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Best Proximity Point of Generalized $(F-\tau)$-Proximal Non-Self Contractions in Generalized Metric Spaces

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Abstract: In this manuscript, we introduce the idea of generalized (F,τ)-proximal contraction and prove new best proximity results for these contractions in the context of a generalized metric space. Our findings extend and generalize a number of previous results from the literature. Also, We present an example to illustrate the applicability of our results.

Keywords: P-property, best proximity point, generalized (F,τ)-proximal contraction.

1 Introduction

The well-known fixed point finding, often known as the Banach contraction principle, is one of the most significant outcomes of mathematical analysis [1]. It is the most often used fixed point result in various disciplines of mathematics and is generalizable in a wide range of ways (see [2,3,4]). The fixed point result was defined in the context of whole metric spaces by Wardowski [5], who generalized the Banach contraction principle in metric spaces.

On the other hand, the importance of the FP technique lies in the fact that it presents a unified process and an important tool in solving equations that do not have to be linear. In the case of $d(x,Tx)\neq 0$, that is, a contraction mapping T does not possess a fixed point, it became necessary to search a point x that makes $d(x,Tx)$ is minimum with meaning the point x is close proximity to T.

The point x is called the best proximity $(BPP(T)$ of $T:A\to B$, if $d(x,Tx)=d(A,B))$, where $\{d(A,B)=\inf d(x,y): x\in A, y\in B\}$. Various best proximity point results were established on such spaces, for example, see [6,7,8] and references therein.

Sankar Raj [9] and Zhang et al. [10] defined the notion of P-property and weak P-property respectively. Beg et al. [11] defined the concept of generalized F-proximal non-self contraction mappings and obtained some best proximity point results.

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved under these spaces. In particular, generalized metric spaces were introduced by Branciari [12] , in such a way that triangle inequality is replaced by the rectangular inequality

$$d(x,y)\leq d(x,u)+d(u,v)+d(v,y),$$

for all pairwise distinct points x,y,u,v. Any metric space is a generalized metric space but in general, generalized metric space might not be a metric space. Various fixed point results were established on such spaces, the readers can refer to (see [13,14,15,16,17,18,19]).

Motivated by the above results, in this paper, we prove a new existence of best proximity point for generalized $(F-\tau)$-proximal contraction defined on a closed subset of a complete generalized metric space. Our theorems extend, generalize, and improve many existing results.

2 Preliminaries

Definition 1.[20] *Let X be a non-empty set and $d: X\times X\to\mathbb{R}^+$ be a mapping such that for all $x,y\in X$ and for all distinct points $u,v\in X$, we have*

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(i) $d(x,y)=0$ *if and only if* $x=y$;
(ii) $d(x,y)=d(y,x)$ *for all distinct points* $x,y\in X$;
(iii) $d(x,y)\leq d(x,u)+d(u,v)+d(v,y)$.

Then (X,d) is called an generalized metric space.

Definition 2.[20] *Let* (X,d) *be a generalized metric space and* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence in* X, *and* $x\in X$. *Then*
(i)the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges to* x *if and only if*

$$\lim_{n\to+\infty} d(x,x_n)=0.$$

(ii)the sequence $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy if*

$$\lim_{n,m\to+\infty} d(x_n,x_m)=0.$$

Lemma 1.[20] *Let* (X,d) *be an generalized metric space and* $\{x_n\}_n$ *be a Cauchy sequence with pairwise disjoint elements in* X. *If* $\{x_n\}_{n\in\mathbb{N}}$ *converges to* $x,y\in X$, *then* $x=y$.

Definition 3.[20]. *Let* (X,d) *be a generalized metric space.* X *is said to be complete if every Cauchy sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *converges to* $x\in X$.

Definition 4.[5]. *Let* Γ *be the family of all functions* $F:\mathbb{R}^+\to\mathbb{R}$ *such that*

(i) F *is strictly increasing.*
(ii)For each sequence $(x_n)_{n\in\mathbb{N}}$ *of positive numbers,*

$$\lim_{n\to\infty} x_n=0 \text{ if and only if } \lim_{n\to\infty} F(x_n)=-\infty.$$

(iii) F *is continuous.*

Definition 5.[21]. *Let* (A,B) *be a pair of non-empty subsets of a metric space* (X,d). *The following notions hold*

$$d(A,B)=\{\inf d(a,b): a\in A, b\in B\},$$
$$A_0=\left\{\begin{array}{c} a\in A \text{ there exists } b\in A \\ \text{such that } d(a,b)=d(A,B)\end{array}\right\},$$
$$B_0=\left\{\begin{array}{c} b\in B \text{ there exists } a\in A \\ \text{such that } d(a,b)=d(A,B)\}\end{array}\right\}.$$

Definition 6.[21] *Let* $T:A\to B$ *be a given mapping. An element* x^* *is said to be a best proximity point of* T *if*

$$d(x^*,Tx^*)=d(A,B).$$

Definition 7.[9]. *Let* (A,B) *be a pair of non empty subsets of a metric space* (X,d) *such that* A_0 *is non-empty. Then the pair* (A,B) *is to have P-property if and only if*

$$\begin{cases} d(x_1,y_1)=d(A,B) \\ d(x_2,y_2)=d(A,B)\end{cases}$$
$$\Rightarrow d(x_1,x_2)=d(y_1,y_2),$$

for all $x_1,x_2\in A_0$ *and* $y_1,y_2\in B_0$.

Definition 8.[22]. *A set* B *is called approximately compact with respect to* A *if every sequence* $\{x_n\}$ *of* B *with* $d(y,x_n)\to d(y,B)$ *for some* $y\in A$ *has a convergent subsequence.*

Definition 9.[5] *Let* Γ *be the family of all functions* $F:(0,+\infty)\to(1,+\infty)$ *such that*

(F_1) F *is strictly increasing;*
(F_2) *For each sequence* $x_n\in(0,+\infty)$,

$$\lim_{n\to 0} x_n=0, \text{ if and only if } \lim_{n\to\infty} F(x_n)=-\infty;$$

(F_3) *There exists* $k\in(0,1)$ *such that* $\lim_{x\to 0} x^k F(x)=0$.

