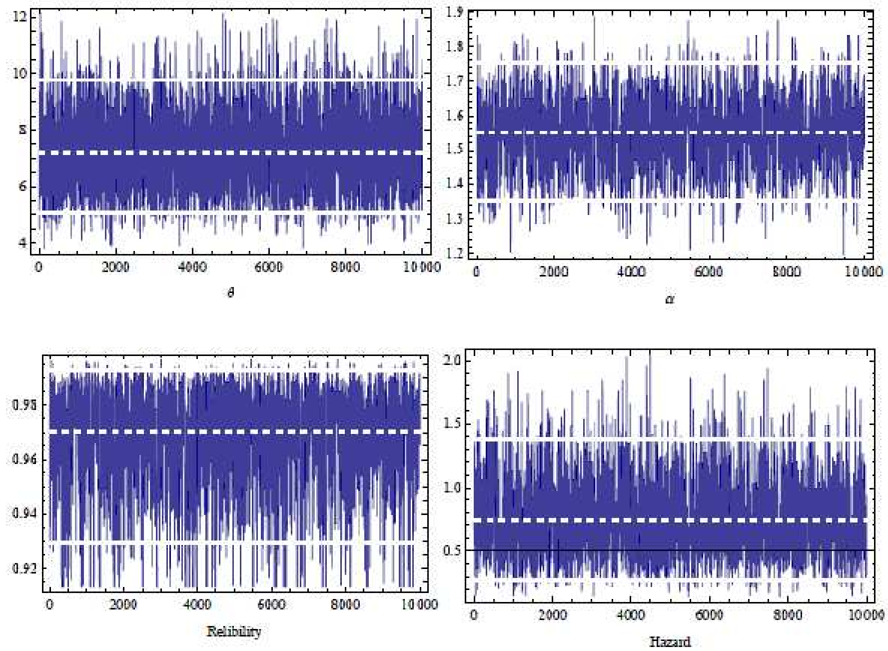

Fig. 3: MCMC output of α, θ, $S(t)$ and $H(t)$. Dashed lines (...) represent the posterior means and soled lines (—) represent lower, and upper bounds 95% probability interval.

- **(a)** Using $x_1, x_2, \ldots, x_m$ from step 2, the MLE of θ and α, say $\hat{\theta}_{ML}$ and $\hat{\alpha}_{ML}$ were computed by solving Equations (10) and (11) numerically. Substituting the $\hat{\theta}_{ML}$ and $\hat{\alpha}_{ML}$ into (3) and (4), we obtain the MLEs of the reliability and hazard rate functions, say $\hat{S}_{ML}(t)$ and $\hat{H}_{ML}(t)$ at some t.
- **(b)** Using algorithms, given in Sections 4, We computed the bootstrap estimates and 95% credible intervals of θ, α, $S(t)$ and $H(t)$ based on 1000 bootstrap samples.
- **(c)** We computed the Bayes estimates and 95% credible intervals of θ, α $S(t)$ and $H(t)$ according to the hybrid algorithm discussed in Subsection 5.1, based on $10,000$ MCMC samples, using Eqs. (38), (39) and (40).

(4).We repeated the previous steps 1,000 times, and computed the means and the MSEs for different censoring sizes m and censoring schemes where

$$\text{MSE} = \frac{1}{1000}\sum_{i=1}^{1000}\left[g\left(\hat{\lambda}\right)^{i} - g\left(\lambda_0\right)\right]^2$$

and $g(\lambda_0)$ are true values and the $g\left(\hat{\lambda}\right)^{i}$ are the ith estimates of $g(\lambda_0)$ evaluated at $\hat{\lambda}$.

