Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180216

Exact solutions and conservation laws of a new fourth-order nonlinear (3+1)-dimensional Kadomtsev-Petviashvili-like equation

Chaudry Masood Khalique[1,2,3,*], *Oke Davies Adeyemo* [1,2], *and Isaac Mohapi*[1]

[1]Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X2046, South Africa
[2]African Institute for Mathematical Sciences Research Centre, 6 Melrose Road, Muizenberg, Cape Town 7945, South Africa
[3]Department of Mathematics and Informatics, Azerbaijan University, Jeyhun Hajibeyli str. 71, Baku AZ1007, Azerbaijan

Received: 12 Aug. 2023, Revised: 22 Sep. 2023, Accepted: 22 Oct. 2023
Published online: 1 Mar. 2024

Abstract: In this paper, we investigate an inclusive innovative fourth-order nonlinear Kadomtsev-Petviashvili-like model, a three-dimensional nonlinear partial differential equation. The focus is on utilizing the Lie symmetry method to derive exact solutions that demonstrate significant advancements in the model. Initially, a systematic approach is employed to compute the Lie point symmetries of the equation. These symmetries play a crucial role in identifying a diverse range of group invariant results for the model. The obtained solutions encompass logarithmic, exponential, and hyperbolic functions, as well as elliptic integral functions, with the latter being the most general solutions. Additionally, several noteworthy algebraic function solutions are also discovered. This research distinguishes itself by presenting a wealth of results that exhibit substantial variation. Furthermore, the dynamics of the solutions are thoroughly explored through diagrammatic analysis using computer software. Towards the conclusion, Ibragimov's theorem is applied to construct various conservation laws for the underlying model. This technique yields a multitude of conservation laws, which are subsequently discussed and highlighted.

Keywords: Nonlinear (3+1)-Dimensional Kadomtsev-Petviashvili-like equation; Lie Symmetry Analysis; Group-invariant and Exact Solutions; Innovation Support.

1 Introduction

Nonlinear partial differential equations (NLNPADEs) remain the subject of much research that is done today. This is due to their unquestionable role in attempting to model natural and man-made relationships between physical quantities. In recent times, significant inroads have been made in coming up with algorithms for handling NLNPADEs, with much credit due to the advancement of computers and their computational power. Nevertheless, great minds have had to lay the theoretical foundations upon which these technologies are built.

Lately, many researchers who have a keen interest in the nonlinear physical phenomena have delved into examining exact solutions of NLNPADEs due to their relevance in analyzing the outcome of any given model. Therefore, it is germane that the research into closed-form solutions to NLNPADEs serves a very crucial purpose in observing certain physical circumstances. Besides, the diversity of solutions of NLNPADEs occupies an essential position in a variety of areas of sciences inclusive of optical fibres, chemical physics, geochemistry, biology, hydrodynamics, chemical kinematics, meteorology, heat flow, plasma physics together with electromagnetic theory. Given the aforementioned and for emphasis, having realized that sizable scientists have contemplated nonlinear science as the most outstanding borderline for fundamental cognition of nature, we present some pertinent models that include an investigation in [1] was carried out on the modified as well as generalized Zakharov-Kuznetsov model, delineating the ion-acoustic meandering solitary waves resident in a magneto-plasma and possessive of electron-positron-ion observable in the autochthonous universe. This model was utilized in representing

* Corresponding author e-mail: Masood.Khalique@nwu.ac.za

dust-magneto-acoustic, and ion-acoustic, together with dust-ion-acoustic waves in the laboratory dusty plasmas. Additionally, the bright solitons, alongside their various interaction attribute related to the coupled Fokas-Lenells system was studied in reference [2].

Femtosecond optical pulses embedded in a double-refractive optical fibre, modeled into an NLNPADE, was further investigated. Moreover, the Boussinesq-Burgers-type system recounting shallow water waves and also emerging near ocean beaches and lakes was given attention in the paper [3]. We can continue with the list, but we mention a few. See more, e.g., in [4–16].

Now, having established the fact that no general technique in achieving various exact traveling wave results of NLNPADEs has been found, researchers have come up with some sound, effective, and efficient techniques so that the seemingly nagging problem could be nibbed in the bud. Some of these techniques include bifurcation technique [17], Painlevé expansion [18], homotopy perturbation technique [19], tanh-coth approach [20], extended homoclinic test approach [21], Cole-Hopf transformation technique [22], Adomian decomposition approach [23], Bäcklund transformation [24], Lie symmetry analysis [25, 26], F-expansion technique [27], rational expansion technique [28], tan-cot technique [29], Hirota technique [30], Darboux transformation [31], tanh-function technique [32], the $(G'/G)-$expansion technique [33], sine-Gordon equation expansion technique [34], generalized unified technique [35], exponential function technique [36], and so on and so forth.

Take for example, Sophus Lie (1842-1899), with his quintessential work on Lie Algebras [25, 26] is essentially a unified approach for the treatment of a wide class of differential equations (DEs). More recent methods of solving DEs include Hirota's bilinear method [37], simplest equation method [38], multiple exponential function method [39], Kudryashov's method [40], extended simplest equation approach [41], just to mention a few. Since the inception of Kadomtsev and Petviashvili's hierarchy of equations a little more than half a century ago, dozens of research papers have emerged, each exploring an aspect of this rich domain of equations, see for example, [42–48]. Basically, the standard Kadomtsev-Petviashvili model (KPm) is given as [49]:

$$(u_t + 6uu_x + u_{xxx})_x + u_{yy} = 0. \tag{1}$$

The KPm is established as a common generalization of the well-recognized Korteweg-de Vries equation (KdVe). In the KdVe, waves are stringently unidimensional. Nonetheless, in the KP model, this stringency is slackened. Both the KdVe as well as the KPm are completely integrable. The KP hierarchy has an infinite number of bilinear equations [50]. These hierarchies contain the extended form:

$$(u_t - 6u\,u_x + u_{xxx})_x + a\,u_{tt} + b\,u_{ty} - u_{yy} = 0, \tag{2}$$

where $a \neq 0,\ b \neq 0$.

In particular, this model furnishes an extension of the KPm (1). We remark that its KPm extended version (eKPmm) emerges in [51] as non-integrable which is an example revealing the extent to which Hirota's technique can be applied with regards to systems that are not integrable. In addition, as asserted in [51], the eKPmm model is non-integrable due to the reason that it possesses no three-soliton solution unless we have $a = b^2/12$ contingent upon the fact that it is transformable to (1). Therefore, upon introducing the additional terms presented as $a\,u_{tt}$ alongside $b\,u_{ty}$ engenders the extended model to be non-integrable in comparison with various other extended KPm where the added terms do not suppress or terminate as it were, the integrability property of the equation (see for example [52]). Moreover, it is from this hierarchy that we also obtained the B-type KP equations, BKP in short. Additionally, there are other derived varieties of models of KP referred as the extended KP-Like equation (exKp-Likeq) that reads [53]:

$$w_{tx} + 3w_x w_{xx} + 3ww_x^2 + \frac{3}{2}w^3 w_x + \frac{3}{2}w^2 w_{xx} + w_{yy} = 0, \tag{3}$$

where $w = w(t, x, y)$.

