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Characterizations of Hasimoto Surfaces in Euclidean 3-spaces $\mathbb{E}^3$

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Abstract: The position vector of the surface $r = r(s,t)$ is called Hasimoto surface if the relation $r_t = r_s \wedge r_{ss}$ hold. In this paper Hasimoto surfaces in Euclidean space $\mathbb{E}^3$ will be introduced. Hasimoto surfaces are investigated by using the Darboux frame and discuss the geometric properties. The position vector of W-curve is stated by a linear combination of its Frenet frame with differentiable functions.

Keywords: Hasimoto surface, Darboux frame, W-curve, Euclidean space

1 Introduction

In the theory of curves in Riemannian manifolds, one of the most and to give characterizations of a regular curve.

Let $r = r(s,t)$ be a position vector of a moving curve ϕ on surface M^2 in $\mathbb{E}^3$ such that $r(s,t)$ is a unit speed curve for all t. If the surface M^2 is a Hasimoto surface, then, the position vector r satisfy the following condition

$$r_t = r_s \wedge r_{ss}. \tag{1}$$

This equation is called the vortex filament or smoke ring equation.

The geometric properties of Hasimoto surfaces are investigated in detail by [1,2]. In 1972, Hasimoto showed that vortex filament equation is equivalent non-linear Schrodinger equation [3,4].

In 1965, R. Betchov [5] transformed (1) into a coupled system of intrinsic equations for the curvature and torsion with the aid of the Serret-Frenet formula.

2 Preliminaries

Let $\phi : I \to \mathscr{M}^2$ be a regular unit speed curve on the orientiable surface $\mathscr{M}^2$. Let $\{T,N,B\}$ be an orthonormal Frenet frame along a moving curve ϕ in $\mathscr{M}^2$ such that $T = \phi'$ is the unit vector field tangent to ϕ, N is the unit vector field in the direction T' normal to ϕ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2}$$

where the functions k and τ are called the curvature and the torsion of the curve ϕ, respectively. Find the curvature of the curve as follows

$$k^2 = g_{\mathbb{E}^3}(T',\, T').$$

The planes spanned by $\{T,N\}$, $\{T,B\}$ and $\{N,B\}$ are respectively known as the osculating, the rectifying and the normal plane.

Introduce a new frame, called Darboux frames $\{T,\, \eta,\, g\}$ with

$$\begin{pmatrix} T \\ \eta \\ g \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3}$$

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where $g = \eta \wedge T$ and β is the angle between the vector fields N and η.
The derivative formulas of (3) can be given as follows:

$$\begin{pmatrix} T' \\ \eta' \\ g' \end{pmatrix} = \begin{pmatrix} 0 & k_\eta & k_g \\ -k_\eta & 0 & -t_r \\ -k_g & t_r & 0 \end{pmatrix} \begin{pmatrix} T \\ \eta \\ g \end{pmatrix}, \tag{4}$$

where k_g is the geodesic curvature, k_η is the normal curvature, t_r is the geodesic torsion of the curve ϕ and $' = \frac{d}{ds}$. Here Darboux curvatures are defined by

$$\begin{cases} k_\eta(s) = k(s)\cos\beta(s) \\ k_g(s) = -k(s)\sin\beta(s) \\ t_r(s) = -\tau(s) - \beta'(s). \end{cases} \tag{5}$$

Theorem 2.1.[14] Suppose $r = r(s,t)$ is a NLS surface such that $r = r(s,t)$ is a unit speed curve with normal vector field for all t. Then the following is satisfied:

$$\begin{pmatrix} T_t \\ \eta_t \\ g_t \end{pmatrix} = \begin{pmatrix} 0 & \alpha & \lambda \\ -\alpha & 0 & -\gamma \\ -\lambda & \gamma & 0 \end{pmatrix} \begin{pmatrix} T \\ \eta \\ g \end{pmatrix}, \tag{6}$$

where α, λ and γ are smooth functions given by

$$\begin{cases} \alpha = k_g' - k_\eta t_r \\ \lambda = -k_\eta' - k_g t_r \\ k^2\gamma = (kk')' - \alpha^2 - \lambda^2 + \delta, \end{cases} \tag{7}$$

