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The Cordiality of Cone and Lemniscate Graphs

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Abstract: A graph is said to be cordial if it has 0-1 labeling that satisfies certain conditions $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$. In this paper we present necessary and sufficient conditions for cone and lemniscate graphs to be cordial.

Keywords: cone, lemniscate, cordial graph, sum graph

1 Introduction

Labeling graphs are used widely in different subjects including astronomy, coding theory and communication networks. The concept of graph labeling was introduced during the sixties' of the last century by Rosa [1]. Many researches have been working with different types of labeling graphs [2,3,4]. An excellent reference for this purpose is the survey written by Gallian [5]. All graphs considered in this theme are finite, simple and undirected. The original concept of cordial graphs is due to Chait[4]. He showed that each tree is cordial; an Euerlian graph is not cordial if its size is congruent to $2(mod\ 4)$. In [6] a graph G with n vertices and m edges, and every vertex has odd degree, then G is not cordial when $n+m \equiv 2(mod\ 4)$. Let $G = (V, E)$ be a graph and let $f : V \to \{0, 1\}$ be a labeling of its vertices, and let the induced edge labeling $f^* : E \to \{0, 1\}$ be given by $f^*(e) = (f(u) + f(v))(mod 2)$, where $e = uv$ and $u, v \in V$. Let v_0 and v_1 be the numbers of vertices that are labeled by 0 and 1, respectively, and let e_0 and e_1 be the corresponding numbers of edges. Such a labeling is called *cordial* if both $|v_0 - v_1| \leq 1$ and $|e_0 - e_1| \leq 1$ hold. A graph is called *cordial* if it has a cordial labeling. The *sum* or *join* of G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set and edge set given by $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup J$, respectively, where J consists of edges join each vertex of G_1 to every vertex of G_2. In [2,3], Diab reported several results concerning the sum and union of the cycles C_n and paths P_m together and with other specific graphs. In [7], Kirchherr discussed the cordiality of a cactus which is a connected graph all whose blocks are cycles and similar result is given by S. M. Lee and A. Liu [6]. In this paper, we define the lemniscate graph denoted by $L_{n,m} := C_n * C_m$, as the graph which is formed from two cycles C_n and C_m by merging a vertex of C_m. It easy to see that order of $L_{n,m}$ is $n+m-1$ and its size is $n+m$, $L_{n,m}$ is an Euerlian graph and $L_{n,m}$ is isomorphic to $L_{m,n}$. In section 3, we prove that any lemniscate graph is cordial if and only if its size is not congruent to $2(mod\ 4)$. Finally, we show that the cone graph,$\overline{K}_n + C_m$, is cordial for all $n \geq 1$ and $m \geq 3$.

2 Terminologies and Notations

We define the lemniscate graph denoted by $L_{n,m} := C_n * C_m$, as the graph which is formed from two cycles C_n and C_m by merging a vertex of C_m. It easy to see that order of $L_{n,m}$ is $n+m-1$ and its size is $n+m$, $L_{n,m}$ is an Euerlian graph and $L_{n,m}$ is isomorphic to $L_{m,n}$. We consider the common vertex of $L_{n,m} := C_n * C_m$ to be the last vertex of C_n and first vertex of C_m. A cone graph, also called the n-point suspension of C_m, is defined by the graph join $\overline{K}_n + C_m$, where C_m is a cyclic graph and $\overline{K}_n$ is isolated graph. Given one cycle of the lemniscate graph with $4r$ vertices, we let L_{4r} denote the labeling 0011...0011 (repeated r-times), let L'_{4r} denote the labeling 1100...1100 (repeated r times). The denote the labeling 1001 1001...1001 (repeated r times) and labeling 0110...0110 (repeated r times) are written S_{4r} and S'_{4r}. Sometimes, we modify this by adding symbols at one end or the other (or both), thus $L_{4r}101$ denotes the labeling

