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Moment Properties of Dual Generalized Order Statistics Based on Fréchet-Weibull Distribution

M. I. Khan

Department of Mathematics, Faculty of Science, Islamic University of Madinah, Madinah, Saudi Arabia

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Abstract: The dual generalized order statistics or sometimes called lower generalized order statistics is a combined structure of examining random variables in decreasing order. In this paper, some simple recurrence relations for single and product moments of dual generalized order statistics from Fréchet-Weibull distribution have been derived and its special cases are discussed. The characterization results are also presented based on the recurrence relations.

Keywords: Recurrence relations, Fréchet-Weibull distribution, Characterization.

1 Introduction

The Fréchet-Weibull distribution was introduced by [1] as a generalization of some of the commonly used distributions for modeling lifetime data, such as the generalized new extended Weibull distribution, Lindley Weibull distribution, the half-logistic generalized Weibull distribution and Weibull distribution. The tremendous application of Fréchet-Weibull distribution has not been found in extreme values (earthquakes, floods,) but also in several areas such as quality control, engineering, physics, and medicine.

The probability density function (pdf) of Fréchet-Weibull distribution is given in (1)

$$f(x) = \alpha\gamma\beta^{\lambda}\lambda^{\alpha\gamma}x^{-1-\alpha\gamma}\exp\left\{-\beta^{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha\gamma}\right\}, \quad x>0, \quad \alpha,\beta,\lambda,\gamma>0 \tag{1}$$

where α,γ and λ,β are shape, scale parameters.
The cumulative density function (cdf) is given by

$$F(x) = \exp\left\{-\beta^{\alpha}\left(\frac{\lambda}{x}\right)^{\alpha\gamma}\right\}, \quad x>0, \quad \alpha,\beta,\lambda,\gamma>0. \tag{2}$$

We note that from (1) and (2)

$$F(x) = \frac{x^{1+\alpha\gamma}}{\alpha\gamma\beta^{\lambda}\lambda^{\alpha\gamma}}f(x). \tag{3}$$

[2] introduced and extensively studied the model of reversed ordered variables, popularly known as dual (lower) generalized order statistics (dgos). This model unifies to study the properties of decreasing order variables, (from highest to lowest life length arranged of an electric bulb). Lower record values and reversed order statistics are of main interest to this technique.

The joint pdf of n dgos is given by

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i}f(x_i)\right)[F(x_n)]^{k-1}f(x_n), \tag{4}$$

* Corresponding author e-mail: izhar.stats@gmail.com

In literature, the moments properties of some specific distribution based on dgos are investigated by several authors. Notable references include, [3,4,5,6,7,8].
Characterization technique plays an essential role in statistics and probability distribution. Many characterization techniques are available in the literature. One of them is recurrence relations. Several authors have characterized the different distribution through different approaches based on dgos. For the detailed discussion see, [9,10,11,12,13,14].
In this article, the rest of the findings are outlined as follows. Relationship for moments of dgos for Fréchet-Weibull distribution have been discussed in Section 2 and Section 3. Characterization results based on recurrence relations are presented in Section 4. The conclusion is given in Section 5.

2 Single Moments

Here assume two cases:
Case I $\gamma_i \neq \gamma_j, i \neq j, i, j = 1,2,\cdots,n-1$.
In view of (4), *pdf* of the r^{th} dgos is

$$f_{X_d(r,n,\tilde{m},k)}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1}, \quad -\infty < x < \infty, \tag{5}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i,$$

and

$$a_i(r) = \prod_{j(\neq i)=1}^{r} \frac{1}{(\gamma_i - \gamma_j)}, \quad \gamma_i \neq \gamma_j, \, 1 \leq i \leq r \leq n$$

Case II $m_i = m_j = m, \quad i, j = 1,2,\cdots,n-1$.
The *pdf* of the r^{th} dgos is

$$f_{X_d(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}[F(x)], \quad -\infty < x < \infty, \tag{6}$$

where,

$$\gamma_i = k + (n-i)(m+1), \qquad h_m^d(x) = \begin{cases} -\frac{1}{m+1}x^{m+1}, & m \neq -1 \\ -\log(x), & m = -1 \end{cases}$$

and

$$g_m^d(x) = h_m^d(x) - h_m^d(1), \quad x \in [0,1).$$

Theorem 2.1. The single moments of r^{th} dgos $(1 \leq r \leq n)$ for Fréchet-Weibull distribution is related as

$$E[X^j_{d(r,n,\tilde{m},k)}] = E[X^j_{d(r-1,n,\tilde{m},k)}] - \frac{j}{\alpha\gamma\beta^{\lambda}\lambda^{\alpha\gamma}\gamma_r} E[X^{j+\alpha\gamma}_{d(r,n,\tilde{m},k)}]. \tag{7}$$