where $\delta = k_{g_t} k_\eta - k_{\eta_t} k_g$.
Lemma 2.2. From (5), we obtain

$$\delta = -\beta_t k^2, \tag{8}$$

$$\alpha^2 + \lambda^2 = k^2\tau^2 + k'^2, \tag{9}$$

$$\alpha k_g - \lambda k_\eta = kk'. \tag{10}$$

Using compatibility conditions $T_{st} = T_{ts}$, $\eta_{st} = \eta_{ts}$ and $g_{st} = g_{ts}$, we get

$$\begin{pmatrix} \alpha' \\ \lambda' \\ \gamma' \end{pmatrix} = \begin{pmatrix} 0 & -t_r & k_g \\ t_r & 0 & -k_\eta \\ -k_g & k_\eta & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \lambda \\ \gamma \end{pmatrix} + \begin{pmatrix} k_{\eta_t} \\ k_{g_t} \\ t_{r_t} \end{pmatrix}. \tag{11}$$

The mean curvature H_{mean} and the Gaussian curvature K_G are, respectively, defined by

$$H_{mean} = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

and

$$K_G = \frac{LN - M^2}{EG - F^2}.$$

The Laplace-Beltrami operator of a smooth function $\varphi : M^2 \to \mathbb{R}, (s,t) \mapsto \varphi(s,t)$ with respect to the first fundamental form of the surface M^2 is the operator Δ, defined as [6]

$$\Delta\varphi = \frac{-1}{W}\left[\left(\frac{G\varphi_s - F\varphi_t}{W}\right)_s + \left(\frac{E\varphi_t - F\varphi_s}{W}\right)_t\right], \tag{12}$$

where $\varphi = \varphi(s,t)$ and $W = \sqrt{EG - F^2}$.

3 Hasimoto surface

In this section, Hasimoto surface are investigated by using the Darboux frame and discuss the geometric properties of Hasimoto surface.
Lemma 3.1.[1]

$$\beta_t = \frac{k''}{k} - \tau^2 - \gamma, \tag{13}$$

where β is the angle between the vector fields N and η.

Moreover, the evolution equations for curvature and torsion are

$$k_t = k\tau' + 2\tau k', \quad \tau_t = -\left(\frac{k''}{k}\right)' + 2\tau\tau' - kk'. \tag{14}$$

The coefficients of the first fundamental form of the surface $r = r(s,t)$ are

$$E = 1, \ F = 0, \ G = k^2. \tag{15}$$

The unit normal vector of the Hasimoto surface is given by

$$\mathbf{N} = -\frac{k_g}{k}g - \frac{k_\eta}{k}\eta. \tag{16}$$

Then the coefficients of the second fundamental form of the surface $r = r(s,t)$ are

$$L = -\frac{k_\eta^2 + k_g^2}{\sqrt{k_\eta^2 + k_g^2}} = -k,$$

$$M = \frac{(k_\eta^2 + k_g^2)t_r + k_g k_\eta' - k_\eta k_g'}{\sqrt{k_\eta^2 + k_g^2}}$$
$$= \frac{\beta' k^2 + t_r k^2}{k} = -\tau k,$$

$$N = \frac{(k_g k_g' + k_\eta k_\eta')' - (t_r k_g + k_\eta')^2 - (t_r k_\eta - k_g')^2}{\sqrt{k_\eta^2 + k_g^2}}$$
$$= -k\tau^2 + k'',$$

where $' = \frac{\partial}{\partial s}$.

Thus, one can find that the mean curvature H_{mean} and the curvature K_G of $r = r(s,t)$ as:

Theorem 3.2.[1] Let $r = r(s,t)$ be a Hasimoto surface, then the Gaussian curvature K_G and mean curvature H_{mean} are given by

$$K_G = -\frac{k''}{k}, \quad (17)$$

$$H_{mean} = \frac{k'' - k(k^2+\tau^2)}{2k^2}, \quad (18)$$

respectively, where $k \neq 0$.

Corollary 3.3. Let $r = r(s,t)$ be a Hasimoto surface. There are no developable and minimal Hasimoto surface in $\mathbb{E}^3$.

Proof.

If $r = r(s,t)$ is a developable and minimal Hasimoto surface, then $K_G = 0$ and $H_{mean} = 0$. From (17) and (18) we have that $k = 0$, which is a contradiction. Hence there are no developable and minimal Hasimoto surface in $\mathbb{E}^3$.