Table 3:MLE and Bayes MCMC estimates under BSEL and BLINEX for Wu et al. [1].

parameters	MLEs	ω	BSEL	BLINEX		
				$c=-1$	$c=0.0001$	$c=1$
α	1.5525	0	1.5437	1.5492	1.5437	1.5382
		0.3	1.5464	1.5502	1.5464	1.5425
		0.9	1.5516	1.5522	1.5516	1.5510
θ	7.1899	0	7.1747	8.0616	7.1747	6.5312
		0.3	7.1792	7.8698	7.1792	6.6876
		0.9	7.1884	7.3201	7.1884	7.1007
$S(t)$	0.9734	0	0.9690	0.9691	0.9690	0.9688
		0.3	0.9703	0.9704	0.9703	0.9702
		0.9	0.9730	0.9730	0.9730	0.9730
$H(t)$	0.7082	0	0.7558	0.8042	0.7558	0.7138
		0.3	0.7415	0.7764	0.7415	0.7121
		0.9	0.7130	0.7182	0.7130	0.7088

Table 4. Average mean of the different estimators and the corresponding MSEs of θ and α, when $\omega = 0$

n	m	Scheme		MLE	Boot-p	Boot-t	MCMC		
							BSEL	BLINEX	
								$c=-4$	$c=4$
40	35	$(5,34^0)$	θ	1.5372	1.5674	1.5229	1.4509	1.5849	1.3449
				0.0864	0.0978	0.0886	0.0552	0.0789	0.0649
			α	1.0260	1.0663	1.0459	1.0248	1.0532	0.9964
				0.0169	0.0225	0.0192	0.0154	0.0192	0.0124
		$(17^0,5,17^0)$	θ	1.5502	1.5817	1.5376	1.4633	1.5950	1.3578
				0.0750	0.0866	0.0765	0.0462	0.0687	0.0547
			α	1.0321	1.0730	1.0533	1.0286	1.0553	0.9995
				0.0195	0.0262	0.0222	0.0174	0.0214	0.0137
		$(34^0,5)$	θ	1.5236	1.5549	1.5109	1.4415	1.5683	1.3386
				0.0824	0.0929	0.0850	0.0537	0.073	0.0647
			α	1.0370	1.0793	1.0595	1.0329	1.0599	1.0032
				0.0200	0.0270	0.0231	0.018	0.0220	0.0142
40	18	$(22,17^0)$	θ	1.5878	1.6881	1.5895	1.4122	1.6449	1.2511
				0.1614	0.2211	0.1702	0.0685	0.1322	0.1005
			α	1.0317	1.0772	1.0529	1.0262	1.065	0.9897
				0.021	0.0281	0.0237	0.0174	0.0241	0.0143
		$(9^0,22,8^0)$	θ	1.6403	1.7885	1.66	1.4338	1.6802	1.2673
				0.2135	0.3432	0.2363	0.0689	0.1649	0.0925
			α	1.0419	1.0956	1.0765	1.0115	1.0452	0.9779
				0.0241	0.0339	0.0298	0.0179	0.0229	0.0157
		$(17^0,22)$	θ	1.6381	1.8171	1.6634	1.4202	1.6751	1.2528
				0.2489	0.4435	0.2588	0.0744	0.1838	0.1005
			α	1.0555	1.1205	1.0986	1.015	1.0526	0.9777
				0.0264	0.0407	0.0266	0.0183	0.024	0.0157
50	45	$(5,44^0)$	θ	1.5244	1.5458	1.5120	1.4587	1.5632	1.3713
				0.0631	0.0695	0.0643	0.0442	0.0586	0.0509
			α	1.0244	1.0560	1.0408	1.0236	1.0462	1.0010
				0.0136	0.0175	0.0152	0.0125	0.0151	0.0102
		$(22^0,5,22^0)$	θ	1.5192	1.5415	1.5097	1.4548	1.5583	1.3694
				0.0602	0.0661	0.0617	0.0429	0.0557	0.0501
			α	1.0312	1.0631	1.0478	1.0291	1.0505	1.0061
				0.0143	0.0185	0.0161	0.0132	0.0159	0.0108
		$(44^0,5)$	θ	1.5433	1.5671	1.5326	1.4747	1.5789	1.3873
				0.0657	0.0731	0.0664	0.0444	0.0622	0.0482
			α	1.0286	1.0605	1.0459	1.0264	1.0477	1.0037
				0.0143	0.0184	0.0161	0.0132	0.0156	0.0109

