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Similarity Solutions for Solving Riesz Fractional Partial Differential Equations

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Abstract: In this work, we use the similarity method to solve fractional order partial differential equations where the fractional derivative is defined in Riesz sense. Two examples are presented to illustrate how problems are reduced from two-variable fractional partial differential equations to ordinary ones. Fourier transform method is used for solving the ordinary problems.

Keywords: Fractional partial differential equations, Riesz fractional derivatives, similarity methods.

1 Introduction

The field of fractional calculus has attracted the interest of researchers in many fields of applied sciences such as mathematics, physics, chemistry, engineering, finance and social sciences. This is the result of the availability of several definitions for fractional derivatives that are utilized to present more accurate models for real life phenomena. These definitions include Riemann-Liouville [1], Caputo [2], Riesz [3], Riesz-Feller [4], and Jumarie [5]. These definitions have been employed for generalizing many models into the form of fractional partial differential equations (FPDEs).

Analytic solutions to FPDEs were generally obtained either by using Laplace transform with Fourier transform or by applying the separation of variables technique [1]. Recently, some semi-analytic methods have been also utilized to obtain series solution to FPDEs. These methods include Adomian decomposition method [6] and [7], homotopy analysis method [8] and [9], homotopy perturbation method [10] and [11], variational iteration method [12] and [13], and fractional differential transformation method [14] and [15].

Riesz fractional derivative definition has been studied by authors either in finite or in infinite domains. Examples of the research that considered Riesz definition on infinite domains include the work in [16] where the authors obtained the fundamental solutions of the space Riesz FPDE and the space-time Riesz FPDE using methods of Fourier series expansion and Laplace transform. They also include the series solution obtained to such problems via homotopy analysis method [8] or via the variational iteration method [13]. Whereas the work on Riesz definition on finite domains include obtaining an approximate solution for the fractional diffusion equation with the Riesz fractional derivative by utilizing the McCormack numerical method [17]. Also, the work in [18] where the authors obtained the analytical solutions of two types of FPDEs with Riesz space fractional derivatives; fractional diffusion equation and fractional advection–dispersion equation.

Both linear and nonlinear partial differential equations have been tackled by symmetry methods. Yet, the application of these methods for obtaining solutions of FPDEs is still in the initial stage. The work reported in this area include for example the derivation of scaling transformations to reduce time-fractional heat equation with Riemann-Liouville fractional derivative to a fractional differential equation (FDE) but with Erdelyi-Kober fractional differential operator [19]. Also, similarity solutions are presented in [20] for the time-fractional nonlinear conduction equations to reduce them to ordinary FDEs that are solved by analytic and numerical techniques. Whereas for the fractional derivative defined in Caputo sense, symmetry properties of fractional diffusion equations are studied in [21]. Finally, the Lie group method is applied in [22] to a space-time fractional diffusion equation where the fractional derivative given by Jumarie sense.

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In this work, we solve space-fractional PDEs with fractional derivative in Riesz sense. We illustrate a direct approach to use similarity methods to reduce FPDEs to FDEs in the same fractional derivative. Two examples are presented where the resulting FDE is solved by Fourier transform method. The graphs of the solution, shows the continuation of the solution of the obtained FDE with the exact solutions of the corresponding integer order problem.

This article is arranged as follows. The definition and properties of Riesz fractional derivative are listed in Section 2. In Section 3, similarity method technique is illustrated to transform FPDEs into FDEs motivated by two examples. In Section 4, Fourier transform method is employed to obtain the solution to the FDE and the graphs of the solution are presented. The conclusion of this work is summarized in Section 5.

2 The Riesz Fractional Derivative

The Riesz fractional derivative R_x^α is defined as [13,23,24]

$$R_x^\alpha u(x) = -\frac{[D_+^\alpha u(x) + D_-^\alpha u(x)]}{2\cos(\alpha\pi/2)}, \qquad 0<\alpha<2, \quad \alpha\neq 1, \tag{1}$$

where $D_\pm^\alpha u(x)$ are the Weyl fractional derivatives defined by

$$f(x) = \begin{cases} \pm\frac{d}{dx} I_\pm^{1-\alpha} u(x), & 0<\alpha<1; \\ \frac{d^2}{dx^2} I_\pm^{2-\alpha} u(x), & 1<\alpha<2, \end{cases} \tag{2}$$

where $I_\pm^\beta$ denote the Weyl fractional integrals of order $\beta>0$, given by

$$I_+^\beta u(x) = \frac{1}{\Gamma(\beta)}\int_{-\infty}^{x} (x-z)^{\beta-1} u(z)dz,$$

$$I_-^\beta u(x) = \frac{1}{\Gamma(\beta)}\int_{x}^{\infty} (z-x)^{\beta-1} u(z)dz. \tag{3}$$