Definition 10.[5] *Let* (X,d) *be a metric space and* $T:X\to X$ *be a self-mapping.* T *is called an* $(F,\tau)-$*contraction if there exist* $F\in\Gamma$ *and* $\tau>0$ *such that for any* $x,y\in X$,

$$d(Tx,Ty)>0$$
$$\Rightarrow F[d(Tx,Ty)]+\tau\leq F(d(x,y)),$$

3 Main results

In this section, inspired by the notion of F-proximal contraction of the first and second kind, we introduce new generalized (F,τ)-proximal first and second kind on complete generalized metric space.

We begin with the following definition:

Definition 11. *We say that a mapping* $T:A\to B$ *is a generalized* (F,τ)*-proximal contraction of first kind if there exist* $F\in\Gamma$, $\tau>0$ *and* $a,b,c,h\geq 0$ *with* $a+b+c+2h\leq 1$, $c\neq 1$ *such that*

$$\begin{cases} d(u_1,Tv_1)=d(A,B) \\ d(u_2,Tv_2)=d(A,B)\end{cases}$$
$$\Rightarrow F(d(u_1,u_2))+\tau$$
$$\leq F\left[\begin{array}{c} ad(v_1,v_2)+bd(u_1,v_1) \\ +cd(u_2,v_2)+h(d(v_2,u_1))\end{array}\right],$$

for all $u_1,u_2,v_1,v_2\in A$ *and* $u_1\neq v_1$.

Definition 12. *We say that a mapping* $T:A\to B$ *is a generalized* (F,τ)*-proximal contraction of second kind, if there exist* $F\in\gamma$, $\tau>0$ *and* $a,b,c,h\geq 0$ *with* $a+b+c+h\leq 1$, $c\neq 1$ *such that*

$$\begin{cases} d(u_1,Tv_1)=d(A,B) \\ d(u_2,Tv_2)=d(A,B)\end{cases}$$
$$\Rightarrow F(d(Tu_1,Tu_2))+\tau$$
$$\leq F\left[\begin{array}{c} ad(Tv_1,Tv_2)+bd(Tu_1,Tv_1) \\ +cd(Tu_2,Tv_2)+hd(Tv_2,Tu_1)\end{array}\right]$$

for all $u_1,u_2,v_1,v_2\in A$ *and* $Tu_1\neq Tv_1$.

Theorem 1. *Let* (X,d) *be a complete generalized metric space and* (A,B) *be a pair of non-void closed subsets of* X. *If* B *is approximately compact with respect to* A *and* $T:A\to B$ *satisfy the following conditions:*

(i)$T(A_0) \in B_0$ and the pair (A,B) satisfies the weak P-property;
(ii)T is a generalized (F,τ)-proximal contraction of first kind.

Then there exists a unique $u \in A$ such that $d(u,Tu) = d(A,B)$. In addition, for any fixed element $u_0 \in A_0$, the sequence $\{u_n\}$ defined by

$$d(u_{n+1},Tu_n) = d(A,B),$$

converges to the proximity point.

*Proof.*Choose an element $u_0 \in A_0$. As, $T(A_0) \in B_0$, therefore there is an element $u_1 \in A_0$ satisfying

$$d(u_1,Tu_0) = d(A,B).$$

Since $T(A_0) \in B_0$, there exists $u_2 \in A_0$ such that

$$d(u_2,Tu_1) = d(A,B).$$

Again, since $T(A_0) \in B_0$, there exists $u_3 \in A_0$ such that

$$d(u_3,Tu_2) = d(A,B).$$

Continuing this process, by induction, we construct a sequence $x_n \in A_0$ such that

$$d(u_{n+1},Tu_n) = d(A,B), \forall n \in \mathbb{N}.$$

Since (A,B) satisfies the P property, we conclude that

$$d(u_n,u_{n+1}) = d(Tu_n,Tu_{n+1}), \forall n \in \mathbb{N}.$$

If $u_{n_0} = u_{n_0+1}$ for some $n_0 \in \mathbb{N}$, one obtains

$$d\left(u_{n_0},Tu_{n_0}\right) = d\left(u_{n_0+1},Tu_{n_0}\right) = d(A,B),$$

that is, $u_{n_0} \in BPP$. Thus, we suppose that $d(u_n,u_{n+1}) > 0$ for all $n \in \mathbb{N}$.

We shall prove that the sequence u_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = 0.$$

As T is generalized (F,τ)-proximal contraction of the first kind, we have that

$$\begin{aligned}
&F(d(u_n,u_{n+1})) + \tau \\
&\leq F\begin{bmatrix} ad(u_{n-1},u_n) + bd(u_{n-1},u_n) \\ +cd(u_n,u_{n+1}) + h(d(u_n,u_n)) \end{bmatrix} \\
&= F[ad(u_{n-1},u_n) + bd(u_{n-1},u_n) + cd(u_n,u_{n+1})] \\
&= F[(a+b)d(u_{n-1},u_n) + cd(u_n,u_{n+1})].
\end{aligned}$$

Since F is strictly increasing, continuous function and $\tau > 0$, we deduce

$$d(u_n,u_{n+1}) < (a+b)d(u_{n-1},u_n) + cd(u_n,u_{n+1}).$$

Thus

$$d(u_n,u_{n+1}) < \frac{a+b}{1-c}(d(u_{n-1},u_n)).$$

If $a+b+c+h = 1$, we have $0 < 1-c$ and so, for each $\forall n \in \mathbb{N}$,

$$d(u_n,u_{n+1}) \leq \frac{a+b}{1-c}(d(u_{n-1},u_n)) \leq d(u_{n-1},u_n).$$

Consequently

$$F(d(u_n,u_{n+1})) < F(d(u_{n-1},u_n)).$$

If $a+b+c+h < 1$, we have $0 < 1-c$ and so

$$d(u_n,u_{n+1}) < d(u_{n-1},u_n), \forall n \in \mathbb{N},$$

Consequently

$$F(d(u_n,u_{n+1})) \leq F(d(u_{n-1},u_n)) - \tau.$$

It implies that

$$\begin{aligned}
F(d(u_n,u_{n+1})) &\leq F(d(x_{n-1},u_n) - \tau \\
&\leq F(d(u_{n-2},u_{n-1}) - 2\tau \\
&\leq \cdots \leq F(d(u_0,u_1) - n\tau.
\end{aligned}$$