Some researchers have examined exKp-Likeq (3), intending to achieve some solutions to the model. For instance, the authors in [53] studied (3) via the generalized bilinear differential equation related to KP type given as [30]

$$\begin{aligned}&\left(D_{ttt}D_{xxx} + D_{xxx}^4 + D_{yyy}^2\right) f \cdot f = \\ &2f_{tx}f - 2f_t f_x + 6f_{xx}^2 + 2ff_{yy} - 2f_y^2 = 0,\end{aligned} \tag{4}$$

to seek various results in polynomial structures. Eventually, nine classes of rational solutions were secured through the use of Maple symbolic computation.

Moreover, we have the three dimensional KP-like equation that reads [54]:

$$\begin{aligned}w_{tx} + \frac{3}{2}w^3 w_x + \frac{3}{2}w^2 w_{xx} + \\ 3ww_x^2 + 3w_x w_{xx} + w_{yy} + w_{zz} = 0,\end{aligned} \tag{5}$$

which is observed to be an extended version of (3). So (5) is called the extended Kadomtsev-Petviashvili-like equation (extKP-Lke). Lü et al. adopted the generalized bilinear operators with $p = 3$ to generate eighteen classes of rational solutions to extKP-Lke (5). They were able to accomplish a search for polynomial solutions associated with generalized bilinear equations via symbolic computations. Later, in [55], the authors engaged the concept of Lie group theory to compute exact solutions of (5). Moreover, Kudryashov as well as power series techniques, were employed to achieve more solutions to the equation. Conservation laws of (5) were constructed via Ibragimov's method. Besides, in [56] the authors obtained various copious invariant solutions associated with extKP-Lke (5) using the symmetry method.

One other member of KP hierarchy is the Bousinesque-Kadomtsev-Petviashvili, abbreviated as BKP and one of the families of these BKP equations is

$$u_{tz} - u_{xxxy} - 3(u_x u_y)_x + 3u_{xx} = 0. \tag{6}$$

This equation has been the subject of much research [43–46]. Studies have shown that in [57], the author articulated the multiple exponential-function approaches in handling BKP (6) using symbolic computation in dealing with the involved computational algebraic systems. As a result, the shape-changing character of the anti-kink solution of (6) was explored. Besides, Ma and Fan in [44] utilized the analyzed linear superposition principle associated with exponential travelling waves for Hirota bilinear equations. In consequence, the construction of a specific sub-class of $N-$soliton solutions was achieved via the linear combinations of these exponential travelling waves which were later applied to secure some particular $N-$wave solutions. In their recent work, researchers in [43] successfully derived two new NLNPADEs from equation (6) and simultaneously established the validity of the equations by using the simplified linear superposition principle [44]. Furthermore, there are other derived varieties of models of KP referred to as KP-like equations. One such is the derived equation in [43], that is the (3+1)-dimensional Kadomtsev-Petviashvili like equation (3D-KPLike)

$$\begin{aligned} a u_{tx} + b\, u_{ty} + c\, u_{tz} - d\, u_{xxxy} - \\ 3u_x u_{xy} - 3u_{xx}u_y + e u_{xx} = 0, \end{aligned} \tag{7}$$

with arbitrary non-zero constants a, b, c, d and e. The authors went on to obtain generalized resonant multi-solitons whose existence, according to the authors, justifies the validity of the equation. We state categorically here for novelty, that model (7) is a new equation which has not been comprehensively examined using Lie symmetry analysis. Therefore, in this paper, for the first time, a detailed and comprehensive study of equation (7) is performed using its Lie algebras with a view to generating various exact solutions of the equation. We intend to use the symmetries to find group-invariant solutions of equation (7). Therefore in Section 2, stepwise computations of Lie point symmetries associated with (7) are outlined. Section 3 reveals the reductions of the underlying model using the obtained symmetries so that various possible exact solutions could be found. Besides, Section 4 is dedicated to the display of various graphical representations of the obtained solutions. Finally, we compute conservation laws using Ibragimov's theorem in Section 5. The concluding remarks are made thereafter.

2 Lie symmetries of 3D-KPLike (7)

We begin by extracting the Lie symmetries associated with the 3D-KPLike (7). A generic infinitesimal generator of (7) takes the form:

$$X = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \xi^3 \frac{\partial}{\partial y} + \xi^4 \frac{\partial}{\partial z} + \eta \frac{\partial}{\partial u}, \tag{8}$$

where ξ^1, ξ^2, ξ^3, ξ^4 and η are functions of (t, x, y, z). Generator (8) must, however, conform to the invariance condition:

$$\begin{aligned} X^{[4]}(a\, u_{tx} + b\, u_{ty} + c\, u_{tz} - d\, u_{xxxy} \\ - 3u_x u_{xy} - 3u_{xx}u_y + e\, u_{xx})|_{(7)} = 0. \end{aligned} \tag{9}$$

Here $X^{[4]}$ is the third extension of the generator X given by:

$$\begin{aligned} X^{[3]} =& X + \zeta_x \frac{\partial}{\partial u_x} + \zeta_y \frac{\partial}{\partial u_y} \\ &+ \zeta_{tx} \frac{\partial}{\partial u_{tx}} + \zeta_{ty} \frac{\partial}{\partial u_{ty}} + \zeta_{tz} \frac{\partial}{\partial u_{tz}} \\ &+ \zeta_{xx} \frac{\partial}{\partial u_{xx}} + \zeta_{xy} \frac{\partial}{\partial u_{xy}} + \zeta_{xxxy} \frac{\partial}{\partial u_{xxxy}}, \end{aligned} \tag{10}$$

with ζ_x, ζ_y, ζ_{tx}, ζ_{ty}, ζ_{tz}, ζ_{xx}, ζ_{xy} and ζ_{xxxy} defined by:

$$\begin{aligned} \zeta_x &= D_x(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - \\ &\quad u_y D_x(\xi^3) - u_z D_x(\xi^4), \\ \zeta_y &= D_y(\eta) - u_t D_x(\xi^1) - u_x D_x(\xi^2) - \\ &\quad u_y D_x(\xi^3) - u_z D_x(\xi^4), \\ \zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi^1) - u_{tx} D_x(\xi^2) - \\ &\quad u_{ty} D_x(\xi^3) - u_{tz} D_x(\xi^4), \\ \zeta_{ty} &= D_y(\zeta_t) - u_{tt} D_y(\xi^1) - u_{tx} D_y(\xi^2) - \\ &\quad u_{ty} D_y(\xi^3) - u_{tz} D_y(\xi^4), \\ \zeta_{tz} &= D_z(\zeta_t) - u_{tt} D_z(\xi^1) - u_{tx} D_z(\xi^2) - \\ &\quad u_{ty} D_z(\xi^3) - u_{tz} D_z(\xi^4), \\ \zeta_{xx} &= D_x(\zeta_x) - u_{xt} D_x(\xi^1) - u_{xx} D_x(\xi^2) - \\ &\quad u_{xy} D_x(\xi^3) - u_{xz} D_x(\xi^4), \\ \zeta_{xy} &= D_y(\zeta_x) - u_{xt} D_y(\xi^1) - u_{xx} D_y(\xi^2) - \\ &\quad u_{xy} D_y(\xi^3) - u_{xz} D_y(\xi^4), \\ \zeta_{xxx} &= D_x(\zeta_{xx}) - u_{xxt} D_x(\xi^1) - u_{xxx} D_x(\xi^2) - \\ &\quad u_{xxy} D_z(\xi^3) - u_{xxz} D_x(\xi^4), \\ \zeta_{xxxy} &= D_y(\zeta_{xxx}) - u_{xxxt} D_y(\xi^1) - u_{xxxx} D_y(\xi^2) - \\ &\quad u_{xxxy} D_y(\xi^3) - u_{xxxz} D_y(\xi^4), \end{aligned}$$

where D_t, D_x, D_y and D_z are the total derivatives, which are given by:

$$\begin{aligned}
D_t &= \frac{\partial}{\partial t}+u_t\frac{\partial}{\partial u}+u_{tt}\frac{\partial}{\partial u_t}+u_{tx}\frac{\partial}{\partial u_x}+\\
&\qquad u_{ty}\frac{\partial}{\partial u_y}+u_{tz}\frac{\partial}{\partial u_z}+\cdots,\\
D_x &= \frac{\partial}{\partial x}+u_x\frac{\partial}{\partial u}+u_{xx}\frac{\partial}{\partial u_x}+u_{xt}\frac{\partial}{\partial u_t}+\\
&\qquad u_{xy}\frac{\partial}{\partial u_y}+u_{xz}\frac{\partial}{\partial u_z}+\cdots,\\
D_y &= \frac{\partial}{\partial y}+u_y\frac{\partial}{\partial u}+u_{yy}\frac{\partial}{\partial u_y}+u_{yt}\frac{\partial}{\partial u_t}+\\
&\qquad u_{yx}\frac{\partial}{\partial u_x}+u_{yz}\frac{\partial}{\partial u_z}+\cdots,\\
D_z &= \frac{\partial}{\partial z}+u_z\frac{\partial}{\partial u}+u_{zz}\frac{\partial}{\partial u_z}+u_{zt}\frac{\partial}{\partial u_t}+\\
&\qquad u_{zy}\frac{\partial}{\partial u_y}+u_{zx}\frac{\partial}{\partial u_x}+\cdots.
\end{aligned} \tag{11}$$

Expanding (9) and splitting leads to the following Lie symmetries:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\ X_2=\frac{\partial}{\partial x},\ X_3=\frac{\partial}{\partial y},\ X_4=\frac{\partial}{\partial z},\\
X_5 &= c\,t\frac{\partial}{\partial t}-a\,z\frac{\partial}{\partial x}-b\,z\frac{\partial}{\partial y}-c\,z\frac{\partial}{\partial z},\\
X_6 &= 3c\,t\frac{\partial}{\partial t}-3(b\,z-c\,y)\frac{\partial}{\partial y}+c\,e\,y\frac{\partial}{\partial u},\\
X_7 &= 2c\,t\frac{\partial}{\partial t}-(a\,z-c\,x)\frac{\partial}{\partial x}+\\
&\qquad (b\,z-c\,y)\frac{\partial}{\partial y}-c\,u\frac{\partial}{\partial u},\\
X_G &= G(t)\frac{\partial}{\partial u},\ X_F=F(z)\frac{\partial}{\partial u},
\end{aligned} \tag{12}$$

where G and F are arbitrary functions of t and z respectively.

3 Symmetry reductions and exact solutions of (7)

In this section, we obtain different types of exact solutions of the 3D-KPLike (7) by utilizing the symmetries (12).

3.1 Invariant solutions under the symmetry $X_1 + X_G$

First we take into account symmetry $X_1 + X_G$ which furnishes invariants:

$$\begin{aligned}
Q(\xi,\eta,\zeta)+\int G(t)dt &= u(t,x,y,z),\\
\xi=x,\ \eta=y,\ \zeta&=z.
\end{aligned} \tag{13}$$

Inserting the value of u in the equation (7) yields the NLNPADE

$$e\,Q_{\xi,\xi}-3Q_\eta Q_{\xi\xi}-3Q_\xi Q_{\xi\eta}-d\,Q_{\xi\xi\xi\eta}=0, \tag{14}$$

which can be easily integrated using Maple. Thus, we gain the solution of 3D-KPLike (7) in this regard as

$$\begin{aligned}
u(t,x,y,z) &= \int G(t)dt+A_4+\\
&\quad 2d\,A_2\tanh\left(A_2x+\frac{e}{4dA_2}y+A_3z+A_1\right),
\end{aligned} \tag{15}$$

where arbitrary constants $A_1, A_2, \ldots, A_4$ are involved. Furthermore, we observe that (14) possesses the symmetries:

$$\begin{aligned}
M_1 &= \frac{\partial}{\partial \zeta},\qquad M_2=\frac{\partial}{\partial \xi},\qquad M_3=\frac{\partial}{\partial \eta},\\
M_4 &= \frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}+\frac{\partial}{\partial \zeta},\\
M_5 &= \xi\frac{\partial}{\partial \xi}+\zeta\frac{\partial}{\partial \zeta}+\left(\frac{1}{3}e\eta+\zeta-Q\right)\frac{\partial}{\partial Q}.
\end{aligned}$$