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0011 ...0011 101 when $r \geq 1$ and 101 when $r = 0$. Similarly, $1L'_{4r}$ is the labeling 1 1100...1100 when $r \geq 1$ and 1 when $r = 0$. The labeling 01...01 (repeated r times) denoted by M_r if r is even and 01...010 if r is odd. Likewise 10...10, is denoted by M'_r if r is even and 10...101, if r is odd. For a given labeling of the lemniscate graph $L_{n,m} := C_n * C_m$, we let v_i and e_i (for $i = 0,1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for C_n, and we let y_i and b_i be those for C_m. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) - 1$, and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b1)$. For a given labeling of the join $G + H$, we let v_i and e_i (for $i = 0,1$) be the numbers of labels that are i as before, we let x_i and a_i be the corresponding quantities for G, and we let y_i and b_i be those for H. It follows that $v_0 = x_0 + y_0; v_1 = x_1 + y_1; e_0 = a_0 + b_0 + x_0y_0 + x_1y_1$ and $e_1 = a_1 + b_1 + x_0y_1 + x_1y_0$, thus, $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1)$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b1) + (x_0 - x_1)(y_0 - y_1)$. The labeling of $L_{n,m}$ or $\overline{K}_n + C_m$ is denoted by $[A;B]$; where the labeling A is taken for C_n or $\overline{K}_n$ and the labeling B is taken for C_m.

3 The cordially of lemniscate graph

In this section, we show that the lemniscate $L_{n,m}$ is cordial for all $n,m \geq 3$. We give a proof which is not the same one given by [6] and [7]. Through out this section the labeling of the common vertex is considered as a part of the first cycle,C_n. Also, the labeling of the first cycle is starting from the common vertex and the labeling of the second cycle C_m is starting from the vertex that follows the common vertex. This target will be achieved after the following series of lemmas.

Lemma 3.1. The lemniscate graph $L_{3,m}$ is cordial if and only if $m \neq [3(mod4)]$.
Proof. We prove the easy direction first. Let $m = 3(mod4)$, then it is obvious that $L_{3,4t+3}$, $t \geq 0$, is an Eulerian graph with size congruent to $2(mod4)$; and consequently $L_{3,4t+3}$ is not cordial [2]. Now, let $m \neq [3(mod4)]$. Then we consider three cases:
Case (1) Suppose that $m \equiv 0(mod4)$, i.e. $m = 4t, t \geq 1$, then, we choose the labeling $[100;011L_{4t-4}]$ for $L_{3,4t}$. Therefore
$x_0 = 2, x_1 = 1, a_0 = 1, a_1 = 2, y_0 = 2t - 1, y_1 = 2t, b_0 = 2t$ and $b_1 = 2t$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = -1$. As an examples, Figure (1) illustrate the cordiality of $L_{3,4}$. Thus $L_{3,4t}$ is cordial for all $t \geq 1$.
Case (2) Suppose that $m \equiv 1(mod4)$, i.e. $m = 4t + 1, t \geq 1$, then, we choose the labeling $[110;L'_{4t}]$ for $L_{3,4t+1}$. Therefore $x_0 = 1, x_1 = 2, a_0 = 1, a_1 = 2, y_0 = 2t, y_1 = 2t, b_0 = 2t + 1$ and $b_1 = 2t$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = -1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. As an examples, Figure (2) illustrate the cordiality of $L_{3,5}$. Thus $L_{3,4t+1}$ is cordial for all $t \geq 1$.
Case (3) Suppose that $m \equiv 2(mod4)$, i.e. $m = 4t + 2, t \geq 1$, then, one can choose the labeling $[010;L_{4t}1]$ for $L_{3,4t+2}$. Therefore $x_0 = 2, x_1 = 1, a_0 = 1, a_1 = 2, y_0 = 2t, y_1 = 2t + 1, b_0 = 2t + 2$ and $b_1 = 2t$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 1$. As an examples, Figure (3) illustrate the cordiality of $L_{3,6}$. Thus $L_{3,4t+2}$ is cordial for all $t \geq 1$. Thus the lemma is proved.■