4 $I-$ Harmonic Hasimoto surfaces in $\mathbb{E}^3$

Theorem 4.1. The Laplacian Δ of the Hasimoto surface $r = r(s,t)$ can be expressed as

$$\Delta r(s,t) = \frac{-1}{k}\left[Q(s,t)\eta + P(s,t)g\right], \quad (19)$$

where

$$Q(s,t) = -\frac{k_t k_g}{k^2} + kk_\eta + \frac{k_{g_t}}{k} - \frac{\gamma k_\eta}{k},$$

$$P(s,t) = \frac{k_t k_\eta}{k^2} + kk_g - \frac{k_{\eta_t}}{k} - \frac{\gamma k_g}{k},$$

$k_{g_t} = \frac{\partial k_g}{\partial t}$, $k_{\eta_t} = \frac{\partial k_\eta}{\partial t}$.

Proof. By (12), the Laplacian operator Δ of r can be expressed as

$$\Delta r(s,t) = \frac{-1}{k}\left[\frac{\partial}{\partial s}\left(\frac{k^2 r_s}{k}\right) + \frac{\partial}{\partial t}\left(\frac{r_t}{k}\right)\right]. \quad (20)$$

From (1), we have

$$r_s = T,\ r_{ss} = k_\eta \eta + k_g g,\ r_t = k_g \eta - k_\eta g. \quad (21)$$

$$r_{tt} = (-\alpha k_g + \lambda k_\eta)T + (k_{g_t} - \gamma k_\eta)\eta - (k_{\eta_t} + \gamma k_g)g. \quad (22)$$

Then using (21) and (22), we have

$$\Delta r(s,t) = \frac{-1}{k}\left[R(s,t)T + Q(s,t)\eta + P(s,t)g\right],$$

where

$$R(s,t) = k_s - \frac{\alpha k_g}{k} + \frac{\lambda k_\eta}{k},$$
$$Q(s,t) = -\frac{k_t k_g}{k^2} + kk_\eta + \frac{k_{g_t}}{k} - \frac{\gamma k_\eta}{k},$$
$$P(s,t) = \frac{k_t k_\eta}{k^2} + kk_g - \frac{k_{\eta_t}}{k} - \frac{\gamma k_g}{k}.$$

Now we take the derivative with respect to s of k^2, that is

$$kk_s = k_g k_{g_s} + k_\eta k_{\eta_s}.$$

Substituting the latter in $R(s,t)$, we have $R(s,t) = 0$, we obtain the Laplacian Δ of the Hasimoto surface $r = r(s,t)$ given as in (20). Thus, the proof is completed □.

Remark 4.2.

$$k_\eta Q(s,t) + k_g P(s,t) = k(k^2+\tau^2) - k''.$$

Corollary 4.3. Therefore, r is harmonic if and only if $H_{mean} = 0$.

5 Hasimoto surfaces having pointwise 1-Type Gauss Map in $\mathbb{E}^3$

Let $r = r(s,t)$ be a Hasimoto surface. From (12), (15) and (16), we write the Laplacian operator of the Gauss map as

$$\Delta N = -\frac{k'}{k}N_s - N_{ss} - \frac{1}{k^2}N_{tt} + \frac{k'}{k^3}N_t, \quad (23)$$

where

$$N_s = kT - \tau\sin\beta\,\eta - \tau\cos\beta\,g,$$
$$N_t = k\tau T + \vartheta\sin\beta\,\eta + \vartheta\cos\beta\,g,$$

$$N_{ss} = k'T + ((k^2+\tau^2)\cos\beta - \tau'\sin\beta)\eta$$
$$- ((k^2+\tau^2)\sin\beta + \tau'\cos\beta)\eta$$

$$N_{tt} = (k_t\tau + k\tau_t - \vartheta\alpha\sin\beta - \vartheta\lambda\cos\beta)T +$$
$$(\alpha k\tau + \vartheta\beta_t\cos\beta + \vartheta_t\sin\beta + \vartheta\gamma\cos\beta)\eta$$
$$+ (\lambda k\tau - \vartheta\gamma\sin\beta - \vartheta\beta_t\sin\beta + \vartheta_t\cos\beta)g,$$

where $\vartheta = \frac{k''}{k} - \tau^2$.

