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Sumudu Transform for Some Implementations with a New Kernel

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Abstract: In this essay, we examine real practical problems' Sumudu transform solutions. We take into account the economic models that rely on a constant proportionate Caputo derivative and market equilibrium. By using intriguing implementations, we demonstrate the Sumudu transform's precision and strength.

Keywords: Sumudu transforms, constant proportional Caputo derivative, modelling.

1 Introduction

Fractional derivatives and the bivariate Mittag-Leffler function have both been the subject of several investigations. Here are a few of them: In their study of the Mittag-Leffler-type function of two variables, Garg et al. [1] Bivariate Mittag-Leffler functions that appear in solutions of convolution integral equations with 2D-Laguerre-Konhauser polynomials in the kernel have been explored by Zarslan and Kürt [2], among others. According to Bonfanti et al. [3], research has been done on the unified rheological model for cells and cellularized materials. A specific bivariate Mittag-Leffler function has been researched by Kürt et al. [4]. The multivariate analogue of the generalized Mittag-Leffler function has been looked for by Saxena et al. Jayakumar et al. [5] has worked the generalization to bivariate Mittag–Leffler and bivariate discrete Mittag–Leffler autoregressive processes. Mainardi [6] has investigated some properties of the Mittag-Leffler function. Abdeljawad et al. [7,8,9,10,11] have solved many fractional differential equations with different technicques.

Sumudu transformation is a promising transformation due to its basic formula and features resulting from this formula. It can contribute to the solution of many complex problems. Weerakoon [12] has studied the partial difference equations for Sumudu transformations. Then Watinal [13] has constructed some works related to this transform. The study of Watugala's user [14] showed that the Sumudu transformation is effective for solving differential equations. Watugala [15] extended this transformation to bivariate partial differential equations. In this study, firstly we used the Sumudu transform with constant propotional Caputo derivative for some examples. Secondly we deal with economic model which is solved by Sumudu transform with new derivative.

2 Main theorems and applications

Definition 1. *We define [16]:*

$$A = \{g(x) \mid \exists N, \tau_1, \tau_2 > 0, \mid g(x) \mid < N \exp(\mid x \mid / \tau_k),\ if\, x \in (-1)^k \times [0,\infty)\}, \tag{1}$$

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and

$$G(s) = S[g(x)] = \int_0^\infty g(sx)\exp(-x)dx, \quad s \in (-\tau_1, \tau_2). \tag{2}$$

Authors showed the Sumudu transform which has units preserving properties, and moreover they can utilize to investigate the problems without resorting to the frequency domain. There are many strength points of this new transform, particularly with respect to implementations in problems with physical dimensions. Actually, the Sumudu transform is linear, protects linear functions, and also especially does not alter units [14].

Lemma 1. *Let $g(x)$ be in A, and let $G_k(s)$ indicates the Sumudu transform of the kth derivative, $g^{(k)}(x)$ of $g(x)$, then for $j \geq 1$, [16]*

$$G_k(s) = \frac{G(s)}{s^k} - \sum_{l=0}^{k-1} \frac{g^{(l)}(0)}{s^{k-l}}. \tag{3}$$

Lemma 2. *We obtain the Sumudu transform of the power series function as [16],*

$$g(x) = \sum_{k=0}^{\infty} a_k x^k, \tag{4}$$

its power series function is as:

$$G(s) = \sum_{k=0}^{\infty} k! a_k s^k. \tag{5}$$

Definition 2. *The Caputo fractional derivative is given as [17]:*

$${}_a^C D^\sigma g(x) = \frac{1}{\Gamma(1-\sigma)} \int_a^x (x-\tau)^{-\sigma} g'(\tau) d\tau. \tag{6}$$

Definition 3. *The derivative of the constant proportional Caputo (CPC) is defined as [17]:*

$${}_0^{CPC} D_x^\alpha g(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \left(k_1(\alpha) g(\tau) + k_0(\alpha) g'(\tau)\right) (x-\tau)^{-\alpha} d\tau. \tag{7}$$

Definition 4. *Let $\xi, \eta : [0,\infty) \to \Re$, then the convolution of ξ, η is presented as:*

$$(\xi * \eta) = \int_0^x \xi(x-u)\eta(u)du, \tag{8}$$

then we have:

$$S\{(\xi * \eta)(x)\} = sS\{\xi(x)\}S\{\eta(x)\}. \tag{9}$$

Definition 5. *The classical Mittag-Leffler function which has one parameter $E_\alpha(\nu)$ is presented by:*

$$E_\alpha(\nu) = \sum_{l=0}^{\infty} \frac{\nu^l}{\Gamma(\alpha l + 1)} \quad (\nu \in \mathbb{C}, Re(\alpha) > 0), \tag{10}$$