Taking the limit as $n \to \infty$, we have

$$F(d(u_n,u_{n+1})) \leq \lim_{n\to\infty} F(d(u_0,u_1)) - n\tau = -\infty.$$

by (F_2), we obtain that

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = 0. \tag{1}$$

Now, we shall prove that

$$\lim_{n\to\infty} d(u_n,u_{n+2}) = 0.$$

As T is generalized (F,τ)-proximal contraction of the first kind then, we get

$$\begin{aligned}
&F(d(u_n,u_{n+2})) + \tau \\
&\leq F\begin{bmatrix} ad(u_{n-1},u_{n+1}) + bd(u_{n-1},u_n) \\ +cd(u_{n+1},u_{n+2}) + h(d(u_n,u_{n+1})) \end{bmatrix} \\
&\leq F\begin{bmatrix} ad(u_{n-1},u_{n+1}) + \\ (b+c+h)(d(u_{n-1},u_n)) \end{bmatrix} \\
&\leq F[\max\{d(u_{n-1},u_{n+1}), d(u_{n-1},u_n)\}].
\end{aligned}$$

Take $a_n = d(u_n,u_{n+2})$ and $b_n = d(u_n,u_{n+1})$. Hence

$$F(a_n) \leq F(\max\ a_{n-1},b_{n-1}) - \tau.$$

Since F is a decreasing and continuous function, then, we get

$$a_n < \max\{a_{n-1},b_{n-1}\}.$$

and

$$b_n \leq b_{n-1} \leq \max\{a_{n-1},b_{n-1}\}.$$

which implies that

$$\max\{a_n,b_n\} \le \max\{a_{n-1},b_{n-1}\},\ \forall n\in\mathbb{N}.$$

Therefore, the sequence $\max\{a_{n-1},b_{n-1}\}_{n\in\mathbb{N}}$ is monotone non increasing, this implies that, there exists $\lambda \ge 0$ so that

$$\lim_{n\to\infty}\max\{a_n,b_n\} = \lambda.$$

Let $\lambda > 0$, then

$$\lim_{n\to\infty}\sup a_n = \lim_{n\to\infty}\sup\max\{a_n,b_n\} = \lim_{n\to\infty}\max\{a_n,b_n\}.$$

Taking the $\lim\sup_n \to\infty$ in (3.7), and using the properties of F_3, we obtain

$$F\left(\lim_{n\to\infty}\sup a_n\right) \le F\left(\lim_{n\to\infty}\max\{a_{n-1},b_{n-1}\}\right)-\tau.$$

Therefore

$$F(\lambda) \le F(\lambda)-\tau.$$

By (F_1), we get $\lambda<\lambda$, a contradiction. Therefore

$$\lim_{n\to\infty} d(u_{n+2,}u_n) = 0. \tag{2}$$

Next, we shall prove that $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d(u_{n,}u_m)=0$, for all $n\in\mathbb{N}$. If otherwise there exists an $\varepsilon>0$ for which we can find sequence of positive integers $\left\{u_{n_{(k)}}\right\}_k$ and $\left\{u_{m_{(k)}}\right\}_k$ of $\{u_n\}$ such that, for all positive integers k, $n_{(k)} > m_{(k)} > k$,

$$d\left(u_{m_{(k)}},u_{n_{(k)}}\right) \ge \varepsilon \text{ and } d\left(u_{m_{(k)}},u_{n_{(k)-1}}\right) < \varepsilon. \tag{3}$$

Now, using (3) and the rectangular inequality, one has

$$\begin{aligned}\varepsilon &\le d\left(u_{m_{(k)}},u_{n_{(k)}}\right)\\ &\le d\left(x_{m_{(k)}},x_{m_{(k)+1}}\right)+d\left(u_{m_{(k)+1}},u_{m_{(k)-1}}\right)\\ &+d\left(u_{m_{(k)-1}},u_{n_{(k)}}\right)\\ &< d\left(u_{m_{(k)}},u_{m_{(k)+1}}\right)+d\left(u_{m_{(k)+1}},u_{m_{(k)}-1}\right)+\varepsilon.\end{aligned}$$

Then

$$\lim_{k\to\infty} d\left(u_{m_{(k)}},u_{n_{(k)}}\right) = \varepsilon. \tag{4}$$

By rectangular inequality, we have

$$\begin{aligned}d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right) &\le d\left(u_{m_{(k)+1}},u_{m_{(k)}}\right)+d\left(u_{m_{(k)}},u_{n_{(k)}}\right)\\ &+d\left(u_{n_{(k)}},u_{n_{(k)+1}}\right).\end{aligned}$$

$$\begin{aligned}d\left(u_{m_{(k)}},u_{n_{(k)}}\right) &\le d\left(u_{m_{(k)}},u_{m(k)+1}\right)+d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right)\\ &+d\left(u_{n_{(k)+1}},u_{n_{(k)}}\right).\end{aligned}$$

$$\begin{aligned}\varepsilon &\le d\left(u_{m_{(k)}},u_{n_{(k)}}\right) \le d\left(u_{m_{(k)}},u_{n_{(k)-1}}\right)\\ &+d\left(u_{n_{(k)-1}},u_{n_{(k)+1}}\right)+d\left(u_{n_{(k)+1}},u_{n_{(k)}}\right).\end{aligned}$$

Letting $k\to\infty$ in the above inequalities, and using (1), (2) and (3), one can write

$$\lim_{k\to\infty} d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right) = \varepsilon, \tag{5}$$

and

$$\lim_{k\to\infty} d\left(u_{m_{(k)}},u_{n_{(k)-1}}\right) = \varepsilon. \tag{6}$$