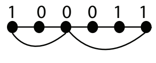

Fig. 1: The cordiality of $L_{3,4}$

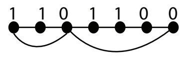

Fig. 2: The cordiality of $L_{3,5}$

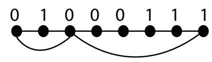

Fig. 3: The cordiality of $L_{3,6}$

Lemma 3.2. The lemniscate graph $L_{n,m}$, where $n = 0(mod4)$ is cordial if and only if $m \neq 2(mod4)$.
Proof. We prove the easy direction first. Let $n = 4r$, $r \geq 1$, and let $m = 4t + j$, $t \geq 1$, and $j = 2$. $L_{4r,4t+2}$ is not cordial since it is an Eulerian graph with size congruent to $2(mod4)$ [8]. Now, we consider three cases:
Case (1) At $j = 0$ i.e. $m = 4t, t \geq 1$. One can select the labeling $[L_{4r}; 100L'_{4t-4}]$ for $L_{4r,4t)}$. Therefore $x_0 = x_1 = a_0 = a_1 = 2r, y_0 = 2t, y_1 = 2t - 1, b_0 = 2t$ and $b_1 = 2t$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. As an examples, Figure (4) illustrate the cordiality of $L_{4,4}$. Thus $L_{4r,4t}$ is cordial for all $r,t \geq 1$.
Case (2) At $j = 1$ i.e. $m = 4t + 1, t \geq 1$. One can select the labeling $[L_{4r}; L'_{4t}]$ for $L_{4r,4t+1}$. Therefore $x_0 = x_1 = a_0 = a_1 = 2r, y_0 = 2t, y_1 = 2t, b_0 = 2t + 1$ and $b_1 = 2t$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 1$. As an examples, Figure (5) illustrate the cordiality of $L_{4,5}$. Thus $L_{4r,4t+1}$ is cordial for all $r,t \geq 1$.
Case (3). At $j = 3$ i.e. $m = 4t + 3, t \geq 1$. One can select the labeling $[L_{4r}; 0L'_{4t}1]$ for $L_{4r,4t+3}$. Therefore $x_0 = x_1 = a_0 = a_1 = 2r, y_0 = 2t + 1, y_1 = 2t + 1, b_0 = 2t + 1$ and $b_1 = 2t + 2$. It follows that $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and

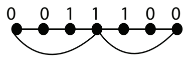

Fig. 4: The cordiality of $L_{4,4}$

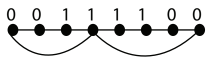

Fig. 5: The cordiality of $L_{4,5}$

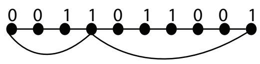

Fig. 6: The cordiality of $L_{4,7}$

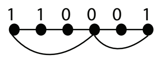

Fig. 7: The cordiality of $L_{4,3}$

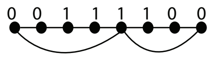

Fig. 8: The cordiality of $L_{5,4}$

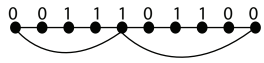

Fig. 9: The cordiality of $L_{5,6}$

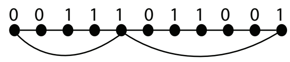

Fig. 10: The cordiality of $L_{5,7}$

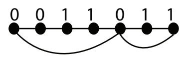

Fig. 11: The cordiality of $L_{5,3}$

$e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = -1$. As an examples, Figure (6) illustrate the cordiality of $L_{4,7}$. Thus $L_{4r,4t+3}$ is cordial for all $r,t \geq 1$. Finally at $t = 0$; $L_{4r,3}$ is isomorphic to $L_{3,4r}$. Using Lemma (3.1), we conclude that $L_{4r,3}$ is cordial. As an examples, Figure (7) illustrate the cordiality of $L_{4,3}$. Thus the lemma is proved.■

Lemma 3.3. The lemniscate graph $L_{n,m}$, where $n = 1(mod 4)$ is cordial if and only if $m \neq 1(mod 4)$.

Proof. Let $n = 4r + 1$, $r \geq 1$ and $m = 4t + j$, $0 \leq j \leq 3$. We prove the easy direction first. $L_{4r+1,4t+1}$ is not cordial since this is an Eulerian graph with size congruent to $2(mod 4)$ [8]. Now, we consider three cases:

Case (1) At $m = 4t, t \geq 1$. We select the labeling $[L_{4r}1; 100L'_{4t-4}]$ for $L_{4r+1,4t}$. Therefore $x_0 = 2r, x_1 = 2r+1, a_0 = 2r+1, a_1 = 2r, y_0 = 2t, y_1 = 2t-1, b_0 = 2t$ and $b_1 = 2t$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 1$. As an examples, Figure (8) illustrate the cordiality of $L_{5,4}$. Thus $L_{4r+1,4t}$ is cordial for all $r,t \geq 1$.

Case (2) At $j = 2$ i.e. $m = 4t+2, t \geq 1$. One can select the labeling $[L_{4r}1; S'_{4t}0]$ for $L_{4r+1,4t+2}$. Therefore $x_0 = 2r, x_1 = 2r+1, a_0 = 2r+1, a_1 = 2r, y_0 = 2t+1, y_1 = 2t, b_0 = 2t$ and $b_1 = 2t+2$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = -1$. As an examples, Figure (9) illustrate the cordiality of $L_{5,6}$. Thus $L_{4r+1,4t+2}$ is cordial for all $r,t \geq 1$.