Also, we have:

$$E_{\alpha,\beta}(\nu) = \sum_{l=0}^{\infty} \frac{\nu^l}{\Gamma(\alpha l + \beta)} \quad (\nu, \beta \in \mathbb{C}, Re(\alpha) > 0). \tag{11}$$

Theorem 1. *The Sumudu transform is presented by:*

$$S\{{}_0^{CPC} D_x^\alpha g(x)\} = k_1(\alpha) S\{g(x)\} s^{1-\alpha} + k_0(\alpha)[S\{g(x)\} - g(0)] s^{-\alpha}. \tag{12}$$

Proof. We have

$$\begin{aligned} {}_0^{CPC} D_x^\alpha g(x) &= \frac{1}{\Gamma(1-\alpha)} \int_0^x \left(k_1(\alpha) g(\tau) + k_0(\alpha) g'(\tau)\right) (g-\tau)^{-\alpha} d\tau \\ &= k_1(\alpha) \left\{ g(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \right\} + k_0(\alpha) \left\{ g'(x) * \frac{x^{-\alpha}}{\Gamma(1-\alpha)} \right\}, \end{aligned}$$

then we apply the Sumudu transform to the above equation:

$$
\begin{aligned}
S\left[{}_{0}^{CPC}D_{x}^{\alpha}g(x)\right] &= S\left[k_1(\alpha)\left\{g(x)*\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}+k_0(\alpha)\left\{g'(x)*\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}\right]\\
&= s\left[k_1(\alpha)S\{g(x)\}S\left\{\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}+k_0(\alpha)S\{g'(x)\}S\left\{\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right\}\right]\\
&= s\left[k_1(\alpha)S\{g(x)\}s^{-\alpha}+k_0(\alpha)\left(\frac{S\{g(x)\}-g(0)}{s}\right)s^{-\alpha}\right]\\
&= k_1(\alpha)S\{g(x)\}s^{1-\alpha}+k_0(\alpha)\left[S\{g(x)\}-g(0)\right]s^{-\alpha}.
\end{aligned}
$$

We take into account:

$$ {}_{0}^{CPC}D_{x}^{\alpha}g(x)=0, \quad g(0)=A, \tag{13}$$

Using the Sumudu transform to this equation, we can write

$$ S\{{}_{0}^{CPC}D_{x}^{\alpha}g(x)\}=0. \tag{14}$$

If the expression Theorem 1 is utilized, then we obtain as:

$$ k_1(\alpha)S\{g(x)\}s^{1-\alpha}+k_0(\alpha)[S\{g(x)\}-g(0)]s^{-\alpha}=0, \tag{15}$$

and

$$ S\{g(x)\}=\frac{Ak_0(\alpha)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}}=\frac{A}{1+\frac{k_1(\alpha)}{k_0(\alpha)}s}, \tag{16}$$

the result of applying the inverse Sumudu transform on the previous equation is presented as:

$$ g(x)=AS^{-1}\left\{\frac{1}{1+\frac{k_1(\alpha)}{k_0(\alpha)}s}\right\}, \tag{17}$$

then the solution is acquired

$$ g(x)=A\exp\left(\frac{-k_1(\alpha)}{k_0(\alpha)}x\right). \tag{18}$$

Let consider another example:

$$ {}_{0}^{CPC}D_{x}^{\alpha}g(x)=\lambda g(x), \quad g(0)=1, \tag{19}$$

The result of applying the Sumudu transform to the previously mentioned equation is

$$ S\{{}_{0}^{CPC}D_{x}^{\alpha}g(x)\}=\lambda S\{g(x)\}, \tag{20}$$

Thus, we reach

$$ k_1(\alpha)S\{g(x)\}s^{1-\alpha}+k_0(\alpha)[S\{g(x)\}-g(0)]s^{-\alpha}=\lambda S\{g(x)\}, \tag{21}$$

and

$$
\begin{aligned}
S\{g(x)\} &= \frac{k_0(\alpha)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}-\lambda}\\
&= \frac{1}{1-\frac{\lambda}{k_0(\alpha)}s^{\alpha}+\frac{k_1(\alpha)}{k_0(\alpha)}s}\\
&= \left[1-\frac{\lambda s^{\alpha}-k_1(\alpha)s}{k_0(\alpha)}\right]^{-1}\\
&= \sum_{b=0}^{\infty}\left[\frac{\lambda s^{\alpha}-k_1(\alpha)s}{k_0(\alpha)}\right]^{b}\\
&= \sum_{b=0}^{\infty}\frac{1}{k_0(\alpha)^b}\sum_{c=0}^{b}\binom{b}{c}[\lambda s^{\alpha}]^{b-c}[-k_1(\alpha)s]^{c}\\
&= \sum_{b=0}^{\infty}\sum_{c=0}^{b}\frac{-k_1(\alpha)^c\lambda^{b-c}}{k_0(\alpha)^b}\binom{b}{c}s^{\alpha b-\alpha c+c}.
\end{aligned}
$$