On the other hand, we get

$$\begin{aligned}&M\left(u_{m_{(k)}},u_{n_{(k)}}\right)\\ &=\min\left\{\begin{array}{c} d\left(u_{m_{(k)}},u_{n_{(k)}}\right),d\left(u_{m_{(k)}},Tu_{m_{(k)}}\right)\\ ,d\left(u_{n_{(k)}},Tu_{n_{(k)}}\right),d\left(u_{m_{(k)}},Tu_{n_{(k)}}\right)\end{array}\right\}\\ &=\min\left\{\begin{array}{c} d\left(u_{m_{(k)}},u_{n_{(k)}}\right),d\left(u_{m_{(k)}},u_{m_{(k)-1}}\right)\\ ,d\left(u_{n_{(k)}},u_{n_{(k)-1}}\right),d\left(u_{m_{(k)}},u_{n_{(k)-1}}\right)\end{array}\right\}.\end{aligned}$$

Letting $k\to\infty$ in the above inequalities and using (4), and (6), we have

$$\lim_{k\to\infty} M\left(u_{m_{(k)}},u_{n_{(k)}}\right) = \varepsilon.$$

By (5) and for each $A=\frac{\varepsilon}{2}>0$, there exists $n_0\in\mathbb{N}$ such that

$$\left|d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right)-\varepsilon\right| \le A,\ \forall n\ge n_0,$$

which leads to

$$d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right) \ge A > 0,\ \forall n\ge n_0.$$

Again by (6) and for each $B=\frac{\varepsilon}{2}>0$, there exists $n_1\in\mathbb{N}$ such that

$$M\left(u_{m_{(k)}},u_{n_{(k)}}\right) \ge B > 0,\ \forall n\ge n_1.$$

Substituting $u_1=u_{m_{(k)+1}}, u_2=u_{n_{(k)+1}}, v_1=u_{m_{(k)}}$ and $v_1=u_{n_{(k)}}$ in assumption of the theorem, we get

$$F\left(d\left(u_{m_{(k)+1}},u_{n_{(k)+1}}\right)\right) \tag{7}$$

$$\le F\left\{\begin{array}{l} ad\left(u_{m_{(k)}},u_{n_{(k)}}\right)\\ +bd\left(u_{m_{(k)}1},u_{n_{(k)}}\right)\\ +cd\left(u_{n_{(k)}+1},u_{n_{(k)}}\right)\\ +hd\left(u_{n_{(k)}},u_{m_{(k)}+1}\right)\end{array}\right\}-\tau.$$

Letting $k\to\infty$ in (7) and using (F_1) and (F_3), we obtain that

$$F(\varepsilon) < [F(a\varepsilon+b\varepsilon+c\varepsilon+h\varepsilon)].$$

Hence $\varepsilon < \varepsilon$, which is a contraction. Thus $\lim_{n,m\to\infty} d(u_n,u_m) = 0$, that is $\{x_n\}$ is a Cauchy sequence, then there exists $z \in A$ such that

$$\lim_{n\to\infty} d(u_n,u) = 0.$$

Also

$$\begin{aligned} d(u,B) &\le d(u,Tu_n) \\ &\le d(u,u_n) + d(u_n,u_{n+1}) + d(u_{n+1},Tu_n) \\ &= d(u,u_n) + d(u_n,u_{n+1}) + d(A,B) \\ &\le d(u,u_n) + d(u_n,u_{n+1}) + d(u,B). \end{aligned}$$

Therefore, $d(u,Tu_n) \to d(u,B)$. In spite of the fact that B is approximately compact with respect to A, the sequence $\{Tu_n\}$ has a subsequence $\{Tu_{n_k}\}$ converging to some element $v \in B$. So it turns out that

$$d(u,v) = \lim_{n\to\infty} d\left(u_{n_k+1}, Tu_{n_k}\right) = d(A,B). \tag{8}$$

Thus u must be an element of A_0. Again, since $T(A_0) \in B_0$, there exists $t \in A_0$ such that

$$d(t,Tu) = d(A,B), \tag{9}$$

for some element t in A. Using the weak p-property and (8), we have

$$d(u_{n_k+1},t) = d(Pu_{n_k},Pu), \forall n_k \in \mathbb{N}.$$

If for some n_0, $d(t,u_{n_0+1}) = 0$, consequently $d(Pu_{n_0},Tu) = 0$. So $Pu_{n_0} = Tu$, hence $d(A,B) = d(u,Tu)$. Thus the conclusion is immediate. So let for any $n \ge 0$, $d(t,u_{n+1}) > 0$. Since T is a generalized (F,τ)-proximal contraction of the first kind, it follows from this that

$$\begin{aligned} &F(d(t,u_{n+1})) + \tau \\ &\le F\begin{bmatrix} ad(u,u_n) + bd(t,u) \\ +cd(u_n,u_{n+1}) + hd(u_n,t) \end{bmatrix} \end{aligned} \tag{10}$$

Since F is continuous function, by letting $n \to \infty$ in inequality (10), we obtain

$$\begin{aligned} F(d(t,u)) + \tau &\le F[(b+h)(d(u,t))] \\ &\le F[(d(u,t))] \\ &< F(d(t,u)). \end{aligned}$$

It is a contradiction. Therefore, $u = t$, that

$$d(u,Tu) = d(t,Tu) = d(A,B).$$

Uniqueness: Suppose that there is another best proximity point z of the mapping T such that

$$d(z,Tz) = d(A,B).$$

Since T is a generalized (F,τ)-proximal contraction of the first kind, it follows from this that

$$\begin{aligned} F(d(z,u)) + \tau &\le F\begin{bmatrix} ad(z,u) + bd(z,z) \\ +cd(u,u) + h(d(z,u)) \end{bmatrix} \\ &= F[(a+h)d(z,u)], \end{aligned}$$

which is a contradiction. Thus, z and u must be identical. Hence T has a unique best proximity point.

Assume that $\ell \in \Omega$. It is obvious that $\ell \in \Re_{d_\Xi}(\ell,\rho)$ for $\rho > \vartheta_\Xi$. This yields $\rho \in \Re_{d_\Xi}(\ell,\rho) \subseteq \cup_{\ell\in\Omega,\rho>\vartheta_\Xi} \Re_{d_\Xi}(\ell,\rho)$.

Next, we state and prove the best proximity point theorem for non-self generalized (F,τ)-proximal contraction of the second kind.

Theorem 2. *Let (X,d) be a complete generalized metric space and (A,B) be a pair of non-void closed subsets of (X,d). If A is approximately compact with respect to B and $T : A \to B$ satisfies the following conditions :*

(i) $T(A_0) \in B_0$ and the pair (A,B) satisfies the weak P-property;
(ii) T is continuous generalized (F,τ)-proximal contraction of second kind.