When $\alpha=0$ the Weyl fractional derivative degenerates into the identity operator

$$D_\pm^0 u(x) = Iu(x) = u(x). \tag{4}$$

For continuity we get

$$D_\pm^1 u(x) = \pm\frac{d}{dx}u(x), \qquad D_\pm^2 u(x) = \frac{d^2}{dx^2}u(x). \tag{5}$$

Evidently, in the case $\alpha=2$, Riesz fractional derivative takes the form of the second-order derivative operator

$$R_x^2 u(x) = \frac{d^2}{dx^2}u(x). \tag{6}$$

For the case $\alpha=1$, we have

$$R_x^1 u(x) = \frac{d}{dx}Hu(x) = \frac{d}{dx}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(z)}{z-x}dz, \tag{7}$$

where H is the Hilbert transform and the integral is understood in the Cauchy principal value sense.

3 Similarity Method Solution

In this section, we illustrate the technique for using similarity methods in solving FPDEs with Riesz definition of fractional derivative.

Problem 1. Consider the following problem

$$t^{1-\frac{\alpha}{2}}\frac{\partial u}{\partial t} = R_x^\alpha u, \qquad u=u(x,t). \tag{8}$$

To solve equation (8), first we perform its scaling transformation using similarity methods, see [20] and [25]. Consider the new independent and dependent variables denoted by $\bar{t}$, $\bar{x}$, and $\bar{u}$ defined in the following way

$$t=\lambda^{n}\bar{t}, \qquad x=\lambda^{p}\bar{x}, \qquad u=\lambda^{q}\,\bar{u}, \tag{9}$$

where λ is called the scaling parameter, p, q, and n are arbitrary constants to be determined such that equation (8) remains invariant under this transformation. From Riesz definition for the case $1<\alpha<2$, it can be easily verified that

$$\begin{aligned} R_x^{\alpha}u(x,t) &= \frac{-1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)}\frac{d^2}{dx^2}\left[\int_{-\infty}^{x}(x-z)^{1-\alpha}u(z)dz+\int_{x}^{\infty}(z-x)^{1-\alpha}u(z)dz\right] \\ &= \frac{-1}{2\cos(\alpha\pi/2)}[\lambda^{-2p}\frac{d^2}{d\bar{x}^2}\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{\bar{x}}(\lambda^{p}\bar{x}-\lambda^{p}\bar{z})^{1-\alpha}\lambda^{q}\,\bar{u}\lambda^{p}d\bar{z}+ \\ &\quad \lambda^{-2p}\frac{d^2}{d\bar{x}^2}\frac{1}{\Gamma(2-\alpha)}\int_{\bar{x}}^{\infty}(\lambda^{p}\bar{z}-\lambda^{p}\bar{x})^{1-\alpha}\lambda^{q}\,\bar{u}\lambda^{p}d\bar{z}] \\ &= \lambda^{q-p\alpha}R_{\bar{x}}^{\alpha}\bar{u}(\bar{x},\bar{t}). \end{aligned} \tag{10}$$

where $z=\lambda^{p}\bar{z}$. Also we have

$$\frac{\partial u}{\partial t}=\lambda^{q-n}\frac{\partial\bar{u}}{\partial\bar{t}}. \tag{11}$$

Hence, by substituting equations (10) and (11) into equation (8), we get

$$\lambda^{n(1-\frac{\alpha}{2})}\bar{t}\lambda^{q-n}\,\frac{\partial^{\alpha}\bar{u}}{\partial\bar{t}^{\alpha}}=\lambda^{q-p\alpha}\,R_{\bar{x}}^{\alpha}\,u(\bar{x},\bar{t}). \tag{12}$$