Case (3). At $j = 3$ i.e. $m = 4t+3, t \geq 1$. We choose the labeling $[L_{4r}1; 0L'_{4t}1]$ for $L_{4r+1,4t+3}$. Therefore $x_0 = 2r, x_1 = 2r+1, a_0 = 2r+1, a_1 = 2r, y_0 = 2t+1, y_1 = 2t+1, b_0 = 2t+1$ and $b_1 = 2t+2$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = -1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. As an examples, Figure (10) illustrate the cordiality of $L_{5,7}$. Thus $L_{4r+1,4t+3}$ is cordial for all $r,t \geq 1$. Finally; since $L_{4r+1,3}$ is isomorphic to $L_{3,4r+1}$, $L_{4r+1,3}$ is cordial. As an examples, Figure (11) illustrate the cordiality of $L_{5,3}$. Thus the lemma follows.■

Lemma 3.4. The lemniscate graph $L_{n,m}$, where $n = 2(mod 4)$ is cordial if and only if $m \neq 0(mod 4)$.

Proof. Let $n = 4r+2$, $r \geq 1$ and $m = 4t + j$, $0 \leq j \leq 3$. We prove the easy direction first. Let $L_{4r+2,4t}$ i.e. $j = 0$. This graph is not cordial since it is an Eulerian graph and its size congruent to $2(mod 4)$ [8]. Now, we consider three cases:

Case (1) At $j = 1$, i.e. $m = 4t+1, t \geq 1$. We choose the labeling $[0S_{4r}1; L'_{4t}]$ for $L_{4r+2,4t+1}$. Therefore $x_0 = 2r+1, x_1 = 2r+1, a_0 = 2r, a_1 = 2r+2, y_0 = 2t, y_1 = 2t, b_0 = 2t+1$ and $b_1 = 2t$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 0$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = -1$. As an examples, Figure (12) illustrate the cordiality of $L_{6,5}$. Here, we conclude that the lemniscate graph $L_{4r+2,4t+1}$ is isomorphic to $L_{4t+1,4r+2}$ which is cordial by lemma 3.3. Thus $L_{4r+1,4t+2}$ is cordial for all $r,t \geq 1$.

Case (2) At $j = 2$, i.e. $m = 4t+2, t \geq 1$. We choose the labeling $[L_{4r}10; S_{4t}0]$ for $L_{4r+2,4t+2}$. Therefore $x_0 = 2r+1, x_1 = 2r+1, a_0 = 2r+2, a_1 = 2r, y_0 = 2t+1, y_1 = 2t, b_0 = 2t$ and $b_1 = 2t+2$. Hence $v_0 - v_1 = (x_0 - x_1) + (y_0 - y_1) = 1$ and $e_0 - e_1 = (a_0 - a_1) + (b_0 - b_1) = 0$. As an examples,