Then, we apply the inverse Sumudu transform and obtain:

$$
g(x)=\sum_{b=0}^{\infty}\sum_{c=0}^{b}\frac{-k_1(\alpha)^c\lambda^{b-c}}{k_0(\alpha)^b}\binom{b}{c}\frac{x^{\alpha b-\alpha c+c}}{\Gamma(\alpha b-\alpha c+c+1)}. \tag{22}
$$

When we take $a=b-c$, we will get

$$
\begin{aligned}
g(x) &= \sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{-k_1(\alpha)^c\lambda^{a}}{k_0(\alpha)^{c+a}}\frac{(c+a)!}{c!a!}\frac{x^{\alpha a+c}}{\Gamma(\alpha a+c+1)}\\
&= \sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{(c+a)!}{c!a!}\left[\frac{-k_1(\alpha)}{k_0(\alpha)}x\right]^{c}\left[\frac{\lambda}{k_0(\alpha)}x^{\alpha}\right]^{c}\frac{x^{\alpha a+c}}{\Gamma(\alpha a+c+1)}.
\end{aligned}
$$

We can write this series as [18]:

$$
g(x)=E^{1}_{\alpha,1,1}\left(\frac{\lambda}{k_0(\alpha)}x^{\alpha},\frac{-k_1(\alpha)}{k_0(\alpha)}x\right). \tag{23}
$$

3 Applications of the economic model

A wide variety of applications such as optimization problems, balance or static analysis, comparative static have been encountered recently. The authors have begun using a number of new tools in differential analysis for all these fields to develop economic models. It will be understood that economists use mathematical economic models to get strong predictions about maximum profit. Each economic model has maximum benefit for buyers, maximum profit for sellers, and limited pricing in a balance model. How to reach balance point in economy, price, supply and demand, price supply and demand and supply and demand curves. A competitive market is based on the competitive balance. We have

$$
q_d(x)=d_0-d_1g(x),\quad q_s(x)=-s_0+s_1g(x) \tag{24}
$$

where g the price of goods, d_0,s_0,d_1,s_1 are positive constants. For $q_d(x)=q_s(x)$, we have $g^*=\frac{d_0+s_0}{d_1+s_1}$. Now, considering [19]:

$$
g'(x)=k(q_d-q_s), \tag{25}
$$

where $K>0$. Then, we acquire

$$
g'(x)+k(d_1+s_1)g(x)=k(d_0+s_0). \tag{26}
$$

Then, we obtain

$$
g(x)=\frac{d_0+s_0}{d_1+s_1}-\left[g(0)+\frac{d_0+s_0}{d_1+s_1}\right]\exp(-k(d_1+s_1)x). \tag{27}
$$

Let us consider the following example:

$$
{}^{CPC}_{0}D^{\alpha}_{x}g(x)+k(d_1+s_1)g(x)=k(d_0+s_0), \tag{28}
$$

If we apply the Sumudu transform to the equation, we obtain

$$S\{{}_0^{CPC}D_x^{\alpha}g(x)\}+k(d_1+s_1)S\{g(x)\}=k(d_0+s_0), \tag{29}$$

If we use the expression Theorem 1, then we find:

$$k_1(\alpha)S\{g(x)\}s^{1-\alpha}+k_0(\alpha)[S\{g(x)\}-g(0)]s^{-\alpha}+k(d_1+s_1)S\{g(x)\}=k(d_0+s_0), \tag{30}$$