From equation (12), it is clear that by setting

$$n=2p, \tag{13}$$

equation (8) is invariant under transformation (9). The characteristic equation associated with transformation (9) is given by

$$\frac{du}{qu}=\frac{dx}{px}=\frac{dt}{nt}. \tag{14}$$

At $q=0$, this shows that $u\,(x,t)$ can be expressed as

$$u\,(x,t)=f\,(\zeta), \tag{15}$$

where $\zeta=xt^{\frac{-p}{n}}$.
By using formula (15), we have

$$\begin{aligned} R_x^{\alpha}u(x,t) &= \frac{-1}{2\cos(\alpha\pi/2)\Gamma(2-\alpha)}\frac{d^2}{dx^2}\left[\int_{-\infty}^{x}(x-z)^{1-\alpha}u(z)dz+\int_{x}^{\infty}(z-x)^{1-\alpha}u(z)dz\right] \\ &= \frac{-1}{2\cos(\alpha\pi/2)}[t^{\frac{-2p}{n}}\frac{d^2}{d\zeta^2}\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{\zeta}(\frac{\zeta}{t^{\frac{-p}{n}}}-\frac{y}{t^{\frac{-p}{n}}})^{1-\alpha}f(\zeta)t^{\frac{p}{n}}dy+ \\ &\quad t^{\frac{-2p}{n}}\frac{d^2}{d\zeta^2}\frac{1}{\Gamma(2-\alpha)}\int_{\zeta}^{\infty}\frac{y}{t^{\frac{-p}{n}}}-\frac{\zeta}{t^{\frac{-p}{n}}})^{1-\alpha}f(\zeta)t^{\frac{p}{n}}dy] \\ &= t^{\frac{-p}{n}\alpha}R_{\zeta}^{\alpha}f(\zeta), \end{aligned} \tag{16}$$

where $y=zt^{\frac{-p}{n}}$ and $\frac{d^2}{dx^2}=t^{\frac{-2p}{n}}\frac{d^2}{d\zeta^2}$.
Formula (15) yields

$$\frac{\partial u}{\partial t}=(\frac{-p}{n})x\,t^{\frac{-p}{n}-1}\frac{df}{d\zeta}. \tag{17}$$

From equation (13), the resulting FDE is given by:

$$\frac{-1}{2}\zeta\frac{df}{d\zeta}=R^{\alpha}_{\zeta}f(\zeta). \tag{18}$$

Problem 2. Consider the following problem

$$t\frac{\partial u}{\partial t}+x\frac{\partial u}{\partial x}=-t^{\frac{\alpha}{2}}\,R^{\alpha}_{x}u(x,t), \qquad u=u(x,t). \tag{19}$$

On the same manner, we use the similarity transformation variables defined by equation (9) we have

$$R^{\alpha}_{x}u(x,t)=\lambda^{q-p\alpha}R^{\alpha}_{\bar{x}}\bar{u}.(\bar{x},\bar{t}). \tag{20}$$

and

$$\begin{aligned}\frac{\partial u}{\partial x}&=\lambda^{q-p}\frac{\partial \bar{u}}{\partial \bar{x}},\\ \frac{\partial u}{\partial t}&=\lambda^{q-n}\frac{\partial \bar{u}}{\partial \bar{t}}.\end{aligned} \tag{21}$$

By substituting equations (20) and (21) into equation (19) we get

$$\lambda^{n}\bar{t}\lambda^{q-n}\frac{\partial \bar{u}}{\partial \bar{t}}+\lambda^{p}\bar{x}\lambda^{q-p}\frac{\partial \bar{u}}{\partial \bar{x}}=-\lambda^{\frac{n\alpha}{2}}\lambda^{q-p\alpha}R^{\alpha}_{\bar{x}}\bar{u}(\bar{x},\bar{t}). \tag{22}$$

From equation (22), it is clear that

$$n+q-n=p+q-p=\frac{n\alpha}{2}+q-p\alpha. \tag{23}$$

Equation (23) means that by setting $\frac{p}{n}=\frac{1}{2}$ then equation (19) is invariant under transformation (9). The characteristic equation becomes

$$\frac{du}{qu}=\frac{dx}{px}=\frac{dt}{nt}. \tag{24}$$

By solving the characteristic equation (24), and setting $q=0$, $u(x,t)$ can be expressed as

$$u\,(x,t)=f\,(\zeta),\ \zeta=x\,t^{\frac{-1}{2}}. \tag{25}$$

From formula (25) we have

$$R^{\alpha}_{x}u(x,t)=t^{\frac{-\alpha}{2}}R^{\alpha}_{\zeta}f(\zeta). \tag{26}$$

Finally, the FPDE (19) is reduced to the ordinary FDE (18).