We invoke M_1 and this furnishes the invariants $W(r,s)=Q(\xi,\eta,\zeta)$, $r=\xi$ and $s=\eta$. Using these invariants, (14) reduces to:

$$e\,W_{rr}-3W_sW_{rr}-3W_rW_{rs}-d\,W_{rrrs}=0. \tag{16}$$

Solving (16), we achieve the solution of 3D-KPLike (7) as:

$$\begin{aligned}
u(t,x,y,z) &= 2d\,C_1\tanh\left(C_1x+\frac{e}{4d\,C_1}y+C_0\right)\\
&\qquad +\int G(t)dt+C_2,
\end{aligned} \tag{17}$$

with arbitrary constants C_0, C_1 and C_2. Further exploration of (16) gives its symmetries as:

$$\begin{aligned}
J_1 &= \frac{\partial}{\partial r}+F_1(s)\frac{\partial}{\partial s}+\frac{1}{3}e\,F_1(s)\frac{\partial}{\partial W},\\
J_2 &= F_2(s)\frac{\partial}{\partial s}+\left(\frac{1}{3}e\,F_1(s)+1\right)\frac{\partial}{\partial W},\\
J_3 &= r\frac{\partial}{\partial r}+F_3(s)\frac{\partial}{\partial s}+\left(\frac{1}{3}e\,F_3(s)+\frac{1}{3}e\,ss-W\right)\frac{\partial}{\partial W}.
\end{aligned}$$

By taking $F_i=1$, $i=1,2,3$, the symmetry J_1 yields the invariants $H(w)=W(r,s)$, $w=s-r$ and making use of them further reduces (7) to the nonlinear ordinary differential equation (NLNODE) given as:

$$2eH''(w)-6H'(w)H''(w)+dH''''(w)=0. \tag{18}$$

Now, for symmetry J_2, we have $H(w)+\left(\frac{1}{3}e+1\right)s=W(r,s)$, where $w=r$, hence transforming (7) further to linear ordinary differential equation (LNODE) $H''(w) = 0$. This, then gives a solution of the equation (7) as:

$$u(t,x,y,z) = B_1 x + \left(\frac{1}{3}e + 1\right) y + \int G(t)dt + B_2, \tag{19}$$

where B_1 and B_2 are arbitrary constants. In the case of symmetry J_3, we have invariants $\frac{1}{3r}\{3H(w) + (er+1)s\} = W(r,s)$, $w = -\ln(r)$, thus reducing (7) to:

$$\begin{aligned} &11dH''(w) - 12H'(w)^2 - 6H'(w)H''(w) \\ &\quad - 9H(w)H'(w) + 6dH'(w) - 3H(w)H''(w) \\ &\quad + 6dH'''(w) + dH''''(w) = 0. \end{aligned} \tag{20}$$

Exploring J_2 gives a trivial solution of (7) whereas J_3 purveys $W(r,s) = Q(\xi,\eta,\zeta)$, where $r = \xi$ and $s = \zeta$. Hence, (7) gives $W_{rr} = 0$ and so we have a solution of 3D-KPLike (7) in this regard as:

$$u(t,x,y,z) = f_1(z)x + f_2(z) + \int G(t)dt, \tag{21}$$

where $f_1(z)$ and $f_2(z)$ are arbitrary functions of z.

Now, we consider the symmetry M_4 and so, we have $W(r,s) = Q(\xi,\eta,\zeta)$, where $r = \eta - \xi$ and $s = \zeta - \xi$. Substituting this in (7), one achieves the NLNPADE:

$$\begin{aligned} &e\,W_{rr} + eW_{ss} + 2eW_{rs} - 6W_rW_{rr} - 9W_rW_{rs} \\ &\quad - 3W_rW_{ss} - 3W_sW_{rr} - 3W_sW_{rs} + d\,W_{rrrs} \\ &\quad + 3dW_{rrss} + dW_{rsss} + dW_{rrrr} = 0. \end{aligned} \tag{22}$$

Thus, we achieve a solution of the 3D-KPLike (7) as:

$$\begin{aligned} u(t,x,y,z) = &\int G(t)dt \\ &+ C_3 + C_2 \tanh\left[C_1(y-x) - C_1(z-x) + C_0\right], \end{aligned} \tag{23}$$

where arbitrary constants C_0, C_1, C_2 and C_3 exist. Further exploration of equation (22) yields three symmetries presented as:

$$I_1 = \frac{\partial}{\partial r} + \frac{\partial}{\partial s} + \frac{1}{3}e\frac{\partial}{\partial W},\quad I_2 = \frac{\partial}{\partial r} + \left(\frac{1}{3}e + 1\right)\frac{\partial}{\partial W},$$
$$I_3 = (r+1)\frac{\partial}{\partial r} + s\frac{\partial}{\partial s} + \left\{\frac{1}{3}e + \frac{2}{3}e(r-s) - W\right\}\frac{\partial}{\partial W}.$$

Using symmetry I_1 purveys a trivial solution of (7). However, in case of I_2, following the above procedure, one secures $\frac{1}{3}er + H(w) + r = W(r,s)$, $w = s$, which reduces (7) to LNODE $H''(w) = 0$. Solving this equation, one achieves the solution:

$$\begin{aligned} u(t,x,y,z) = &A_1(z-x) + \left\{\frac{1}{3}e + 1\right\}(y-x) \\ &+ \int G(t)dt + A_2, \end{aligned} \tag{24}$$

which satisfies 3D-KPLike (7) with A_1 and A_2 serving as arbitrary integration constants. Meanwhile, symmetry generator I_3 gives no result of interest.

Now, we examine the symmetry M_5. This furnishes $\frac{1}{6\xi}\{2e\xi\eta + 3\xi\zeta + 6W(r,s)\} = Q(\xi,\eta,\zeta)$ with $r = \eta$ and $s = \zeta/\xi$. Invoking these invariants, (7) transforms into NLNPADE:

$$\begin{aligned} &18d\,s\,W_{r,s} - 3s^2W_rW_{ss} - 3s^2W_sW_{rs} \\ &\quad + 9d\,s^2W_{rss} - 15sW_rW_s - 3sWW_{r,s} \\ &\quad - 9WW_r + 6dW_r + ds^3W_{rsss} = 0, \end{aligned} \tag{25}$$

which can be integrated using Maple. Thus, the 3D-KPLike (7) has the solution:

$$\begin{aligned} u(t,x,y,z) = &\frac{2exy + 3xz + 6f_1(y)f_2\left(\frac{z}{x}\right)}{6x} \\ &+ \int G(t)dt, \end{aligned} \tag{26}$$

with arbitrary functions f_1 and f_2 depending on their respective arguments. Moreover, the symmetries of (25) are:

$$K_1 = \frac{\partial}{\partial r} + s\frac{\partial}{\partial s},\qquad K_2 = \frac{\partial}{\partial r} + \frac{1}{s}\frac{\partial}{\partial W},$$
$$K_3 = \frac{\partial}{\partial r} + s^2\frac{\partial}{\partial s} - sW\frac{\partial}{\partial W}.$$