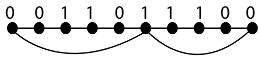

Fig. 12: The cordiality of $L_{6,5}$

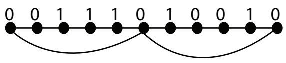

Fig. 13: The cordiality of $L_{6,6}$

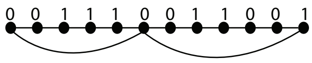

Fig. 14: The cordiality of $L_{6,7}$

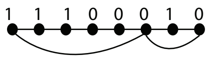

Fig. 15: The cordiality of $L_{6,3}$

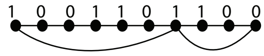

Fig. 16: The cordiality of $L_{7,4}$

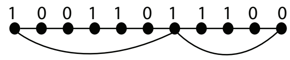

Fig. 17: The cordiality of $L_{7,5}$

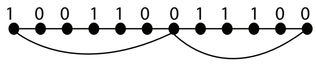

Fig. 18: The cordiality of $L_{7,6}$

Figure (13) illustrate the cordiality of $L_{6,6}$. Thus $L_{4r+2,4t+2}$ is cordial for all $r,t \geq 1$.
Case (3) At $j=3$ i.e. $m=4t+3, t\geq 1$. We choose the labeling $[L_{4r}10;0L'_{4t}1]$ for $L_{4r+2,4t+3}$. Therefore $x_0=2r+1, x_1=2r+1, a_0=2r+2, a_1=2r, y_0=2t+1, y_1=2t+1, b_0=2t+1$ and $b_1=2t+2$. Hence $v_0-v_1=(x_0-x_1)+(y_0-y_1)=0$ and $e_0-e_1=(a_0-a_1)+(b_0-b_1)=1$. As an examples, Figure (14) illustrate the cordiality of $L_{6,7}$. Since $L_{4r+2,3}$ is isomorphic to $L_{3,4r+2}$, $L_{4r+2,3}$ is cordial. As an examples, Figure (15) illustrate the cordiality of $L_{6,3}$. Thus $L_{4r+2,4t+3}$ is cordial for all $r,t\geq 1$. Thus the lemma is proved.■

Lemma 3.5. The lemniscate graph $L_{n,m}$, where $n=3(mod4)$ is cordial if and only if $m\neq 3(mod4)$.
Proof. Let $n=4r+3$ and $m=4t+j$, $0\leq j\leq 3$. We prove the easy direction first $L_{4r+3,4t+3}$ is not cordial since it is an Eulerian graph and its size congruent to $2(mod4)$ [8]. Now, we consider three cases:
Case (1) At $j=0$ i.e. $m=4t, t\geq 1$. We choose the labeling $[1L_{4r}01;100L'_{4t-4}]$ for $L_{4r+3,4t}$. Therefore $x_0=2r+1, x_1=2r+2, a_0=2r+1, a_1=2r+2, y_0=2t, y_1=2t-1, b_0=2t$ and $b_1=2t$. Hence $v_0-v_1=(x_0-x_1)+(y_0-y_1)=0$ and $e_0-e_1=(a_0-a_1)+(b_0-b_1)=-1$. As an examples, Figure (16) illustrate the cordiality of $L_{7,4}$. The lemniscate graph $L_{4r+3,4t}$ is isomorphic to $L_{4t,4r+3}$ which is cordial by lemma 3.2. Thus $L_{4r+3,4t}$ is cordial for all $r,t\geq 1$.
Case (2) At $j=1$ i.e. $m=4t+1, t\geq 1$. We choose the labeling $[1L_{4r}00;L'_{4t}]$ for $L_{4r+3,4t+1}$. Therefore $x_0=2r+2, x_1=2r+1, a_0=2r+1, a_1=2r+2, y_0=2t, y_1=2t, b_0=2t+1$ and $b_1=2t$. Hence $v_0-v_1=(x_0-x_1)+(y_0-y_1)=1$ and $e_0-e_1=(a_0-a_1)+(b_0-b_1)=0$. As an examples, Figure (17) illustrate the cordiality of $L_{7,5}$. The lemniscate graph $L_{4r+3,4t+1}$ is isomorphic to $L_{4t+1,4r+3}$ which is cordial by lemma 3.3. Thus $L_{4r+3,4t+1}$ is cordial for all $r,t\geq 1$.
Case (3) At $j=2$ i.e. $m=4t+2, t\geq 1$. We choose the labeling $[1L_{4r}00;1L'_{4t}]$ for $L_{4r+3,4t+2}$. Therefore $x_0=2r+2, x_1=2r+1, a_0=2r+1, a_1=2r+2, y_0=2t, y_1=2t+1, b_0=2t+2$ and $b_1=2t$. Hence $v_0-v_1=(x_0-x_1)+(y_0-y_1)=0$ and $e_0-e_1=(a_0-a_1)+(b_0-b_1)=1$. As an examples, Figure (18) illustrate the cordiality of $L_{7,6}$. The lemniscate graph $L_{4r+3,4t+2}$ is isomorphic to $L_{4t+2,4r+3}$ which is cordial by lemma 3.4. Thus $L_{4r+3,4t+2}$ is cordial for all $r,t\geq 1$. Finally; since $L_{4r+3,3}$ is isomorphic to $L_{3,4r+3}$. By lemma(3.1), $L_{4r+1,3}$ is not cordial and thus the lemma follows.■

Theorem 3.1. The lemniscate graph $L_{n,m}$ is cordial for all n and all m if and only if $L_{n,m}$ is not an Eulerian graph with size congruent to $2(mod4)$.
Proof. The proof follows directly from lemma 3.1, lemma 3.2, lemma 3.3, lemma 3.4 and lemma 3.5.■

4 The cordially of cone graphs

In this section, we show that the cone graphs $\overline{K}_n + C_m$ is cordial for all $n\geq 1, m\geq 3$. The following result was shown in [9]. Here we present another method for the proof. This target will be achieved after the following series of lemmas.