4 Fourier Transform

To find the solution of equation (18), we consider the FDE of the form

$$kt\frac{d\,y}{d\,t}=R^{\alpha}_{t}y(t). \tag{27}$$

Apply Fourier transform, as adopted by [23] and [24], to both sides of equation (27), we get

$$-k(\omega\frac{dY}{d\omega}+Y(\omega))=-|\omega|^{\alpha}Y(\omega). \tag{28}$$

Equation (28) is a separable first order ordinary differential and its solution is given by

$$Y(\omega)=\frac{1}{\omega}C\,e^{\frac{|\omega|^{\alpha}}{k\alpha}}, \tag{29}$$

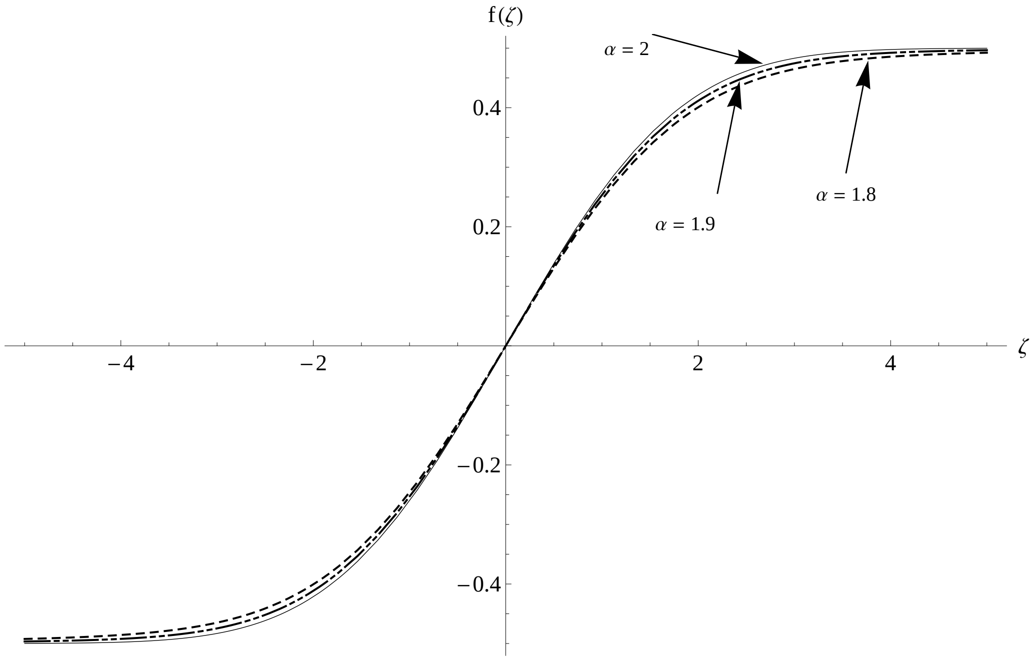

Fig. 1: The function $f(\zeta)$ described by equation (32) at different values of α.

where C is a constant.
Consider the case $C = -\frac{1}{j}$. Then

$$Y(\omega) = -\frac{1}{j\omega} e^{\frac{|\omega|^{\alpha}}{k\alpha}}, \tag{30}$$

and the solution of equation (27) takes the form

$$y(t) = \frac{1}{2\pi} \int_0^t \int_{-\infty}^{\infty} e^{\frac{|\omega|^{\alpha}}{k\alpha}} e^{-j\omega\tau} \, d\omega \, d\tau. \tag{31}$$

So, the solution of equation (18) becomes

$$f(\zeta) = \frac{1}{2\pi} \int_0^{\zeta} \int_{-\infty}^{\infty} e^{\frac{-2|\omega|^{\alpha}}{\alpha}} e^{-j\omega\tau} \, d\omega \, d\tau. \tag{32}$$

Evidently, when $\alpha = 2$, $f(\zeta)$ represents the classical error function which is the solution to the corresponding integer order differential equation.

Figure 1 shows the effect of changing the order of fractional derivative α on the behavior of the solution function $f(\zeta)$ given by equation (32). The figure also illustrates that as α approaches two, the graph takes the form of the graph of the classical error function $\mathrm{erf}(\frac{\zeta}{2})$ which is the solution of the integer-order differential equation corresponding to FDE (18).

5 Conclusions

The similarity method is used to solve FPDEs where the fractional derivative is given in Riesz sense. Because the similarity methods decreases the number of independent variables of the equation by one variable, we use it to transform the considered FPDE with two independent variables into ordinary FDE in the same fractional derivative. The ordinary FDE obtained is solved using Fourier transform. The graph of the solution function at different values of α indicates the continuation of the solution to the solution of the corresponding integer order problem as α tends to 2.

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