and

$$\begin{aligned}
S\{g(x)\} &= \frac{k(d_0+s_0)}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}+k(d_1+s_1)} \\
&+\frac{k_0(\alpha)h(0)s^{-\alpha}}{k_1(\alpha)s^{1-\alpha}+k_0(\alpha)s^{-\alpha}+k(d_1+s_1)} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\left[1-\frac{-k_1(\alpha)s^{1-\alpha}-k_0\alpha s^{-\alpha}}{k(d_0+s_0)}\right]^{-1} \\
&+g(0)\left[1-\frac{-k_1(\alpha)s-k(d_1+s_1)s^{\alpha}}{k_0(\alpha)}\right]^{-1} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{b=0}^{\infty}\left[\frac{-k_1(\alpha)s^{1-\alpha}-k_0(\alpha)s^{-\alpha}}{k(d_0+s_0)}\right]^{b} \\
&+g(0)\sum_{b=0}^{\infty}\left[\frac{-k_1(\alpha)s-k(d_1+s_1)s^{\alpha}}{k_0(\alpha)}\right]^{b} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{b=0}^{\infty}\frac{1}{k^b(d_0+s_0)^b}\sum_{c=0}^{b}\binom{b}{c}[-k_1(\alpha)s^{1-\alpha}]^{b-c}[-k_0(\alpha)s^{-\alpha}]^{c} \\
&+g(0)\sum_{b=0}^{\infty}\frac{1}{k_0(\alpha)^b}\sum_{c=0}^{b}\binom{b}{c}[-k_1(\alpha)s]^{b-c}[-k(d_1+s_1)s^{\alpha}]^{c} \\
&= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{b=0}^{\infty}\sum_{c=0}^{b}(-1)^b\frac{k_1(\alpha)^{b-c}k_0(\alpha)^c}{k^b(d_0+s_0)^b}\binom{b}{c}s^{(1-\alpha)(b-c)-\alpha c} \\
&+g(0)\sum_{b=0}^{\infty}\sum_{c=0}^{b}(-1)^b\frac{k_1(\alpha)^{b-c}k^c(d_1+s_1)^c}{k_0(\alpha)^b}\binom{b}{c}s^{b-c+\alpha}.
\end{aligned}$$

Then, we apply the inverse Sumudu transform and get:

$$\begin{aligned}
g(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{b=0}^{\infty}\sum_{c=0}^{b}(-1)^b\frac{k_1(\alpha)^{b-c}k_0(\alpha)^c}{k^b(d_0+s_0)^b}\binom{b}{c}\frac{x^{(1-\alpha)(b-c)-\alpha c}}{\Gamma((1-\alpha)(b-c)-\alpha c+1)} \\
&+g(0)\sum_{b=0}^{\infty}\sum_{b=0}^{b}(-1)^b\frac{k_1(\alpha)^{b-c}k^c(d_1+s_1)^c}{k_0(\alpha)^b}\binom{b}{c}\frac{x^{(b-c)+\alpha c}}{\Gamma((b-c)+\alpha c+1)},
\end{aligned}$$

when we take $a=b-c$ we will get:

$$\begin{aligned}
g(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{(c+a)!}{c!a!}\frac{(-k_1(\alpha))^a(-k_0(\alpha))^c}{k^{a+c}(d_0+s_0)^{a+c}}\frac{x^{(1-\alpha)a-\alpha c}}{\Gamma((1-\alpha)a-\alpha c+1)} \\
&+g(0)\sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{(c+a)!}{c!a!}\frac{(-k_1(\alpha))^a(-k(d_1+s_1))^c}{k_0(\alpha)^{a+c}}\frac{x^{a+\alpha c}}{\Gamma(a+\alpha c+1)},
\end{aligned}$$

and

$$\begin{aligned}
g(x) &= \frac{(d_0+s_0)}{(d_1+s_1)}\sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{(c+a)!}{c!a!}\left[\frac{-k_0(\alpha)}{k(d_0+s_0)}x^{-\alpha}\right]^c\left[\frac{-k_1(\alpha)}{k(d_0+s_0)}x^{1-\alpha}\right]^a\frac{1}{\Gamma((1-\alpha)a-\alpha c+1)} \\
&+g(0)\sum_{c=0}^{\infty}\sum_{a=0}^{\infty}\frac{(c+a)!}{c!a!}\left[\frac{-k(d_1+s_1)}{k_0(\alpha)}x^{\alpha}\right]^c\left[\frac{-k_1(\alpha)}{k_0(\alpha)}x\right]^a\frac{1}{\Gamma(a+\alpha c+1)}.
\end{aligned}$$

We can write this series as [18]:

$$g(x) = \frac{(d_0+s_0)}{(d_1+s_1)} E^1_{1-\alpha,-\alpha,1}\left(\frac{-k_1(\alpha)}{k(d_0+s_0)}x^{1-\alpha}, \frac{-k_0(\alpha)}{k(d_0+s_0)}x^{-\alpha}\right)$$
$$+g(0)E^1_{1,\alpha,1}\left(\frac{-k_1(\alpha)}{k_0(\alpha)}x, \frac{-k(d_1+s_1)}{k_0(\alpha)}x^{\alpha}\right).$$

4 Conclusion

In this paper, we presented very useful applications of the Sumudu transform. We applied it to the general problems with a novel fractional derivative. We considered the economic models and we investigated this model by the Sumudu transform. We proved the efficiency of the proposed transform by accurate results.

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