In the case of symmetry K_1, one secures $H(w) = W(r,s)$, $w = s/\exp(r)$ as invariants, which reduces (7) to NLNODE:

$$\begin{aligned} &18w^2H'(w)^2 - 24dwH'(w) + 12wH(w)H'(w) \\ &\quad - 36dw^2H''(w) + 3w^2H(w)H''(w) \\ &\quad + 6w^3H'(w)H''(w) - 12d\,w^3H'''(w) \\ &\quad - d\,w^4H''''(w) = 0. \end{aligned} \tag{27}$$

Now, we deal with K_2, which furnishes $\frac{1}{s}\{r + sH(w)\} = W(r,s)$ with $w = s$. Substituting the achieved result in (25) gives the ordinary differential equation (ORDE):

$$6H(w) + 12wH'(w) + 3w^2H''(w) = 0, \tag{28}$$

which solves and yields the solution of 3D-KPLike (7) as:

$$
\begin{aligned}
u(t,x,y,z) = {} & \frac{1}{6x}\left\{2exy + 3xz + \frac{6x}{z}\right. \\
& \left.\left\{\left\{y + \frac{z}{x}\left[\left(\frac{x}{z}\right)^2 C_0 + \left(\frac{x}{z}\right) C_1\right]\right\}\right\}\right. \\
& + \int G(t)dt,
\end{aligned}
\tag{29}
$$

where constants C_0 and C_1 are both integration constants. Next, we take on K_3, which yields $\frac{1}{s}H(w) = W(r,s)$ with $w = (1+rs)/s$. This further reduces (7) to:

$$
6H'(w)H''(w) + dH''''(w) = 0. \tag{30}
$$

Thus, we gain a solution of (7) here with regards to Weierstrass zeta function [58]:

$$
\begin{aligned}
u(t,x,y,z) = {} & \frac{1}{6z}\left\{3z^2 + 2eyz\right. \\
& \left. + 12d\, \text{WeierstrassZeta}\left[\frac{1}{z}(x + yz + C_0 z);0;C_1\right]\right\} \\
& + \int G(t)dt,
\end{aligned}
\tag{31}
$$

where integration constants C_0 and C_1 exist.

3.2 Invariant solutions under the symmetry $X_2 + X_F$

Here, we explore the symmetry $X_2 + X_F$ and it purveys the invariants $Q(\xi,\eta,\zeta) + xF(t) = u(t,x,y,z),\ \ \xi = t,\ \eta = y,\ \zeta = z$, which transforms (7) to yield the partial differential equation $aF(t) + bQ_{\xi\eta} + cQ_{\xi\zeta} = 0$. This can be easily solved and consequently, the 3D-KPLike (7) has the solution:

$$
u(t,x,y,z) = f_1(y,z) + f_2(t, bz - cy) - \frac{a}{b} tyF'(t), \tag{32}
$$

where functions f_1 and f_2 are arbitrary and dependent on their stated arguments.

3.3 Invariant solutions under the symmetry $X_3 + X_F$

The symmetry $X_3 + X_F$ secures the invariants $Q(\xi,\eta,\zeta) + yF(t) = u(t,x,y,z),\ \xi = t,\ \eta = x$, and $\zeta = z$. Invoking these invariants, equation (7) reduces to the NLNPADE:

$$
aQ_{\xi\eta} + bF'(t) + cQ_{\xi\zeta} - 3F(t)Q_{\eta\eta} + eQ_{\eta\eta} = 0. \tag{33}
$$

Solving this equation, we gain a set of solutions of the 3D-KPLike (7) as:

$$
\begin{aligned}
u(t,x,y,z) = {} & f_1(t) + f_2(x) + f_3(z) - \frac{e\,c_0}{c} t\, z \\
& + z\left(\frac{3c_0}{c}\int F(t)dt - \frac{b}{c}F(t)\right),
\end{aligned}
\tag{34a}
$$

$$
\begin{aligned}
u(t,x,y,z) = {} & f_1(t) f_2(x) f_3(z) - \\
& \frac{bF'(t)}{A_0(e - 3F(t))}\left(\frac{1}{2}A_0 x^2 + A_1 x + A_2\right),
\end{aligned}
\tag{34b}
$$

$$
\begin{aligned}
u(t,x,y,z) = {} & yF(t) - \frac{ec_1}{c} tz - \frac{b}{c} zF(t) \\
& + \frac{3c_1}{c} z \int F(t)dt + f_4(t) + f_5(z) \\
& + \frac{1}{2}c_1 x^2 + C_0 x + C_1,
\end{aligned}
\tag{34c}
$$

where functions $f_1(t),\ f_2(x),\ f_3(z),\ f_4(t),\ f_5(z)$ are arbitrary. To secure more robust results we take $F(t) = 1$ in the PDE (33) and then the resulting equation admits the following symmetries:

$$
\begin{aligned}
& M_1 = \frac{\partial}{\partial \xi}, \qquad M_2 = \frac{\partial}{\partial \eta}, \qquad M_3 = \frac{\partial}{\partial \zeta}, \\
& M_4 = Q\frac{\partial}{\partial Q}, \qquad M_5 = \frac{1}{2c}(a\zeta + c\eta)\frac{\partial}{\partial \eta} + \zeta\frac{\partial}{\partial \zeta}, \\
& M_6 = \xi\frac{\partial}{\partial \xi} - \frac{1}{2c}(a\zeta - c\eta)\frac{\partial}{\partial \eta}, \\
& M_7 = \frac{1}{a}(a\zeta - c\eta)\frac{\partial}{\partial \xi} + \frac{2}{a}(e-3)\zeta\frac{\partial}{\partial \eta}, \\
& M_8 = \frac{1}{2}\eta\zeta\frac{\partial}{\partial \eta} + \frac{1}{2}\zeta^2\frac{\partial}{\partial \zeta} - \frac{1}{8c(e-3)}(a\zeta - c\eta)^2\frac{\partial}{\partial \xi} - \\
& \qquad \frac{1}{4}Q\zeta\frac{\partial}{\partial Q}.
\end{aligned}
$$