Then there exists a unique $u \in A$ such that $d(u,Tu) = d(A,B)$ and $u_n \to u$, where u_0 is any fixed point in A_0 and $d(u_{n+1},Tu_n) = d(A,B)$ for $n \ge 0$. Further, if z is another best proximity point of T, then $Tu = Tz$.

Proof. Similar to Theorem 1, we can find a sequence $\{u_n\}$ in A_0 such that

$$d(u_{n+1},Tu_n) = d(A,B). \tag{11}$$

for all non-negative integral values of n. From the p-property and (11), we get

$$d(u_n,u_{n+1}) = d(Tu_{n-1},Tu_n), \forall n \in \mathbb{N}.$$

If for some n_0, $d(u_{n_{0+1}},u_{n_{0+2}}) = 0$, consequently $d(Tu_{n_0},Tu_{n_0+1}) = 0$. So $Tu_{n_0} = Tu_{n_0+1}$, hence $d(A,B) = d(Tu_{n_0},T_{n_0+1})$. Thus the conclusion is immediate. So let for any $n \ge 0$, $d(Tu_n,Tu_{n+1}) > 0$. We shall prove that the sequence u_n is a Cauchy sequence. Let us first prove that

$$\lim_{n\to\infty} d(u_n,u_{n+1}) = 0.$$

As T is generalized (F,τ)-proximal contraction of the second kind, we have

$$\begin{aligned} &F(d(Tu_n,Tu_{n+1})) + \tau \\ &\le F\begin{bmatrix} ad(Tu_{n-1},Tu_n) + bd(Tu_{n-1},Tu_n) \\ +cd(Tu_n,Tu_{n+1}) + hd(Tu_n,Tu_n) \end{bmatrix} \\ &\le F\begin{bmatrix} ad(Tu_{n-1},Tu_n) + bd(Tu_{n-1},Tu_n) \\ +cd(Tu_n,Tu_{n+1}) \end{bmatrix} \\ &= F[(a+b)d(Tu_{n-1},Tu_n) + cd(Tu_n,Tu_{n+1})]. \end{aligned}$$

Since F is strictly increasing, we deduce that

$$\begin{aligned} &d(Tu_n,Tu_{n+1}) \\ &< (a+b)d(Tu_{n-1},Tu_n) + cd(Tu_n,Tu_{n+1}), \end{aligned}$$

which leads to

$$d(Tu_n,Tu_{n+1}) < \frac{a+b}{1-c}(d(Tu_{n-1},Tu_n)).$$

If $a+b+c+h=1$, we have $0<1-c$ and so

$$d(Tu_n,Tu_{n+1}) \leq \frac{a+b}{1-c}(d(Tu_{n-1},Tu_n)) \leq d(Tu_{n-1},Tu_n), \forall n\in\mathbb{N};$$

Consequently

$$F(d(Tu_n,Tu_{n+1})) \leq F(d(Tu_{n-1},Tu_n))-\tau.$$

If $a+b+c+h<1$, we have $0<1-c$ and so

$$d(Tu_n,Tu_{n+1}) < d(Tu_{n-1},Tu_n), \ \forall n\in\mathbb{N}.$$

Consequently

$$F(d(Tu_n,Tu_{n+1})) \leq F(d(Tu_{n-1},Tu_n))-\tau,$$

which implies that

$$\begin{aligned} F(d(Tu_n,Tu_{n+1})) &\leq F(d(Tu_{n-1},Tu_n)-\tau \\ &\leq F(d(Tu_{n-2},Tu_{n-1})-2\tau \\ &\leq \cdots \leq F(d(Tu_0,Tu_1)-n\tau. \end{aligned}$$

Taking the limit as $n\to\infty$, we have

$$F(d(Tu_n,Tu_{n+1})) \leq \lim_{n\to\infty} F(d(Tu_0,Tu_1))-n\tau=-\infty.$$

Since $F\in\Gamma$, we obtain

$$\lim_{n\to\infty} d(Tu_n,Tu_{n+1})=0. \tag{12}$$

We shall prove that

$$\lim_{n\to\infty} d(Tu_n,Tu_{n+2})=0.$$

As T is generalized (F,τ)-proximal contraction of the first kind, we have

$$\begin{aligned} &F(d(Tu_n,Tu_{n+2}))+\tau \\ &\leq F\begin{bmatrix} ad(Tu_{n-1},Tu_{n+1})+bd(Tu_{n-1},Tu_n) \\ +cd(Tu_{n+1},Tu_{n+2})+h(d(Tu_n,Tu_{n+1})) \end{bmatrix} \\ &\leq F\begin{bmatrix} ad(Tu_{n-1},Tu_{n+1}) \\ +(b+c+h)(d(Tu_{n-1},Tu_n)) \end{bmatrix} \\ &\leq F[\max\{d(Tu_{n-1},Tu_{n+1}),d(Tu_{n-1},Tu_n)\}]. \end{aligned}$$

Take $\alpha_n=d(Tu_n,Tu_{n+2})$ and $\beta_n=d(Tu_n,Tu_{n+1})$, we get

$$F(\alpha_n) \leq F(\max\ \alpha_{n-1},\beta_{n-1})-\tau. \tag{13}$$

Since F is decreasing and continuous, then, we have

$$\alpha_n < \max\{\alpha_{n-1},\beta_{n-1}\},$$

and

$$\beta_n \leq \beta_{n-1} \leq \max\{\alpha_{n-1},\beta_{n-1}\}.$$

which yields that

$$\max\{\alpha_n,\beta_n\} \leq \max\{\alpha_{n-1},\beta_{n-1}\}, \ \forall n\in\mathbb{N}.$$

Therefore, the sequence $\max\{\alpha_{n-1},\beta_{n-1}\}_{n\in\mathbb{N}}$ is monotone non increasing, there exists $\mu\geq 0$ such that

$$\lim_{n\to\infty}\max\{\alpha_n,\beta_n\}=\mu.$$

Now, we assume that $\mu>0$, then

$$\limsup_{n\to\infty}\alpha_n=\limsup_{n\to\infty}\max\{\alpha_n,\beta_n\}=\lim_{n\to\infty}\max\{\alpha_n,\beta_n\}.$$