Table 5. Average mean of the different estimators and the corresponding MSEs of S(t) and H(t) when $\omega = 0$.

n	m	Scheme		MLE	Boot-p	Boot-t	MCMC BSEL	MCMC BLINEX $c=-4$	MCMC BLINEX $c=4$
40	35	$(5,34^0)$	$S(t)$	0.1866	0.1827	0.1964	0.2097	0.2153	0.2043
				0.0034	0.0033	0.0036	0.0031	0.0035	0.0029
			$H(t)$	0.5135	0.5438	0.5173	0.4825	0.5003	0.4664
				0.0128	0.0171	0.0139	0.0086	0.0102	0.0079
		$(17^0,5,17^0)$	$S(t)$	0.1807	0.1765	0.1896	0.205	0.2107	0.1995
				0.0024	0.0024	0.0024	0.0021	0.0023	0.0019
			$H(t)$	0.5188	0.5509	0.5246	0.4866	0.5053	0.4699
				0.0094	0.0137	0.0103	0.0059	0.0073	0.0054
		$(34^0,5)$	$S(t)$	0.1866	0.182	0.1954	0.2099	0.2158	0.2044
				0.0028	0.0027	0.003	0.0026	0.0030	0.0024
			$H(t)$	0.5147	0.5475	0.521	0.4829	0.5013	0.4662
				0.0134	0.0183	0.0147	0.0086	0.0104	0.0077
40	18	$(22,17^0)$	$S(t)$	0.1817	0.1738	0.1975	0.2244	0.2353	0.2145
				0.005	0.0046	0.005	0.0045	0.0055	0.0037
			$H(t)$	0.5348	0.5955	0.5467	0.4721	0.5053	0.445
				0.0264	0.0426	0.0302	0.0113	0.0156	0.0103
		$(9^0,22,8^0)$	$S(t)$	0.1734	0.1614	0.1839	0.2239	0.2362	0.2126
				0.0062	0.0059	0.0057	0.0049	0.0061	0.0041
			$H(t)$	0.5637	0.6573	0.597	0.4762	0.5188	0.443
				0.0411	0.0804	0.0535	0.0127	0.0195	0.0112
		$(17^0,22)$	$S(t)$	0.1742	0.1662	0.1939	0.2274	0.2407	0.2153
				0.0069	0.0227	0.0060	0.0053	0.0067	0.0044
			$H(t)$	0.5732	0.6931	0.6171	0.4747	0.5249	0.4382
				0.0528	0.1199	0.0623	0.0133	0.0229	0.0116
50	45	$(5,44^0)$	$S(t)$	0.1865	0.1834	0.1938	0.2049	0.2094	0.2006
				0.0021	0.0021	0.0022	0.002	0.0022	0.0018
			$H(t)$	0.5073	0.5301	0.5102	0.4838	0.4975	0.4712
				0.008)	0.0103	0.0085	0.0057	0.0065	0.0053
		$(22^0,5,22^0)$	$S(t)$	0.1859	0.1825	0.1923	0.2045	0.2089	0.2002
				0.0020	0.0020	0.0021	0.0019	0.0021	0.0018
			$H(t)$	0.5098	0.5337	0.5142	0.4857	0.5001	0.4726
				0.0084	0.0108	0.0090	0.0059	0.0069	0.0055
		$(44^0,5)$	$S(t)$	0.1823	0.1789	0.1889	0.2013	0.2059	0.1971
				0.0024	0.0024	0.0025	0.0021	0.0023	0.0020
			$H(t)$	0.5174	0.5423	0.522	0.4917	0.5064	0.4781
				0.0105	0.0137	0.0112	0.0072	0.0085	0.0064