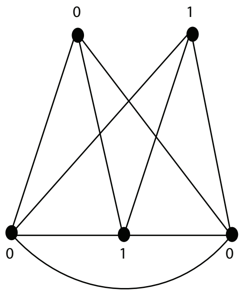

Fig. 19: The cordiality of $\overline{K}_2+C_3$

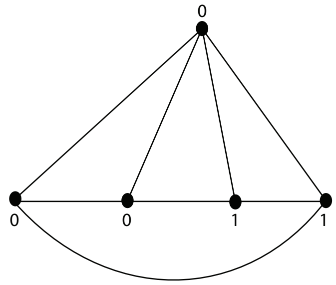

Fig. 21: The cordiality of $\overline{K}_2+C_4$

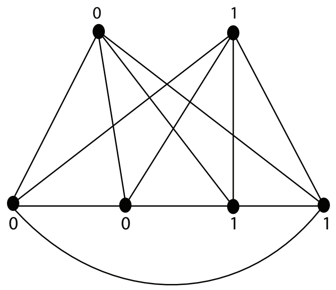

Fig. 20: The cordiality of $\overline{K}_1+C_4$

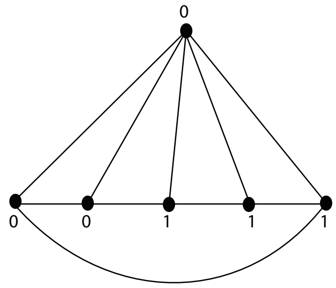

Fig. 22: The cordiality of $\overline{K}_1+C_5$

Lemma 4.1 The cone graph $\overline{K}_n+C_3$ is cordial if and only if n even.

Proof. Suppose that n is odd, then the degree of all vertices of the graph $\overline{K}_n+C_3$ is cordial, and if the sum of order and size of the graph $\overline{K}_n+C_3$ is congruent to $2(mod4)$, then from [2], the graph $\overline{K}_n+C_3$ is not cordial. Conversely, let $n=2r$ where $r\geq 1$, and we label the vertices of C_3 as 001, then if we label the vertices of $\overline{K}_n$ as M_{2r}, then the numbers of vertices of $\overline{K}_n+C_3$ with each label are increase by r, so the difference v_0-v_1 remain the same, and also the numbers of edge of $\overline{K}_n+C_3$ with each label are increase by $3r$, so the difference e_0-e_1 remain the same. As an example, Figure (19) illustrates $\overline{K}_2+C_3$. Therefore $\overline{K}_n+C_3$ is cordial. Thus the lemma follows.

Lemma 4.2 If $m\equiv 0(mod4)$, then the cone graph $\overline{K}_n+C_m$ between isolated vertices $\overline{K}_n$ and cycles C_m is cordial for all $n\geq 1$.

Proof. Suppose that $m=4s$, where $s\geq 1$, then we label the vertices of C_{4s} as $B_0=L_{4s}$, i.e. $y_0=y_1=2s$ and $b_0=b_1=2s$. let $n=2r$ where $r\geq 1$, then if we label the vertices of $\overline{K}_n$ as $A_0=M_{2r}$, i.e. $x_0=x_1=r$ and $a_0=a_1=0$, then from equation (1) in section 2. we have $v_0=v_1=0$ and $e_0=e_1=0$. Conversely, suppose that n odd, $n=2r$ where $r\geq 0$, then if we label the vertices of $\overline{K}_n$ as $A_1=M_{2r+1}$, i.e. $x_0=r+1$, $x_1=r$ and $a_0=a_1=0$, then from equation (1) in section 2. we have $v_0-v_1=1$ and $e_0-e_1=0$. As an examples, Figure (20) and Figure (21) illustrate $\overline{K}_1+C_4$ and $\overline{K}_2+C_4$. Therefore $\overline{K}_n+C_m$ is cordial. Thus the lemma follows.

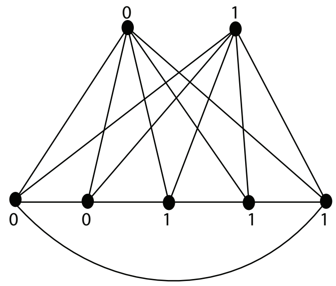

Fig. 23: The cordiality of $\overline{K}_2+C_5$

Lemma 4.3 If $m\equiv 1(mod4)$, then the cone graph $\overline{K}_n+C_m$ between isolated vertices $\overline{K}_n$ and cycles C_m is cordial for all $n\geq 1$.