We engage the combination of $M_1, \ldots, M_4$ and this yields the invariant functions $\exp(\xi)W(r,s) = Q(\xi,\eta,\zeta)$, $r = \eta - \xi$ and $s = \zeta - \xi$. Using these results, (7) reduces to:

$$
\begin{aligned}
& aW_{rr} - eW_{rr} + aW_{rs} + cW_{rs} - \\
& \quad aW_r + cW_{ss} - cW_s + 3W_{rr} = 0,
\end{aligned}
\tag{35}
$$

which admits the two translation symmetries $\partial/\partial r$ and $\partial/\partial s$. Their linear combination $\partial/\partial r + \alpha_1 \partial/\partial s$ provides invariants $H(w) = W(r,s)$ and $w = (s\alpha_0 - r\alpha_1)/\alpha_0$ and these transform (7) to NLNODE:

$$
\begin{aligned}
& a\,\alpha_1^2 H''(w) - a\,\alpha_0\,\alpha_1 H''(w) + c\,\alpha_0^2 H''(w) \\
& - c\,\alpha_0\,\alpha_1 H''(w) - e\,\alpha_1^2 H''(w) + a\alpha_0\,\alpha_1 H'(w) \\
& - c\,\alpha_0^2 H'(w) + 3\alpha_1^2 H''(w) = 0.
\end{aligned}
\tag{36}
$$

This ORDE can be easily solved and consequently, we achieve a solution of the 3D-KPLike (7) as:

$$u(t,x,y,z)=\exp(t)\left\{A_0+A_1\exp\left\{\frac{\alpha_0\,(a\alpha_1-c\alpha_0)}{\Lambda}\times\left[\frac{1}{\alpha_0}\left\{\alpha_0\,(z-t)-\alpha_1\,(x-t)\right\}\right]\right\}\right\}+y, \tag{37}$$

where $\Lambda = a\,\alpha_0\,\alpha_1 - a\,\alpha_1^2 - c\,\alpha_0^2 + c\,\alpha_0\,\alpha_1 + e\,\alpha_1^2 - 3\alpha_1^2$, A_0 and A_1 are arbitrary integration constants. Next, we make use of the symmetry M_5 and see that it produces the invariants $W(r,s)=Q(\xi,\eta,\zeta)$, $r=\xi$ and $s=\frac{1}{c\sqrt{\zeta}}\,(c\eta-a\zeta)$, thus reducing (7) to:

$$2eW_{ss}-csW_{rs}-6W_{ss}=0. \tag{38}$$

Solving the above equation yields the solution of the 3D-KPLike (7) as

$$u(t,x,y,z)=y+f_1(t)+\int f_2\left(\frac{1}{c}\left[c\Omega^2+(4e-12)\,t\right]\right)d\Omega, \tag{39}$$

with $\Omega = \frac{1}{c\sqrt{z}}\,(cx-az)$ as well as arbitrary functions f_1 and f_2 depending on their respective arguments. Now for the symmetry M_6, the invariants are $W(r,s)=Q(\xi,\eta,\zeta)$, $r=\frac{1}{c\sqrt{\xi}}\,(c\eta-a\zeta)$ and $s=\zeta$, which reduces (7) to:

$$2eW_{rr}-crW_{rs}-6W_{rr}=0. \tag{40}$$

In consequence, we secure the solution of (7) presented as:

$$u(t,x,y,z)=y+f_1(z)+\int f_2\left(\frac{1}{4\,(e-3)}\left[c\Delta_0^2+(4e-12)\,z\right]\right)d\Delta_0, \tag{41}$$

where $\Delta_0=\frac{1}{c\sqrt{t}}\,(cx-az)$ and functions f_1 and f_2 are arbitrary, depending on their various arguments. With regards to the symmetry M_7, we obtain invariants:

$$W(r,s)=Q(\xi,\eta,\zeta),$$
$$r=\frac{1}{4\,(e-3)\,\zeta}\left\{2a\eta\zeta-c\eta^2+(12-4e)\,\xi\zeta\right\},$$
$$s=\zeta,$$

and by utilizing them the model (7) transforms to:

$$6cW_r-4cesW_{rs}-a^2sW_{rr}-2ceW_r+12csW_{rs}=0. \tag{42}$$

Eventually, we achieve a solution of 3D-KPLike (7) as:

$$u(t,x,y,z)=f_1(z)+\frac{1}{a\sqrt{z}}f_2\left\{\frac{1}{a^2}\left[a^2z-4c(e-3)\left\{\frac{1}{4\,(e-3)\,z}\left[2axz-cx^2+(12-4e)\,tz\right]\right\}\right]\right\}+y, \tag{43}$$

where functions f_1 and f_2 are arbitrary and depend on their respective arguments. Finally, for the symmetry M_9 invariants are:

$$\frac{1}{\sqrt{\eta}}W(r,s)=Q(\xi,\eta,\zeta),\qquad r=\frac{\zeta}{\eta}$$
$$s=\frac{1}{4c\,(e-3)\,\zeta}\left\{a^2\zeta^2-2ac\eta\zeta+c^2\eta^2+4c\,(e-3)\,\xi\zeta\right\},$$

which when used in equation (7), reduces (7) to:

$$12erW_r-36rW_r+3eW-9W+4er^2W_{rr}-12r^2W_{rr}=0. \tag{44}$$

Therefore, we secure a solution of 3D-KPLike (7) as:

$$u(t,x,y,z)=\frac{1}{\sqrt{x}}\left\{\left(\frac{x}{z}\right)^{1/2}f_1\,(P)+\left(\frac{x}{z}\right)^{3/2}f_2\,(P)\right\}+y. \tag{45}$$

with arbitrary functions f_1 and f_2 depending on their respective arguments $P=\frac{1}{4c(e-3)z}[a^2z^2-2acxz+c^2x^2+4c\,(e-3)\,tz$.

3.4 Invariant solutions under the symmetry X_4+X_F

The Lie symmetry operator X_4+X_F gives the invariants:

$$\xi=t,\qquad \eta=x,\qquad \zeta=y,$$
$$Q(\xi,\eta,\zeta)+\int F(z)dz=u(t,x,y,z). \tag{46}$$

Inserting the value of u in the equation (7) yields the NLNPADE:

$$a\,Q_{\xi\eta}+b\,Q_{\xi\zeta}+e\,Q_{\eta\eta}-3Q_\zeta Q_{\eta\eta}-3Q_\eta Q_{\eta\zeta}-d\,Q_{\eta\eta\eta\zeta}=0. \tag{47}$$