Table 6. Average mean of the different estimators and the corresponding MSEs of θ and α

n	m	Scheme		MCMC ($\omega = 0.3$)			MCMC ($\omega = 0.9$)		
				BSEL	BLINEX		BSEL	BLINEX	
					$c = -4$	$c = 4$		$c = -4$	$c = 4$
40	35	$(5, 34^0)$	θ	1.4768	1.5716	1.3866	1.5286	1.5425	1.5071
				0.0622	0.0802	0.0596	0.0822	0.0852	0.0743
			α	1.0248	1.0454	1.0047	1.0258	1.0289	1.0228
				0.0154	0.0184	0.0134	0.0167	0.0171	0.0163
		$(17^0, 5, 17^0)$	θ	1.4893	1.5828	1.3998	1.5415	1.5552	1.5204
				0.0526	0.0710	(0.0496)	0.0712	0.0742	0.0636
			α	1.0286	1.0486	1.0087	1.0316	1.0345	1.0286
				0.0174	0.0208	0.0149	0.0192	0.0196	0.0186
		$(34^0, 5)$	θ	1.4661	1.5573	1.3791	1.5154	1.5288	1.4949
				0.0601	0.0755	0.0589	0.0786	0.0811	0.0708
			α	1.0329	1.0532	1.0127	1.0364	1.0394	1.0333
				0.0180	0.0214	0.0154	0.0197	0.0202	0.0192
40	18	$(22, 17^0)$	θ	1.4649	1.6329	1.308	1.5703	1.5952	1.5084
				0.0853	0.1422	0.0823	0.1473	0.1581	0.097
			α	1.0278	1.0554	1.0013	1.0311	1.0352	1.027
				0.0184	0.0229	0.0156	0.0206	0.0212	0.0199
		$(9^0, 22, 8^0)$	θ	1.4957	1.6735	1.3262	1.6196	1.6459	1.5404
				0.0955	0.1853	0.0758	0.1918	0.2095	0.1043
			α	1.0206	1.0444	0.9952	1.0389	1.0423	1.0346
				0.0195	0.0232	0.0167	0.0234	0.024	0.0224
		$(17^0, 22)$	θ	1.4856	1.6713	1.3114	1.6163	1.6439	1.5274
				0.1056	0.2137	0.0828	0.2224	0.2441	0.1105
			α	1.0272	1.0538	0.9982	1.0515	1.0554	1.0463
				0.0202	0.0247	0.0169	0.0253	0.0261	0.0242
50	45	$(5, 44^0)$	θ	1.4784	1.5527	1.4071	1.5178	1.5287	1.5029
				0.0486	0.0592	0.0474	0.0606	0.0624	0.0565
			α	1.0236	1.0398	1.0077	1.0243	1.0266	1.0219
				0.0125	0.0145	0.0109	0.0134	0.0137	0.0131
		$(22^0, 5, 22^0)$	θ	1.4741	1.5465	1.4045	1.5127	1.5233	1.4983
				0.0468	0.0564	0.0463	0.0579	0.0595	0.0542
			α	1.0291	1.0451	1.0133	1.0309	1.0332	1.0285
				0.0132	0.0154	0.0116	0.0141	0.0144	0.0138
		$(44^0, 5)$	θ	1.4953	1.5695	1.4236	1.5364	1.5472	1.5213
				0.0494	0.0626	0.0458	0.063	0.0651	0.0584
			α	1.0264	1.0421	1.0109	1.0283	1.0306	1.026
				0.0132	0.0152	0.0116	0.0141	0.0144	0.0139