Proof. Suppose that $m=4s+1$, where $s\geq 1$, and we label the vertices of C_{4s+1} as $B_1=L_{4s}1$, i.e. $y_0=2s$, $y_1=2s+1$, $b_0=2s+1$ and $b_1=2s$. let $n=2r$ where $r\geq 1$, then if we label the vertices of $\overline{K}_n$ as $A_0=M_{2r}$, i.e. $x_0=x_1=r$ and $a_0=a_1=0$, then from equation (1) in section 2. we have $v_0-v_1=-1$ and $e_0-e_1=1$. Conversely, suppose that n odd, $n=2r+1$ where $r\geq 0$, then if we label the vertices of $\overline{K}_n$ as $A_1=M_{2r+1}$, i.e. $x_0=r+1$, $x_1=r$ and $a_0=a_1=0$, then from equation (1) in section 2. we have $v_0-v_1=0$ and $e_0-e_1=0$. As an examples, Figure (22) and Figure (23) illustrate $\overline{K}_1+C_5$ and $\overline{K}_2+C_5$. Therefore $\overline{K}_n+C_m$ is cordial. Thus the lemma follows.

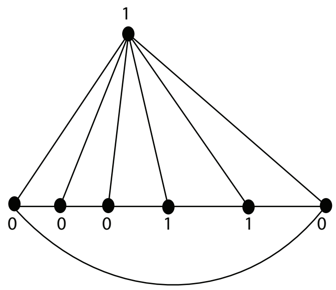

Fig. 24: The cordiality of $\overline{K}_1 + C_6$

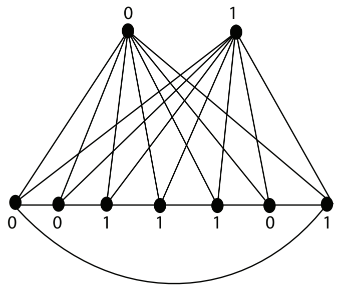

Fig. 25: The cordiality of $\overline{K}_1 + C_6$

Lemma 4.4 If $m \equiv 2(mod4)$, then the cone graph $\overline{K}_n + C_m$ between isolated vertices $\overline{K}_n$ and cycles C_m is cordial for all n odd, $n \geq 3$.
Proof. Suppose that $m = 4s+2$, where $s \geq 1$, then we label the vertices of C_{4s+2} as $B_2 = 0L_{4s}0$, i.e. $y_0 = 2s+2$, $y_1 = 2s$, $b_0 = 2s+2$ and $b_1 = 2s$. let $n = 2r+1$ where $r \geq 1$, then if we label the vertices of $\overline{K}_n$ as $A_1 = M'_{2r+1}$, i.e. $x_0 = r$, $x_1 = r+1$ and $a_0 = a_1 = 0$, then from equation (1) in section 2. we have $v_0 - v_1 = 1$ and $e_0 - e_1 = 0$. As an example, Figure (24) illustrates $\overline{K}_1 + C_6$. Therefore $\overline{K}_n + C_m$ is cordial. Therefore $\overline{K}_n + C_m$ is cordial. Thus the lemma follows.

Lemma 4.5 If $m \equiv 3(mod4)$, then the cone graph $\overline{K}_n + C_m$ between isolated vertices $\overline{K}_n$ and cycles C_m is cordial for all n even, $n \geq 1$.
Proof. Suppose that $m = 4s+3$, where $s \geq 1$, then we label the vertices of C_{4s+3} as $B_3 = L_{4s}101$, i.e. $y_0 = 2s+1$, $y_1 = 2s+1$, $b_0 = 2s+1$ and $b_1 = 2s$. let $n = 2r$ where $r \geq 1$, then if label the vertices of $\overline{K}_n$ as $A_0 = M_{2r}$, i.e. $x_0 = x_1 = r$ and $a_0 = a_1 = 0$, then from equation (1) in section 2. we have $v_0 - v_1 = -1$ and $e_0 - e_1 = -1$. As an example, Figure (25) illustrates $\overline{K}_1 + C_6$. Therefore $\overline{K}_n + C_m$ is cordial. Thus the lemma follows.