Thus, by solving this equation we gain the solution of 3D-KPLike (7) as:

$$u(t,x,y,z)=2d\,C_2\tanh\left(\frac{C_2^2\,(4\,d\,C_2C_3-e)}{aC_2+bC_3}t+C_2x+C_3y+C_1\right)+\int F(z)dz+C_4, \tag{48}$$

where $C_1, C_2, \ldots, C_4$ are arbitrary constants. Further study of (47) reveals that the equation admits the following symmetries:

$$M_1 = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial Q}, \qquad M_2 = \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial Q},$$
$$M_3 = \frac{\partial}{\partial \eta} + \frac{\partial}{\partial Q},$$
$$M_4 = \xi \frac{\partial}{\partial \eta} + \left(1 - \frac{1}{3}a\zeta - \frac{1}{3}b\eta\right)\frac{\partial}{\partial Q},$$
$$M_5 = \xi \frac{\partial}{\partial \xi} + \frac{1}{3}\eta \frac{\partial}{\partial \eta} + \frac{1}{3}\zeta \frac{\partial}{\partial \zeta} + \left(1 + \frac{2}{9}e\zeta - \frac{1}{3}Q\right)\frac{\partial}{\partial Q}.$$

We now explore $M_1, \ldots, M_5$ in obtaining some solutions to (7). Beginning with M_1, we have invariants $\xi + W(r,s) = Q(\xi,\eta,\zeta)$, $r = \eta$ and $s = \zeta$, which transform (7) to:

$$eW_{rr} - 3W_s W_{rr} - 3W_r W_{rs} - dW_{rrrs} = 0. \tag{49}$$

Thus, we have a solution of 3D-KPLike model in this instance as:

$$\begin{aligned} u(t,x,y,z) &= t + \int F(z)dz + C_2 \\ &+ 2dC_1 \tanh\left(C_1 x + \frac{e}{4dC_1}y + C_0\right), \end{aligned} \tag{50}$$

with C_0, C_1, C_2 being arbitrary constants.

Further study of (49) gives

$$J_1 = \frac{\partial}{\partial r} + F_1(s)\frac{\partial}{\partial s} + \frac{1}{3}eF_1(s)\frac{\partial}{\partial W},$$
$$J_2 = F_2(s)\frac{\partial}{\partial s} + \left(\frac{1}{3}eF_2(s) + 1\right)\frac{\partial}{\partial W},$$
$$J_3 = r\frac{\partial}{\partial r} + F_3(s)\frac{\partial}{\partial s} + \left(\frac{1}{3}eF_3(s) + \frac{1}{3}es - W\right)\frac{\partial}{\partial W}.$$

We assume $F_i(s) = s$, $i = 1,2,3$, and then for J_1, one secures the invariants $H(w) + \frac{1}{3}es = W(r,s)$ and $w = s/e^r$. Thus, these invariants further reduce (7) to the NLNODE given as:

$$\begin{aligned} d\,H'(w) &- 6w\,H'(w)^2 + 7d\,wH''(w) \\ &+ 6\,d\,w^2 H'''(w) - 6\,w^2\,H'(w)H''(w) \\ &+ d\,w^3\,H''''(w) = 0. \end{aligned} \tag{51}$$

Next, for J_2, one gains the invariants $H(w) + \frac{1}{3}es + \ln(s) = W(r,s)$ and $w = r$, which further reduce (7) to $H''(w) = 0$. Thus, this gives a solution:

$$\begin{aligned} u(t,x,y,z) &= t + A_1 x + \frac{1}{3}ey + \\ &\ln(y) + \int F(z)dz + A_2, \end{aligned} \tag{52}$$

where A_1 and A_2 are arbitrary constants. In the case of J_3, we have invariants $\frac{1}{3r}\{3H(w) + ers\} = W(r,s)$ and $w = s/r$. Hence, using these invariants equation (7) reduces to:

$$\begin{aligned} 36d\,w\,H''(w) &- 18w\,H'(w)^2 - 6w^2 H'(w)H''(w) \\ &- 3\,w\,H(w)\,H''(w) + 24\,d\,H'(w) \\ &- 12H(w)H'(w) + 12dw^2 H'''(w) \\ &+ dw^3 H''''(w) = 0. \end{aligned} \tag{53}$$

Exploring M_2 purveys $\zeta + W(r,s) = Q(\xi,\eta,\zeta)$, $r = \xi$, $s = \eta$, which reduce (7) to:

$$aW_{rs} - 3W_{ss} + eW_{ss} = 0. \tag{54}$$

This equations is solved easily, and so we have a solution of 3D-KPLike (7) as:

$$\begin{aligned} u(t,x,y,z) =& y + f_1(t) \\ &+ f_2\left[ax - (e-3)\,t\right] + \int F(z)dz. \end{aligned} \tag{55}$$

with arbitrary functions f_1 and f_2.

Now, we consider M_3 and so, we have the invariants $\eta + W(r,s) = Q(\xi,\eta,\zeta)$, $r = \xi$ and $s = \zeta$, which reduce (7) to $W_{rs} = 0$, thus yielding a solution of the 3D-KPLike (7) as:

$$u(t,x,y,z) = x + f_1(t) + f_2\,(y) + \int F(z)dz, \tag{56}$$

with f_1 and f_2 being arbitrary functions. The symmetry M_4 furnishes the invariants $\eta + W(r,s) = Q(\xi,\eta,\zeta)$, $r = \xi$ and $s = \zeta$, thus reducing (7) to:

$$3r\,W_{rs} + 3W_s - e = 0. \tag{57}$$

By solving differential equation (57), one has a solution of (7) as:

$$\begin{aligned} u(t,x,y,z) &= x + \frac{1}{6t}\left\{6x - 2axy - bx^2\right\} \\ &+ 2ey + 6f_1(t) + \frac{6}{t}f_2\,(y) + \int F(z)dz, \end{aligned} \tag{58}$$

with $f_1(t)$ and $f_2(y)$ are arbitrary functions of t and y respectively.

Finally, under symmetry $X_4 + X_F$ we explore M_5, which gives the invariants:

$$\frac{1}{3}\left(e\,\zeta + 9\right) + \frac{1}{\sqrt[3]{\xi}}W(r,s) = Q(\xi,\eta,\zeta),$$
$$r = \frac{\eta}{\sqrt[3]{\xi}} \text{ and } s = \frac{\zeta}{\sqrt[3]{\xi}}.$$

Utilizing these invariants, equation (7) reduces to the NLNPADE:

$$\begin{aligned} &a\,r\,W_{rr} + b\,s\,W_{ss} + b\,r\,W_{rs} + a\,s\,W_{rs} \\ &+ 2aW_r + 2bW_s + 9W_r W_{rs} + 9W_s W_{rr} + 3dW_{rrrs} = 0. \end{aligned}$$

Further exploration of the above equation yields no solution of interest.

3.5 Invariant solutions under the symmetry $X_1, \ldots, X_4$

We will now explore various types of solutions emanating from the translation symmetries.