Taking the $\limsup_n\to\infty$ in (13) and using (F_3), we obtain that

$$F\left(\limsup_{n\to\infty}\alpha_n\right) \leq F\left(\lim_{n\to\infty}\max\{\alpha_{n-1},\beta_{n-1}\}\right)-\tau.$$

Therefore

$$F(\mu)\leq F(\mu)-\tau.$$

By (F_1) we get $\mu<\mu$, a contradiction. Hence

$$\lim_{n\to\infty} d(Tu_{n+2,}Tu_n)=0. \tag{14}$$

Next, we shall prove that $\{Tu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e, $\lim_{n\to\infty} d(Tu_{n,}Tu_m)=0$, for all $n\in\mathbb{N}$. Suppose to the contrary, there exists $\varepsilon>0$ and sequences $\{Tn_{(k)}\}$ and $\{Tm_{(k)}\}$ of natural numbers such that

$$Tm_{(k)} > Tn_{(k)} > k, \tag{15}$$

$$d\left(Tu_{m_{(k)}},Tu_{n_{(k)}}\right) \geq \varepsilon,$$

$$d\left(Tu_{m_{(k)-1}},Tu_{n_{(k)}}\right) < \varepsilon.$$

Using the rectangular inequality, we find that

$$\begin{aligned} \varepsilon \leq d\left(Tu_{m_{(k)}},Tu_{n_{(k)}}\right) &\leq d\left(Tu_{m_{(k)}},Tu_{n(k)-1}\right) \\ &+d\left(Tu_{n(k)-1},Tu_{n_{(k)+1}}\right)+d\left(Tu_{n(k)+1},Tu_{n_{(k)}}\right) \\ &<\varepsilon+d\left(Tu_{n(k)-1},Tu_{n_{(k)+1}}\right) \\ &+d\left(Tu_{n(k)-1},Tu_{n_{(k)}}\right). \end{aligned}$$

It follows from (12) and (14) that

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)}},Tu_{n_{(k)}}\right)=\varepsilon.$$

Using the rectangular inequality, we get

$$\begin{aligned} \varepsilon &\leq d\left(Tu_{m_{(k)}},Tu_{n_{(k)}}\right) \\ &\leq d\left(Tu_{m_{(k)}},Tu_{n(k)-1}\right)+d\left(Tu_{n(k)-1},Tu_{n_{(k)+1}}\right) \\ &+d\left(Tu_{n(k)+1},Tu_{n_{(k)}}\right) \end{aligned}$$

and

$$\begin{aligned} \varepsilon &\leq d\left(Tu_{m_{(k)}},Tu_{n_{(k)+1}}\right) \\ &\leq d\left(Tu_{m_{(k)}},Tu_{n(k)}\right)+d\left(Tu_{n(k)},Tu_{n_{(k)-1}}\right) \\ &+d\left(Tu_{n(k)-1},Tu_{n_{(k)+1}}\right), \end{aligned}$$

which implies that

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)}}, Tu_{n_{(k)+1}}\right) = \varepsilon. \tag{16}$$

Similarly

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)}}\right) = \varepsilon. \tag{17}$$

From the rectangular inequality, we obtain that

$$\begin{aligned} & d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) \\ \leq\ & d\left(u_{m_{(k)+1}}, Tu_{m_{(k)}}\right) + d\left(Tu_{m(k)}, Tu_{n_{(k)}}\right) \\ & + d\left(Tu_{n_{(k)}}, Tu_{n_{(k)+1}}\right). \end{aligned} \tag{18}$$

Also, we can write

$$\begin{aligned} & d\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \\ \leq\ & d\left(Tu_{m_{(k)}}, Tu_{m_{(k)+1}}\right) + d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) \\ & + d\left(Tu_{n_{(k)+1}}, Tu_{n_{(k)}}\right). \end{aligned} \tag{19}$$

Letting $k \to \infty$ in the inequality (18) and (19), we get

$$\lim_{k\to\infty} d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right) = \varepsilon. \tag{20}$$

Substituting $u_1 = Tu_{m_{(k)+1}}, u_2 = Tu_{n_{(k)+1}}, v_1 = Tu_{m_{(k)}}$ and $v_1 = Tu_{n_{(k)}}$ in our assumption of the theorem, we have

$$\begin{aligned} & F\left(d\left(Tu_{m_{(k)+1}}, Tu_{n_{(k)+1}}\right)\right) + \tau \\ \leq\ & F\left\{\begin{array}{l} ad\left(Tu_{m_{(k)}}, Tu_{n_{(k)}}\right) \\ +bd\left(Tu_{m_{(k)}+1}, Tu_{n_{(k)}}\right) \\ +cd\left(Tu_{n_{(k)}+1}, Tu_{n_{(k)}}\right) \\ +hd\left(Tu_{n_{(k)}}, Tu_{m_{(k)}+1}\right) \end{array}\right\}. \end{aligned} \tag{21}$$

Letting $k \to \infty$ in (21) and using (F_1) and (F_3), we get

$$F(\varepsilon) + \tau \leq F(a\varepsilon + b\varepsilon + c\varepsilon + h\varepsilon).$$

which yields that $\varepsilon < \varepsilon$, a contradiction. Hence $\lim_{n,m\to\infty} d(Tu_n, Tu_m) = 0$, thea is $\{Tu_n\}$ is a Cauchy sequence. Then there exists $v \in B$ such that

$$\lim_{n\to\infty} d(Tu_n, v) = 0.$$

Also

$$\begin{aligned} & d(v,A) \\ & \leq d(v, Tu_n) \\ & \leq d(v,u_n) + d(u_n,u_{n+1}) + d(u_{n+1}, Tu_n) \\ & = d(v,u_n) + d(u_n,u_{n+1}) + d(A,B) \\ & \leq d(v,u_n) + d(u_n,u_{n+1}) + d(v,A). \end{aligned}$$

Therefore, $d(v, Tu_n) \to d(v,A)$. Since A is approximately compact with respect to B , the sequence $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to some element $u \in A$. So it turns out that

$$d(u,v) = \lim_{n\to\infty} d\left(u_{n_k+1}, Tu_{n_k}\right) = d(A,B). \tag{22}$$

Because T is a continuous mapping

$$d(u,Tu) = \lim_{n\to\infty} d(u_{n+1}, Tu_n) = d(A,B).$$

For the uniqueness: Suppose that there is another best proximity point z of the mapping T such that

$$d(z,Tz) = d(A,B).$$

Since T is a generalized (F,τ)-proximal contraction of the first second, then we have

$$\begin{aligned} & F(d(Tz,Tu)) + \tau \\ & \leq F\left[\begin{array}{c} ad(Tz,Tu) + bd(Tz,Tz) \\ +cd(Tu,Tu) + h(d(Tz,Tu)) \end{array}\right] \\ & = F[(a+h)d(Tz,Tu)], \end{aligned}$$

which is a contradiction. Thus z and u must be identical. Hence, T has a unique best proximity point.