Proof. [15] have shown that

$$E[\xi\{X_d(r,n,\tilde{m},k)\}] - E[\xi\{X_d(r-1,n,\tilde{m},k)\}] = -C_{r-2}\int_{-\infty}^{\infty} \xi'(x) \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} dx.$$

Let $\xi(x) = x^j$. Then

$$E[X^j_{d(r,n,\tilde{m},k)}] - E[X^j_{d(r-1,n,\tilde{m},k)}] = -jC_{r-2}\int_{0}^{\infty} x^{j-1} \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i} dx.$$

In context of (3), we have

$$E[X^j_{d(r,n,\tilde{m},k)}] - E[X^j_{d(r-1,n,\tilde{m},k)}] = -jC_{r-2}\int_{0}^{\infty} x^{j-1} \sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i - 1} \left[\frac{x^{1+\alpha\gamma}}{\alpha\gamma\beta^{\lambda}\lambda^{\alpha\gamma}}\right] f(x) dx.$$

After simplification posses (7).
This completes the proof of Theorem 2.1.
Corollary 2.1. Let $m_i = m_j = m$, then the single moments of Fréchet-Weibull distribution will be

$$E[X^j_{d(r,n,m,k)}] = E[X^j_{d(r-1,n,m,k)}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r} E[X^{j+\alpha\gamma}_{d(r,n,m,k)}].$$

Remark 2.1. The relation given in (7) reduces to relationship for order statistics as follows,

$$E[X^j_{n-r+1:n}] = E[X^j_{n-r+2:n}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}(n-r+1)} E[X^{j+\alpha\gamma}_{n-r+1:n}]$$

at $m = 0$ and, $k = 1$.
Remark 2.2. The relation given in (7) reduced to single moment of k^{th} lower record value as

$$E[X^j_{L(r)}] = E[X^j_{L(r-1)}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}k} E[X^{j+\alpha\gamma}_{L(r)}], \quad \text{at } m = -1, \text{ and } k \geq 1.$$

3 Product Moments

Case I $\gamma_i \neq \gamma_j$.
The joint *pdf* of the r^{th} and s^{th}–dgos is,

$$f_{X_d(r,n,\tilde{m},k), X_d(s,n,\tilde{m},k)}(x,y) = C_{s-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \left[\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i}\right] \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} \tag{8}$$

where

$$a_i^{(r)}(s) = \prod_{j(\neq i)=r+1}^{s} \frac{1}{(\gamma_i - \gamma_j)}, \qquad \gamma_i \neq \gamma_j,\ r+1 \leq i \leq s \leq n.$$

Case II $m_i = m_j = m$.
The joint *pdf* of the r^{th} and s^{th} –dgos is,

$$\begin{aligned} f_{X_d(r,n,m,k), X_d(s,n,m,k)}(x,y) &= \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1} F(x) \times \\ &\quad [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \quad -\infty < y < x < \infty. \end{aligned} \tag{9}$$

Theorem 3.1. The product moment of r^{th} and s^{th}–dgos ($1 \leq r < s \leq n, i, j > 0$) for Fréchet-Weibull distribution are given as

$$\begin{aligned} E[X^i_{d(r,n,\tilde{m},k)}, X^j_{d(s,n,\tilde{m},k)}] &= E[X^i_{d(r,n,\tilde{m},k)}, X^j_{d(s-1,n,\tilde{m},k)}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_s} \times \\ &\quad E[X^i_{d(r,n,\tilde{m},k)}, X^{j+2\alpha}_{d(s,n,\tilde{m},k)}]. \end{aligned} \tag{10}$$