Lemma 4.6 If $m \equiv 2(mod4)$, then the cone graph $\overline{K}_n + C_m$ is not cordial for all n even.
Proof. It is easy to verify that the degree of all vertices of $\overline{K}_n + C_m$ is odd, and the sum of order of the size of $\overline{K}_n + C_m$ is congruent to $2(mod4)$, then $\overline{K}_n + C_m$ is not cordial[4].

Lemma 4.7 If $m \equiv 3(mod4)$,then the cone graph $\overline{K}_n + C_m$ is not cordial for all n odd.
Proof. It is easy to verify that the graph $\overline{K}_n + C_m$ is an Eulerian graph with size congruent to $2(mod4)$, then from Cahit's theorem [8], $\overline{K}_n + C_m$ is not cordial.

Theorem 4.1 The join of the cycles C_m and an isolated vertices $\overline{K}_n$ is cordial for all n and all m if and only if m is not congruent to $3(mod4)$ and n odd, or when m is not congruent to $2(mod4)$ and n even.
Proof. The proof follows directly from lemma 4.1, lemma 4.2, lemma 4.3, lemma 4.4, lemma 4.5, lemma 4.6 and lemma 4.7 .

Algorithm 1. Algorithm for the cordiality of cone graphs
Cone graph(n,m): joint between cyclic graph and null graph
e: the number of edge
r: random value of 0,1
V: the number of vertex

---

input : $n \longrightarrow$ cyclic graph
$m \longrightarrow$ null graph
output : identify our cone graph is cordial or not
START
using n,m to draw the joint between cyclic graph and null graph denoted by cone graph
$R \longleftarrow (0,1)$
FOR all vertices $\in n$ **do**
define $r \longleftarrow R$
labeling n graph using r
END FOR
FOR all vertices $\in m$ **do**
define $r \longleftarrow R$
labeling n graph using r
END FOR
$e_0 \longleftarrow 0$
$e_1 \longleftarrow 0$
FOR vertex$(i) i = 1$ to n step vertex++ **do**
IF $i < n$
  IF vertex(i) = vertex$(i+1)$
  $e_0 ++$
  ELSE $e_1 ++$
  END IF
ELSE
  IF vertex(i) = vertex(0)
  $e_0 ++$
  ELSE $e_1 ++$
  END IF
END IF
END FOR
FOR vertex m$(i) i = 1$ to m step $i++$ **do**
  FOR vertex n$(j) j = 1$ to n step $j++$ **do**
  IF vertex $m(i)$ = vertex $n(j)$

e_0++
ELSE
e_1++
END IF
END FOR
END FOR
$v_0 \longleftarrow 0$
$v_1 \longleftarrow 0$
FOR vertex $n(i) i = 1$ to n step $i++$ **do**
IF vertex $n(i) = 0$
v_0++
ELSE
v_1++
END IF
END FOR
FOR vertex $m(i) i = 1$ to m step $i++$ **do**
IF vertex $m(i) = 0$
v_0++
ELSE
v_1++
END IF
END FOR
IF$(|v_0 - v_1| <= 1 AND |e0 - e1| <= 1)$
THEN
the cone graph(n,m) is cordial
ELSE IF ((cone graph(n,m) is eulerian graph AND cone graph (n,m) size congruent to $2(mod 4)$)
OR (cone graph(n,m) with size congruent to $2(mod 4)$ AND degree of each vertex of cone graph(n,m) is odd))
THEN
the cone graph(n,m) is not cordial
ELSE
go to **START**
END IF

5 Applications of Cordial Graph Labeling in circuit

An Adder is a device that can add two binary digits.It is a type of digital circuit that performs the operation of additions of two number. It is mainly designed for the addition of binary number, but they can be used in various other applications like binary code decimal address decoding, table index calculation, etc. There are two types of Adder. One is Half Adder, and another one is known as Full Adder. These graphs can be used to design an electric circuit.

6 Conclusion

In this work, we propose two contributions. The first contribution is that we prove two theorems which determine the cordiality for lemniscate graphs and cone graphs. The second contribution is proposing a new algorithm which finds the cordiality for the cone graph. Finally, an application of a cordial graph is introduced.

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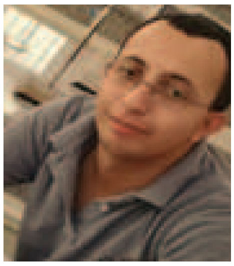

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