(23) can be rewritten as

$$\Delta N = -\frac{1}{k^3}\Lambda_1 T - \frac{1}{k^4}\Lambda_2 \eta - \frac{1}{k^4}\Lambda_3 g, \qquad (24)$$

where

$$\begin{aligned}
\Lambda_1 &= kk'(k^2+\tau^2) - kk'\tau + 3k^2\tau\tau' - kk''' + 2k'k'',\\
\Lambda_2 &= (k^2\tau''' - k^4\tau' - k'k'' + kk'\tau^2 + 4kk''\tau' + 4kk'\tau''\\
&\quad - 4k'k''\tau + 4k\tau k''' - 4k^2\tau^2\tau')\sin\beta +\\
&\quad (k^4(k^2+2\tau^2) + (k''-k\tau^2)^2)\cos\beta,\\
\Lambda_3 &= (k^2\tau''' - k^4\tau' - k'k'' + kk'\tau^2 + 4kk''\tau' + 4kk'\tau''\\
&\quad - 4k'k''\tau + 4k\tau k''' - 4k^2\tau^2\tau')\cos\beta\\
&\quad - (k^4(k^2+2\tau^2) + (k''-k\tau^2)^2)\sin\beta.
\end{aligned}$$

Suppose that the Hasimoto surface has harmonic Gauss map. Then, the vector ΔN given with (24) is zero. Thus, we have $(\Lambda_1,\ \Lambda_2,\ \Lambda_3) = (0,0,0)$.

Hence, the equation

$$\Lambda_2\cos\beta - \Lambda_3\sin\beta = k^4(k^2+2\tau^2) + (k''-k\tau^2)^2$$

implies that $k=0$. It is a contradiction □.

Theorem 5.1. Let $r=r(s,t)$ be a Hasimoto surface. There are no Hasimoto surfaces in $\mathbb{E}^3$, satisfying the condition $\Delta N = 0$.

6 A characterization of involutes of a given curve in $\mathbb{E}^3$

Let ϕ and ϕ^* be two curves in the Euclidean space $\mathbb{E}^3$.

Let $\{T,\eta,g\}$ and $\{T^*,\eta^*,g^*\}$ be Darboux frame of ϕ and ϕ^* respectively. Then the curve ϕ^* is called the involute of the curve ϕ, if the tangent vector of the curve ϕ at the points $\phi(s)$ passes through the tangent vector of the curve ϕ^* at the point $\phi^*(s)$ and

$$g_{\mathbb{E}^3}(T,\ T^*) = 0.$$

Definition 6.1. Let ϕ be a curve in $\mathbb{E}^3$.
1) If both k and τ are constant along ϕ, then is called circular helix with respect to Frenet frame.
2) A curve ϕ such that

$$\frac{\tau}{k} = a, \quad a \in \mathbb{R},$$

is called a general helix with respect to Frenet frame.

If k =constant $\neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

Theorem 6.2. Let the curve ϕ^* be involute of the curve ϕ and let c be a constant real number. Then

$$\phi^*(s) = \phi(s) + (c-s)T(s). \qquad (25)$$

Proof. Assume that ϕ^* is an involute of ϕ. Then ϕ^* can be parameterized by

$$\phi^*(s) = \phi(s) + \mu(s)T(s),$$

where $\mu(s)$ is some differentiable function in s. Differentiating the previous equation with respect to s and using (2), we obtain

$$T^* = (1+\mu'(s))T(s) + \mu(s)(k_\eta\eta + k_g g).$$

Since $g_{\mathbb{E}^3}(T,\ T^*) = 0$. Then, we get

$$\mu(s) = c - s.$$

Thus we get

$$\phi^*(s) = \phi(s) + (c-s)T(s).$$

Theorem 6.3. Let the curve ϕ^* be involute of the curve ϕ, then

$$g_{\mathbb{E}^3}(\phi_s^* \wedge \phi_{ss}^*,\ \phi_{sss}^*) = -(c-s)^3k^3\tau^2\left(\frac{k}{\tau}\right)'. \qquad (26)$$