Proof. [15] have proved that

$$\begin{aligned} &E[\xi\{X_d(r,n,\tilde{m},k), X_d(s,n,\tilde{m},k)\}] - E[\xi\{X_d(r,n,\tilde{m},k), X_d(s-1,n,\tilde{m},k)\}] = \\ &\quad -C_{s-2} \int_{-\infty}^{\infty} \int_0^x \frac{d}{dy} \xi(x,y) \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \left[\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i}\right] \frac{f(x)}{F(x)} dydx. \end{aligned} \tag{11}$$

Consider $\xi(x,y) = \xi_1(x)\xi_2(y) = x^i y^j$ in (11). In context of (3), we get

$$\begin{aligned} &E[X^i_{d(r,n,\tilde{m},k)}, X^j_{d(s,n,\tilde{m},k)}] - E[X^i_{d(r,n,\tilde{m},k)}, X^j_{d(s-1,n,\tilde{m},k)}] = \\ &\quad -\frac{jC_{s-1}}{\gamma_s} \int_0^{\infty} \int_0^x \left\{\frac{y^{1+\alpha\gamma}}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}}\right\} x^i y^{j-1} \sum_{i=r+1}^{s} a_i^{(r)}(s) \left(\frac{F(y)}{F(x)}\right)^{\gamma_i} \left[\sum_{i=1}^{r} a_i(r)[F(x)]^{\gamma_i}\right] \frac{f(x)}{F(x)} \frac{f(y)}{F(y)} dydx, \end{aligned}$$

which refers to (10).
Theorem 3.1 is proved.
Theorem 3.1 corresponds to Theorem 2.1 at $i = 0$.
Corollary 3.1. For $m_i = m_j = m$, product moments of Fréchet-Weibull distribution is given as

$$E[X^i_{d(r,n,m,k)}, X^j_{d(s,n,m,k)}] = E[X^i_{d(r,n,m,k)}, X^j_{d(s-1,n,m,k)}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_s} E[X^i_{d(r,n,m,k)}, X^{j+\alpha\gamma}_{d(s,n,m,k)}]$$

Remark 3.1. At $m = 0, k = 1$ in (10) result reduced for order statistic from Fréchet-Weibull distribution is given as

$$E[X^{i,j}_{n-r+1,n-s+1:n}] = E[X^{i,j}_{n-r+1,n-s+2:n}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}(n-s+1))} E[X^{i,j+\alpha\gamma}_{n-r+1,n-s-1:n}]$$

Remark 3.2. Letting $m = -1, k \geq 1$ in (10) result reduced for k^{th} lower record values from Fréchet-Weibull distribution is given as

$$E[X^{i,j}_{L(r,s)}] = E[X^{i,j}_{L(r,s-1)}] - \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}k} E[X^{i,j+\alpha\gamma}_{L(r,s)}]$$

4 Characterization

The following theorems contain, the characterization of Fréchet-Weibull distribution based on dgos.
Theorem 4.1. Let X be a continuous r.v. having cdf $F(x)$ and pdf $f(x)$. Suppose $0 < F(x) < 1$, for all $x > 0$, then

$$E[X^j_{d(r,n,m,k)}] - E[X^j_{d(r-1,n,m,k)}] = -\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r} E[X^{j+\alpha\gamma}_{d(r,n,m,k)}] \tag{12}$$

if and only if

$$F(x) = \exp\left\{-\beta^\alpha\left(\frac{\lambda}{x}\right)^{\alpha\gamma}\right\}, \quad x > 0,\ \alpha,\beta,\lambda,\gamma > 0. \tag{13}$$

Proof. From Theorem 2.1 necessary part follows with $\tilde{m} = m$. On the contrary, if the relation (12) is satisfied, then (12) can be rearranging as

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx - \frac{C_{r-1}(r-1)}{(r-1)!\gamma_r}\int_0^\infty x^j[F(x)]^{\gamma_r+m}g_m^{r-2}F(x)f(x)dx =$$
$$-\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r}\int_0^\infty x^{j+\alpha\gamma}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx$$

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^j[F(x)]^{\gamma_r}g_m^{r-2}[F(x)]f(x)\left[\frac{g_mF(x)}{F(x)} - \frac{(r-1)[F(x)]^m}{\gamma_r}\right]dx = -\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r}\frac{C_{r-1}}{(r-1)!}\times$$
$$\int_0^\infty x^{j+\alpha\gamma}[F(x)]^{\gamma_r-1}g_m^{r-1}F(x)f(x)dx. \tag{14}$$