Theorem 3. *Let (X,d) be a complete generalized metric space and (A,B) be a pair of non-void closed subsets of (X,d). Let $T: A \to B$ satisfies the following conditions:*

(i) $T(A_0) \in B_0$ and the pair (A,B) satisfies the weak $P-$property;
(ii) T is a generalized (F,τ)-proximal contraction of the first ans second kind.

Then there exists a unique $u \in A$ such that $d(u,Tu) = d(A,B)$ and $u_n \to u$, where u_0 is any fixed point in A_0 and $d(u_{n+1}, Tu_n) = d(A,B)$ for $n \geq 0$.

Proof. Similar to Theorem 1, we find a sequence $\{u_n\}$ in A_0 such that

$$d(u_{n+1}, Tu_n) = d(A,B)$$

for all non-negative integral values of n. Similar to Theorem 1, we can show that the sequence $\{u_n\}$ is a Cauchy sequence. Thus it converges to some element u in A. As in Theorem 2, it can be shown that the sequence $\{Tu_n\}$ is a Cauchy sequence and converges to some element v in B. Therefore

$$d(u,v) = \lim_{n\to\infty} d(u_{n+1}, Tu_n) = d(A,B). \tag{23}$$

Eventually, u becomes an element of A_0. Using the fact $T(A_0) \in B_0$, we have

$$d(t,Tu) = d(A,B).$$

for some $t \in A$. From the $P-$property and (23), we get

$$d(u_{n+1},t) = d(Tu_n, Tu), \forall n \in \mathbb{N}.$$

If for some n_0, $d(t,u_{n_0+1}) = 0$, consequently $d(Tu_{n_0},Tu) = 0$. So $Tu_{n_0} = Tu$, hence $d(A,B) = d(u,Tu)$. Thus the conclusion is immediate. So let for any $n \geq 0$, $d(t,u_{n+1}) > 0$. Since T is a generalized (F,ϕ)-proximal contraction of the first kind, it can be seen that

$$\begin{aligned} &F(d(t,u_{n+1}))+\tau \\ &\leq F\left(\begin{array}{c} ad(u,u_n)+bd(t,u) \\ +cd(u_n,u_{n+1})+hd(u_n,t) \end{array}\right). \end{aligned}$$

Since F is continuous function, by letting $n \to \infty$ in the inequality (**??**), we obtain that $d(u,Tu) = d(t,Tu) = d(A,B)$. Also, as in the theorem 1, we can obtain the uniqueness of the best proximity point of mapping T.

The following example support the theortical results

*Example 1.*Let $X = \{0,1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5},\frac{1}{6},\frac{1}{7},\frac{1}{8}\}$. Define the function $d : X \times X \to (0,+\infty)$ by

$$d\left(\frac{1}{4},\frac{1}{6}\right) = d\left(\frac{1}{5},\frac{1}{7}\right) = 1,$$

$$\begin{aligned} d\left(\frac{1}{2},\frac{1}{7}\right) &= d\left(\frac{1}{4},\frac{1}{7}\right) \\ &= d\left(\frac{1}{6},1\right) = d\left(\frac{1}{6},\frac{1}{7}\right) \\ &= 7, \end{aligned}$$

$$d\left(\frac{1}{3},\frac{1}{7}\right) = d\left(\frac{1}{2},\frac{1}{6}\right) = 2,$$

$$d\left(0,\frac{1}{2}\right) = d\left(1,\frac{1}{3}\right) = 3,$$

$$d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{5}\right) = 4,$$

$$d\left(0,\frac{1}{4}\right) = d\left(1,\frac{1}{5}\right) = 5,$$

$$\begin{aligned} d(0,1) &= d\left(\frac{1}{2},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) \\ &= d\left(\frac{1}{6},\frac{1}{7}\right) = d\left(0,\frac{1}{5}\right) \\ &= d(0,\frac{1}{7}) = 7, \end{aligned}$$

$$\begin{aligned} d\left(0,\frac{1}{8}\right) &= d\left(\frac{1}{2},\frac{1}{8}\right) = d\left(\frac{1}{4},\frac{1}{8}\right) \\ &= d\left(\frac{1}{6},\frac{1}{8}\right) = d\left(1,\frac{1}{8}\right) \\ &= d\left(\frac{1}{6},\frac{1}{5}\right) = d\left(\frac{1}{4},1\right) \\ &= d\left(\frac{1}{2},1\right) = 8, \end{aligned}$$

$$\begin{aligned} d\left(0,\frac{1}{6}\right) &= d\left(1,\frac{1}{7}\right) = d\left(\frac{1}{6},\frac{1}{3}\right) \\ &= d\left(\frac{1}{4},\frac{1}{3}\right) = d\left(0,\frac{1}{3}\right) \\ &= d\left(\frac{1}{3},\frac{1}{8}\right) = d\left(\frac{1}{5},\frac{1}{8}\right) \\ &= d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{7},\frac{1}{8}\right) = 8. \end{aligned}$$

Clearly, (X,d) is a generalized metric space but not a metric space. Indeed

$$d\left(0,\frac{1}{6}\right) > d\left(0,\frac{1}{2}\right) + d\left(\frac{1}{2},\frac{1}{6}\right).$$