Table 7. Average mean of the different estimators and the corresponding MSEs of $S(t)$ and $H(t)$

n	m	Scheme		MCMC ($\omega=0.3$)			MCMC ($\omega=0.9$)		
				BSEL	BLINEX		BSEL	BLINEX	
					$c=-4$	$c=4$		$c=-4$	$c=4$
40	35	$(5,34^0)$	$S(t)$	0.2027	0.2071	0.1988	0.1889	0.1896	0.1883
				0.0031	0.0033	0.0029	0.0033	0.0034	0.0033
			$H(t)$	0.4918	0.5045	0.4794	0.5104	0.5122	0.5082
				0.0096	0.0110	0.0086	0.0123	0.0126	0.0120
		$(17^0,5,17^0)$	$S(t)$	0.1977	0.2021	0.1937	0.1832	0.1839	0.1826
				0.0020	0.0022	0.0019	0.0023	0.0024	0.0023
			$H(t)$	0.4963	0.5095	0.4834	0.5156	0.5175	0.5134
				0.0067	0.0079	0.0059	0.0090	0.0092	0.0087
		$(34^0,5)$	$S(t)$	0.2029	0.2073	0.1989	0.1889	0.1896	0.1883
				0.0026	0.0028	0.0024	0.0027	0.0028	0.0027
			$H(t)$	0.4924	0.5056	0.4795	0.5115	0.5134	0.5092
				0.0097	0.0113	0.0085	0.0128	0.0131	0.0123
40	18	$(22,17^0)$	$S(t)$	0.2116	0.2205	0.2041	0.186	0.1876	0.1847
				0.0042	0.0047	0.0038	0.0047	0.0047	0.0047
			$H(t)$	0.4909	0.5151	0.4673	0.5285	0.5322	0.5229
				0.0143	0.0189	0.0112	0.0242	0.0253	0.0216
		$(9^0,22,8^0)$	$S(t)$	0.2087	0.2191	0.2001	0.1785	0.1805	0.177
				0.0047	0.0052	0.0044	0.0058	0.0057	0.0059
			$H(t)$	0.5025	0.5351	0.4709	0.555	0.5601	0.5456
				0.0182	0.0269	0.0126	0.0369	0.0392	0.0312
		$(17^0,22)$	$S(t)$	0.2114	0.2227	0.202	0.1795	0.1818	0.1779
				0.0051	0.0057	0.0048	0.0064	0.0063	0.0065
			$H(t)$	0.5043	0.5432	0.4678	0.5634	0.5695	0.5505
				0.0206	0.0343	0.0131	0.0469	0.0504	0.0358
50	45	$(5,44^0)$	$S(t)$	0.1993	0.2027	0.1963	0.1883	0.1888	0.1878
				0.0020	0.0021	0.0019	0.0021	0.0022	0.0021
			$H(t)$	0.4909	0.5005	0.4814	0.505	0.5064	0.5034
				0.0062	0.0070	0.0057	0.0077	0.0079	0.0075
		$(22^0,5,22^0)$	$S(t)$	0.1989	0.2023	0.1958	0.1877	0.1883	0.1873
				0.0019	0.002	0.0018	0.0022	0.0023	0.0021
			$H(t)$	0.4930	0.5030	0.4831	0.5074	0.5088	0.5057
				0.0065	0.0073	0.0059	0.0080	0.0082	0.0078
		$(44^0,5)$	$S(t)$	0.1956	0.199	0.1925	0.1842	0.1847	0.1837
				0.0021	0.0022	0.0021	0.0024	0.0025	0.0024
			$H(t)$	0.4994	0.5099	0.4891	0.5148	0.5163	0.5131
				0.0080	0.0091	0.0071	0.0101	0.0103	0.0098

Table 8. Expected width and coverage probabilities of the 95% ?confidence intervals for θ, α, $S(t)$ and $H(t)$ based on different methods, when $\omega = 0$ and $t = 2.5$.