Let,

$$h(x) = \frac{[F(x)]^{\gamma_r}g_m^{r-1}F(x)}{\gamma_r}. \tag{15}$$

Differentiating (15) both sides, we get

$$h^{'}(x) = -[F(x)]^{\gamma_r}g_m^{r-2}[F(x)]f(x)\left[\frac{g_m[F(x)]}{[F(x)]} - \frac{(r-1)[F(x)]^m)}{\gamma_r}\right].$$

Thus

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^jh^{'}(x)dx = \frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\alpha\gamma}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx. \tag{16}$$

Integrating LHS in (16) by parts and substituting the value of $h(x)$,

$$\frac{C_{r-1}}{(r-1)!\gamma_r}\int_0^\infty jx^{j-1}[F(x)]^{\gamma_r}g_m^{r-1}[F(x)]dx-\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_r}\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j+\alpha\gamma}[F(x)]^{\gamma_r-1}g_m^{r-1}[F(x)]f(x)dx=0$$

which reduces to,

$$\frac{C_{r-1}}{(r-1)!}\int_0^\infty x^{j-1}[F(x)]^{\gamma_r})g_m^{r-1}[F(x)]f(x)\left[\frac{F(x)}{f(x)}-\frac{x^{1+\alpha\gamma}}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}}\right]dx=0. \tag{17}$$

Making use of the Müntz-Szász theorem (see [16]) to (17), we get.

$$F(x,\alpha,\beta,\lambda,\gamma)=\frac{x^{1+\alpha\gamma}}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}}f(x)$$

which is (3) and this relationship holds between *pdf* and *cdf* of Fréchet-Weibull distribution.

Theorem 4.2. As stated in Theorem 4.1. Fix positive integers i and j. A necessary and sufficient condition for Fréchet-Weibull distributed as follows

$$E[X^i_{d(r,n,m,k)},X^j_{d(s,n,m,k)}]=E[X^i_{d(r,n,m,k)},X^j_{d(s-1,n,m,k)}]-\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_s}E[X^i_{d(r,n,m,k)},X^{j+\alpha\gamma}_{d(s,n,m,k)}]. \tag{18}$$

Proof. From Theorem 3.1, necessary part follows from with $\tilde{m}=m$. On the contrary if the relation (18) is satisfied, that is

$$E[X^i_{d(r,n,m,k)},X^j_{d(s,n,m,k)}]-E[X^i_{d(r,n,m,k)},X^j_{d(s-1,n,m,k)}]=-\frac{j}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}\gamma_s}E[X^i_{d(r,n,m,k)},X^{j+\alpha\gamma}_{d(s,n,m,k)}].$$

Now by [15], for $\xi(x,y)=x^iy^j$,

$$-\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^x x^iy^{j-1}[F(x)]^mf(x)g_m^{r-1}[F(x)]\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}[F(y)]^{\gamma_s}dydx=$$
$$-\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}}\times$$
$$\int_0^\infty\int_0^x x^iy^{j+\alpha\gamma}[F(x)]^mf(x)g_m^{r-1}[F(x)]\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}[F(y)]^{\gamma_s}f(y)dydx$$

$$\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}\int_0^\infty\int_0^x x^iy^{j-1}[F(x)]^mf(x)g_m^{r-1}[F(x)]\left[h_m(F(y))-h_m(F(x))\right]^{s-r-1}\times$$
$$[F(y)]^{\gamma_s-1}\left[\frac{F(y)}{f(y)}-\frac{y^{1+\alpha\gamma}}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}}\right]dydx=0. \tag{19}$$

Concerning the extension of Müntz-Szász theorem to (19), we obtain

$$\frac{F(y)}{f(y)}=\frac{y^{1+\alpha\gamma}}{\alpha\gamma\beta^\lambda\lambda^{\alpha\gamma}},\quad y>0,\ \alpha,\beta,\lambda,\gamma>0$$

and hence the result.
Theorem 4.2 reduces to Theorem 4.1 at $i=0$.

5 Conclusion

In this study, moments properties of Fréchet-Weibull distribution have been derived based on dgos. Several deductions are also discussed. The characterization results are presented. The outcomes of this article might be of use for researchers in reversed ordered random variables and applied sciences (industrial).

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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