Proof. If we take the derivative (25), we can write

$$\phi_s^* = (c-s)k_\eta\eta + (c-s)k_g g.$$

$$\phi_{ss}^* = -(c-s)k^2T + \Lambda_1\eta + \Lambda_2 g,$$

$$\phi_{sss}^* = \Gamma_1 T + \Gamma_2\eta + \Gamma_3 g, \qquad (27)$$

where

$$\Lambda_1 = -k_\eta + (c-s)k_\eta' + (c-s)k_g t_r = -k_\eta - (c-s)\lambda,$$

$$\Lambda_2 = -k_g + (c-s)k_g' - (c-s)k_\eta t_r = -k_g + (c-s)\alpha,$$

$$\Gamma_1 = 2k^2 - 3(c-s)kk',$$

$$\Gamma_2 = -(c-s)k^2k_\eta + t_r\Lambda_2 - 2k_\eta' + (c-s)k_\eta'' - k_g t_r + (c-s)t_r k_g' + (c-s)t_r' k_g,$$

$$\Gamma_3 = -(c-s)k^2k_g - t_r\Lambda_1 - 2k_g' + (c-s)k_g'' + k_\eta t_r - (c-s)t_r k_\eta' - (c-s)t_r' k_\eta.$$

Hence, we have

$$\phi_s^* \wedge \phi_{ss}^* = (c-s)(k_\eta\Lambda_2 - k_g\Lambda_1)T - (c-s)^2k^2k_g\eta + (c-s)^2k^2k_\eta g. \qquad (28)$$

Lemma 6.4.

$$k_\eta \Lambda_1 + k_g \Lambda_2 = -k^2 + (c-s)kk'.$$

$$k_\eta \Lambda_2 - k_g \Lambda_1 = k^2 (c-s)\tau.$$

If we take the inner product with (28) on both sides of (27), we have (26)

Corollary 6.5. If the curve ϕ is a general helix, then

$$g_{\mathbb{E}^3}(\phi_s^* \wedge \phi_{ss}^*, \phi_{sss}^*) = 0.$$

7 W - curve in $\mathbb{E}^3$

The aim of this section is to continue the study of W - curves. The curve ϕ is called a W - curve, if its curvature and torsion functions are constant. The simplest examples of W - curves are circles, hyperbolas and helices as non-planar W - curves. The characterizations of W - curves are investigated in [7]. W - curves in Lorentz- Minkowski space are investigated in [8,9,10,11].

The authors in [12,13] examined the curvatures of Hasimoto surface according to Bishop frame and give some characterization of parameter curves of these surfaces.

In this section, we give characterization of W - curve. Then the position vector $\phi(s)$ can be written as linear combinations as follows

$$\phi(s) = x_1(s)T + x_2(s)\eta + x_3(s)g \tag{29}$$

for some differentiable functions x_1, x_2 and x_3 of s.

Taking the derivative of (29) with respect to the arc length parameter and using Serret Frenet formulas which are given by (4), we get

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} 0 & k_\eta & k_g \\ -k_\eta & 0 & -t_r \\ -k_g & t_r & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{30}$$

or, equivalently

$$X' = MX + B,$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & k_\eta & k_g \\ -k_\eta & 0 & -t_r \\ -k_g & t_r & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(M - \lambda I) = -\lambda(\lambda^2 + k^2 + t_r^2),$$

so the spectrum of M is $\sigma(M) = \{\lambda_1 = 0, \lambda_2 = i\omega, \lambda_3 = -i\omega\}$, where

$$\omega = \sqrt{k^2 + t_r^2}.$$

Each eigenvalue λ_1, λ_2 and λ_3 corresponds to an eigenvector:

$$V_1 = \begin{pmatrix} t_r \\ k_g \\ -k_\eta \end{pmatrix}, \quad V_2 = \begin{pmatrix} t_r k_\eta - i\omega k_g \\ k_g k_\eta + i\omega t_r \\ k_g^2 + t_r^2 \end{pmatrix},$$

$$V_3 = \begin{pmatrix} t_r k_\eta + i\omega k_g \\ k_g k_\eta - i\omega t_r \\ k_g^2 + t_r^2 \end{pmatrix}.$$

Form matrices P^{-1} and D

$$P^{-1} = \frac{1}{2i\omega^3(k_g^2 + t_r^2)} G$$

where

$$G = \begin{pmatrix} (k_g^2 + t_r^2)2i\omega t_r & (k_g^2 + t_r^2)2i\omega k_g & -(k_g^2 + t_r^2)2i\omega k_\eta \\ -k_g\omega^2 + i\omega k_\eta t_r & t_r\omega^2 + i\omega k_\eta k_g & i\omega(k_g^2 + t_r^2) \\ k_g\omega^2 + i\omega k_\eta t_r & -t_r\omega^2 + i\omega k_\eta k_g & i\omega(k_g^2 + t_r^2) \end{pmatrix}$$

$$P^{-1}MP = D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i\omega & 0 \\ 0 & 0 & -i\omega \end{pmatrix},$$

where P is the change-of-coordinates matrix (matrix formed from the eigenvectors) and D is the diagonal matrix.