n	m	Scheme		MLE	Boot-p	Boot-t	MCMC
40	35	$(5,34^0)$	θ	1.0659(0.950)	1.1491(0.930)	1.1144(0.940)	0.9421(0.955)
			α	0.4811(0.965)	0.5403(0.930)	0.4774(0.955)	0.4612(0.970)
			$S(t)$	0.2043(0.900)	0.2005(0.905)	0.2432(0.950)	0.2038(0.930)
			$H(t)$	0.3865(0.925)	0.4341(0.905)	0.4220(0.940)	0.3518(0.925)
		$(17^0,5,17^0)$	θ	1.0584(0.950)	1.1380(0.940)	1.1069(0.935)	0.9390(0.955)
			α	0.4738(0.950)	0.5355(0.890)	0.4720(0.935)	0.4566(0.965)
			$S(t)$	0.2055(0.915)	0.2004(0.930)	0.2406(0.965)	0.2054(0.970)
			$H(t)$	0.3979(0.955)	0.4488(0.945)	0.4298(0.970)	0.3616(0.960)
		$(34^0,5)$	θ	1.0424(0.945)	1.1332(0.935)	1.0946(0.930)	0.9258(0.960)
			α	0.4788(0.925)	0.5419(0.870)	0.4751(0.915)	0.4602(0.930)
			$S(t)$	0.2071(0.915)	0.2034(0.925)	0.2437(0.955)	0.9258(0.960)
			$H(t)$	0.3950(0.945)	0.4495(0.915)	0.4296(0.965)	0.4602(0.930)
40	18	$(22,17^0)$	θ	1.4923(0.96)	1.7279(0.94)	1.5329(0.95)	1.1891(0.972)
			α	0.5512(0.966)	0.6127(0.94)	0.5473(0.966)	0.5305(0.966)
			$S(t)$	0.2794(0.906)	0.2636(0.926)	0.3567(0.974)	0.279(0.97)
			$H(t)$	0.5584(0.962)	0.694(0.918)	0.6207(0.972)	0.4626(0.962)
		$(9^0,22,8^0)$	θ	1.5481(0.952)	1.9433(0.908)	1.5554(0.944)	1.2083(0.958)
			α	0.5185(0.938)	0.5954(0.90)	0.5281(0.934)	0.5019(0.948)
			$S(t)$	0.2936(0.904)	0.2699(0.906)	0.3537(0.966)	0.2961(0.96)
			$H(t)$	0.6662(0.956)	0.9036(0.894)	0.7214(0.968)	0.5148(0.954)
		$(17^0,22)$	θ	1.5911(0.94)	2.1339(0.902)	1.5669(0.934)	1.2131(0.944)
			α	0.5561(0.938)	0.6455(0.903)	0.5643(0.912)	0.5299(0.966)
			$S(t)$	0.3062(0.902)	0.279(0.892)	0.3654(0.972)	0.3072(0.952)
			$H(t)$	0.7325(0.956)	1.0834(0.896)	0.7808(0.978)	0.5428(0.956)
50	45	$(5,44^0)$	θ	0.9341(0.925)	0.9884(0.925)	0.9708(0.920)	0.8480(0.955)
			α	0.4256(0.950)	0.4689(0.910)	0.4232(0.955)	0.4116(0.955)
			$S(t)$	0.1820(0.945)	0.1790(0.940)	0.2088(0.965)	0.1814(0.965)
			$H(t)$	0.3361(0.940)	0.3673(0.940)	0.3602(0.965)	0.3128(0.940)
		$(22^0,5,22^0)$	θ	0.9201(0.960)	0.9784(0.935)	0.9515(0.950)	0.8380(0.980)
			α	0.4230(0.950)	0.4659(0.900)	0.4201(0.930)	0.4100(0.955)
			$S(t)$	0.1832(0.945)	0.1806(0.945)	0.2056(0.985)	0.1827(0.975)
			$H(t)$	0.3428(0.980)	0.3761(0.920)	0.3639(0.980)	0.3189(0.975)
		$(44^0,5)$	θ	0.9326(0.950)	0.9921(0.925)	0.9686(0.935)	0.8480(0.970)
			α	0.4195(0.945)	0.4600(0.890)	0.4164(0.930)	0.4064(0.945)
			$S(t)$	0.1813(0.915)	0.1784(0.915)	0.2048(0.965)	0.1813(0.960)
			$H(t)$	0.3489(0.960)	0.3847(0.895)	0.3711(0.960)	0.3236(0.955)