Now, if $A = \{\frac{1}{2},\frac{1}{4},\frac{1}{6}\}$ and $B = \{1,\frac{1}{5},\frac{1}{7}\}$, then $d(A,B) = 7$, $A = A_0$ and $B = B_0$. Define the mapping $T : A \to B$ by $T(\frac{1}{6}) = 1$, $T(\frac{1}{2}) = 5$, $T(\frac{1}{4}) = 1$. Then $T(A_0) \subset B_0$. Describe a function $F : (0,+\infty) \to \mathbb{R}$ by $F(t) = ln(t)$ and $\tau \in (0,1)$. Let $u = \frac{1}{4}$, $v = \frac{1}{6}$, $x = \frac{1}{2}$, $y = \frac{1}{4} \in A$, then

$$d(\frac{1}{4},T(\frac{1}{2})) = d(\frac{1}{6},T(\frac{1}{4})) = 7$$

and

$$\begin{aligned} F[d(u,v)]+\tau &= 0+\tau \\ &\leq F(d(x,y)) = ln(5). \end{aligned}$$

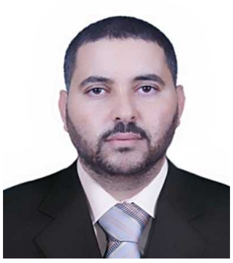
Thus, all conditions of Theorem (1) holds. Moreover, $x = \frac{1}{6}$ is a unique best proximity point of T.

Competing interests:
Authors declare that they have no competing interests

References

[1] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, *Fundam. Math.*, **3(1)**, 133-181 (1922).
[2] A. Kari, M. Rossafi, E. Marhrani, M. Aamri, Fixed-point theorem for Nonlinear $F-$contraction via $w-$distance, *Adv. Math. Phy.*, **2020**, Article ID 6617517 (2020).
[3] H. A. Hammad, M. De la Sen, H. Aydi, Analytical solution for differential and nonlinear integral equations via $F_{\varpi_e}-$Suzuki contractions in modified ϖ_e-metric-like spaces, *J. Func. Spaces*, **2021**, Article ID 6128586, 13 pages.
[4] H. A. Hammad, P. Agarwal, L.G.J. Guirao, Applications to boundary value problems and homotopy theory via tripled fixed point techniques in partially metric spaces, *Mathematics*, 9, (2021), 2012.
[5] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.* **2012** , 94, (2012).
[6] V. Parvaneh, M. R. Haddadi, H. Aydi, On Best Proximity Point Results for Some Type of Mappings, *J. Func. Spaces*, **2020**, Article ID 6298138, 6 pages.

[7] H. A. Hammad, R. A. Rashwan, A. Nafea, F. Jarad, Quadruple best proximity points with applications to functional and integral equations, Adv. Math. Phy., **2022**, Article ID 1849891, 16 pages.
[8] H. Aydi, H. Lakzian, Z. Mitrović, S Radenović, Best Proximity Points of MT-Cyclic Contractions with Property UC, *Numerical Func. Anal. Opt.*, **41(7)**, 871-882, (2019).
[9] V. S. Raj, A best proximity point theorem for weakly contractive non-self-mappings, *Nonlinear Anal.* **74**, 4804-4808, (2011).
[10] J. Zhang, Y. Su, Q. Cheng, A note on A best proximity point theorem for Geraghty-contractions, *Fixed Point Theory Appl*. **2013**, 99 (2013).
[11] I. Beg, G. Mani and A. J. Gnanaprakasam, Best proximity point of generalized $F-$proximal non-self contractions, *J. Fixed Point Theory Appl*. **23**, 49 (2021).
[12] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen. **57**, 31-37, (2000).
[13] M. Jleli, B. Samet, A new generalization of the Banach contraction principle, J Inequal Appl. **2014**, 38 (2014).
[14] R.A. Rashwan, H.A. Hammad and M.G. Mahmoud, Common fixed point results for weakly compatible mappings under implicit relations in complex valued g-metric spaces, *Inform. Sci. Lett.*, **8(3)**, 111-119 (2019).
[15] H. A. Hammad, R. A. Rashwan, A. Nafea, Quadruple coincidence point methodologies for finding a solution to a system of integral equations with a directed graph, *Appl. Math. Inf. Sci.* **16(1)**, 1-14 (2022).
[16] D. M. Albaqeri, H. A. Hammad, Convergence and stability results for new random algorithms in separable Banach spaces, *Appl. Math. Inf. Sci.* **16(3)**, 441-456, (2022).
[17] M. Mudhesh, H. A. Hammad, E. Ameer, A. Ali, Fixed point results under new contractive conditions on closed balls, *Appl. Math. Inf. Sci.* **16(4)**, 555-564, (2022).
[18] H. A. Hammad, P. Agarwal, L.G.J. Guirao, Applications to boundary value problems and homotopy theory via tripled fixed point techniques in partially metric spaces. *Mathematics*, **9**, 2012 (2021).
[19] H. A. Hammad, H. Aydi, M. De la Sen, Solutions of fractional differential type equations by fixed point techniques for multivalued contractions. *Complexity*, **2021**, (2021), ID 5730853, 13 pages.
[20] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, *Fixed Point Theory Appl.*, **2015**, 61 (2015).
[21] A. Eldred, W. A. Kirk, P. Veeramani, Proximal normal structure and relatively nonexpansive mappings, *Stud. Math.* **171(3)**, 283-293 (2005).
[22] Basha, S.S., Veeramani, P.: Best proximity pair theorems for multifunctions with open fibres, *J. Approx. Theory.* **103**, 119-129 (2000).



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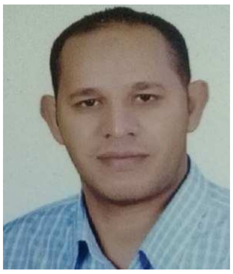

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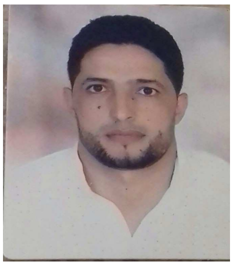

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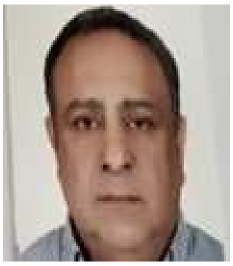

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