We define

$$Y(s) = P^{-1}X(s) = \begin{pmatrix} y_1(s) \\ y_2(s) \\ y_3(s) \end{pmatrix},$$

then

$$Y(s)' = DY(s) + P^{-1}B.$$

The new variables $y_1(s)$, $y_2(s)$ and $y_3(s)$ now are solutions of the decoupled system

$$\begin{cases} y_1'(s) = \frac{t_r}{\omega^2} \\ y_2'(s) = i\omega y_2 + \frac{1}{2\omega^2(k_g^2+t_r^2)}(k_\eta t_r + i\omega k_g) \\ y_3'(s) = -i\omega y_3 + \frac{1}{2\omega^2(k_g^2+t_r^2)}(k_\eta t_r - i\omega k_g). \end{cases}$$

Then the solution to the differential equation

$$Y(s)' = DY(s) + P^{-1}B$$

is

$$\begin{cases} y_1(s) = \frac{t_r}{\omega^2}s + c_0 \\ y_2(s) = \frac{1}{2\omega^3(k_g^2+t_r^2)}(-\omega k_g + ik_\eta t_r) + c_1 e^{i\omega s} \\ y_3(s) = \frac{1}{2\omega^3(k_g^2+t_r^2)}(-\omega k_g - ik_\eta t_r) + c_2 e^{-i\omega s}. \end{cases}$$

Then

$$\begin{cases} x_1(s) = \frac{t_r^2}{\omega^2}s + c_0 t_r + (ak_\eta t_r - b\omega k_g)\cos\omega s + \\ (bk_\eta t_r + a\omega k_g)\sin\omega s \\ x_2(s) = \frac{t_r k_g}{\omega^2}s + c_0 k_g - \frac{k_\eta}{\omega^2} + (ak_\eta k_g + b\omega t_r)\cos\omega s + \\ (bk_\eta k_g - a\omega t_r)\sin\omega s \\ x_3(s) = -\frac{t_r k_\eta}{\omega^2}s - c_0 k_\eta - \frac{k_g}{\omega^2} + ac\cos\omega s + \\ bc\sin\omega s, \end{cases}$$

where $c = k_g^2 + t_r^2$, $a = c_1 + c_2$ and $b = i(c_1 - c_2)$.

Thus, we can state the following theorem:

Theorem 7.1. Let $\phi : J \subset \mathbb{R} \to \mathbb{E}^3$ be a twisted W - curve, then the position vector $\phi(s)$ is obtained with the following differentiable functions

$$\begin{cases} x_1(s) = \frac{t_r^2}{\omega^2}s + c_0 t_r + (ak_\eta t_r - b\omega k_g)\cos\omega s + \\ (bk_\eta t_r + a\omega k_g)\sin\omega s \\ \\ x_2(s) = \frac{t_r k_g}{\omega^2}s + c_0 k_g - \frac{k_\eta}{\omega^2} + (ak_\eta k_g + b\omega t_r)\cos\omega s + \\ (bk_\eta k_g - a\omega t_r)\sin\omega s \\ \\ x_3(s) = -\frac{t_r k_\eta}{\omega^2}s - c_0 k_\eta - \frac{k_g}{\omega^2} + ac\cos\omega s + \\ bc\sin\omega s. \end{cases}$$

8 Conclusions

In this paper, authors obtained the characterization of Hasimoto surfaces in Euclidean space $\mathbb{E}^3$. Hasimoto surfaces are investigated by using the Darboux frame and discuss the geometric properties. The position vector of W-curve is stated by a linear combination of its Frenet frame with differentiable functions.

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Conflict of Interest

The authors declare that they have no conflict of interest.

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