Note: The number outside the bracket is the expected width and the number in the bracket is the coverage probability. Our computational results for the means and MSE are computed in the above steps for the case of unknown θ and α, where the values of the hyperparameters used are $(a_1,b_1) = (4,4)$ and $(a_2,b_2) = (4,4)$ yielding $\theta = 1.4985$ and $\alpha = 0.9864$ (as true values). Also we computed the true values of $S(t)$ and $H(t)$ as $S(t) = 0.1893$ and $H(t) = 0.48798$ at $t = 2.5$. We computed the Bayes estimates and 95% credible intervals based on 11000 MCMC samples and discard the first 1000 values as 'burn-in'. In addition, for the case of balanced loss functions, three different choices such as $0, 0.3, 0.9$ for ω are taken into consideration. For the case of BLINEX loss functions, two different choices such as -4, 4 for c. The MSE values of all estimators are evaluated using Monte Carlo simulations based on 1000 generations of sample of size m ($m = 35, 18, 45$) from a sample of size n ($n = 40$, 50) and different censoring schemes R_i. We will denote, for example, the scheme: ($n = 25$, $m = 5$, $R_i =(0,0,0,0,20)$), by $(4^0,20)$. Tables 4, 5, 6 and 7 display the estimated average mean of MLE, Boot-p, Boot-t and the Bayes estimates relative to BSEL and BLINEX loss functions of the parameters θ, α, $S(t)$ and $H(t)$ and its corresponding MSEs for different censoring schemes. From these tables, comparing the MLE, and approximate Bayes estimators, we observe that Bayes estimators provides the smallest MSE's. Thus, the Bayes estimates relative to BSEL and BLINEX loss functions are better than their corresponding ML estimates, for most cases of n and m. Also, when the effective sample sizes (n,m) are increase the MSEs of the all estimates based on Type-II censored data are decrease. Furthermore; we computed the 95% C.I.'s for θ, α, $S(t)$ and $H(t)$ based on the asymptotic distributions of the MLE. We further compute Boot-p, and Boot-t C.I.'s, and the HPD credible intervals (MCMC intervals). In Table 8, we presented the average confidence credible lengths, and the corresponding coverage percentages. The nominal level for the C.I.'s or the credible intervals is 0.95 in each case. From these tables, we observe that the coverage probabilities of the asymptotic confidence intervals, delta method, bootstrap confidence intervals and credible intervals based on MCMC for θ, α, $S(t)$ and $H(t)$ are close to the desired level of 0.95. We also observe that the HPD intervals provide the smallest average confidence credible lengths for different censoring schemes. It is clear from the Tables $4-8$, when ω goes to one all results of Bayes estimates under both BSEL and BLINEX functions for α, θ, $S(t)$ and $H(t)$ are equal to corresponding MLEs. From Tables $4-7$, we observe that when the value of the shape parameter (c) of LINEX loss function increase the MSEs of the Bayes estimates using the BLINEX loss function are decrease.

8 Conclusions

Based on progressively censored samples, this paper considers estimation of unknown parameters θ and α as well as reliability and hazard functions of an exponentiated Fréchet distribution. We proposed MLEs and Bayes estimators for these unknown parameters. Bayes estimates are computed under different loss functions such as balanced squared error and balanced LINEX. The Bayes estimate of θ and α, and the corresponding credible interval can be obtained using the Gibbs sampling with Metropolis–Hastings technique. We have compared the MLEs and different Bayes estimators in terms of the MSEs for different censoring schemes. We have also compared the confidence intervals obtained using asymptotic distribution of the MLEs, bootstrap methods and credible intervals obtained from the posterior distribution function. We found that Bayes estimates are superior than the corresponding MLEs.

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