Jacobi elliptic function solutions of equation (7)

Using the time and space translations $X_1, \ldots, X_4$, we have the group invariant:

$$u(t,x,y,z) = U(p), \;\; p = \alpha x + \beta y + \nu z - \gamma t, \tag{59}$$

where α, β, ν and γ are arbitrary constants. Using (59), equation (7) is reduced to the fourth-order NLNODE:

$$\begin{aligned}(e\,\alpha^2 - a\,\alpha\,\gamma - b\,\beta\,\gamma - c\,\nu\,\gamma)U''(p)\\ - 6\beta\,\alpha^2 U''(p)U'(p) - d\,\beta\,\alpha^3 U''''(p) = 0.\end{aligned} \tag{60}$$

Integrating (60) once with respect to p gives:

$$\begin{aligned}C_0 + (e\,\alpha^2 - a\,\alpha\,\gamma - b\,\beta\,\gamma - c\,\nu\,\gamma)U'(p)\\ - 3\beta\,\alpha^2 U'(p)^2 - d\,\beta\,\alpha^3 U'''(p) = 0,\end{aligned} \tag{61}$$

where C_0 is an integration constant.

We now let:

$$\begin{aligned}&U'(p) = \alpha\, d\psi(p),\\ &\omega = \frac{1}{d\beta\alpha^3}\left\{e\alpha^2 - a\alpha\gamma - b\beta\gamma - c\nu\gamma\right\}, \quad \text{and}\\ &C_1 = \frac{C_0}{d\beta\alpha^4}.\end{aligned} \tag{62}$$

Equation (61) becomes:

$$\psi'' + 3\psi^2 - \omega\psi + C_1 = 0. \tag{63}$$

Multiplying (63) by ψ' and integrating with respect to p gives:

$$\frac{1}{2}\psi'^2 + \psi^3 - \frac{1}{2}\omega\psi^2 + C_1\psi + C_2 = 0,$$

which when rearranged becomes:

$$\psi'^2 = -(2\psi^3 - \omega\psi^2 + 2C_1\psi + 2C_2), \tag{64}$$

where C_2 is an arbitrary constant of integration.

Now, the right hand side of equation (64) is a cubic function and suppose it has factors $\theta_1 > \theta_2 > \theta_3$ such that $\theta_i \in \Re,\;\; i = 1,2,3$. Thus we have:

$$\psi'^2 = -2(\psi - \theta_1)(\psi - \theta_2)(\psi - \theta_3). \tag{65}$$

The solution for (65) is well-known in terms of the Jacobi elliptic cosine function and is given by [59, 60]:

$$\psi(p) = \theta_2 + (\theta_1 - \theta_2)\mathrm{cn}^2\left(\sqrt{\frac{\theta_1 - \theta_3}{2}}\,p\,\middle|\,S^2\right), \tag{66}$$

where $S^2 = \dfrac{\theta_1 - \theta_2}{\theta_1 - \theta_3}$.

Notice the appearance of the Jacobi elliptic cosine function $\mathrm{cn}(p|S^2)$ with special parameter $S^2 : 0 < S^2 < 1$. The behaviour of S^2 greatly dictates how the function morphs into its trigonometric and hyperbolic counterparts. In general, when $S^2 \to 1$, $\mathrm{cn}(\,p|S^2) \to \mathrm{sech}\,(p)$ and when $S^2 \to 0$, $\mathrm{cn}(p|S^2) \to \cos\,(p)$ [61]. Now, since $U(p) = \alpha d \int \psi(p)$, the solution of the 3D-KPLike (7) is thus:

$$\begin{aligned}u(t,x,y,z) = \alpha\, d &\left\{ \frac{\sqrt{2}\,(\theta_1 - \theta_3)}{\sqrt{\theta_1 - \theta_2}} \right.\\ &\times \mathrm{E}\left\{ \mathrm{sn}\left[\frac{(x\alpha + y\beta - t\gamma + z\nu)\sqrt{\theta_1 - \theta_3}}{\sqrt{2}} \middle| \frac{\theta_1 - \theta_2}{\theta_1 - \theta_3} \right] \right.\\ &\times \left. \left| \frac{\theta_1 - \theta_2}{\theta_1 - \theta_3} \right\} + \theta_2(\alpha x - \gamma t + \beta y + \nu z)\\ &- \frac{(\theta_1 - \theta_3)^2}{\theta_1 - \theta_2}(\alpha x - \gamma t + \beta y + \nu z)\\ &\times \left. \left(1 - \frac{(\theta_1 - \theta_2)^2}{(\theta_1 - \theta_3)^2} \right) \right\},\end{aligned}$$

where sn is the Jacobi elliptic sine function and E is the elliptic integral of the second kind. Solution (67) contains both topological kink when $\{(\theta_1 - \theta_2)\,/\,(\theta_1 - \theta_3)\} \to 1$, and periodic solutions when $\{(\theta_1 - \theta_2)\,/\,(\theta_1 - \theta_3)\} \to 0$.

Hyperbolic function solutions of equation (7)

Multiplying equation (61) by U'' and integrating once with respect to p leads to the second-order NLNODE:

$$\begin{aligned}C_1 + C_0 U'(p) + AU'(p)^2\\ - BU'(p)^3 - EU''(p)^2 = 0,\end{aligned} \tag{67}$$

where $A = (e\alpha^2 - a\alpha\gamma - b\beta\gamma - c\nu\gamma)/2$, $B = \beta\alpha^2$, and $E = 1/2d\beta\alpha^3$.

Now letting $U'(p) = V(p)$, we obtain:

$$\begin{aligned}C_1 + C_0 V(p) + AV(p)^2\\ - BV(p)^3 - EV'(p)^2 = 0.\end{aligned} \tag{68}$$

Letting $C_0 = C_1 = 0$, and solving the resultant equation to get:

$$\begin{aligned}&V(p) = \frac{A}{B}\\ &V(p) = -\frac{A}{B}\left[\left\{\tanh\left(\frac{\sqrt{EA}\,(C_3 - p)}{2E}\right)\right\}^2 - 1\right].\end{aligned} \tag{69}$$

Recall that $U(p) = \int V(p)\mathrm{d}p$. Consequently, reverting to the original variables, the analytic solutions of (7) are:

$$u_1 = \frac{A}{B}p + K_0. \tag{70}$$

and

$$\begin{aligned} u_2 = &-\frac{2EA}{B\sqrt{EA}}\tanh\left\{\frac{\sqrt{EA}(C_3-p)}{2E}\right\} \\ &-\frac{EA}{B\sqrt{EA}}\ln\left\{\tanh\left(\frac{\sqrt{EA}(C_3-p)}{2E}\right)-1\right\} \\ &+\frac{EA}{B\sqrt{EA}}\ln\left(\tanh\left\{\frac{\sqrt{EA}(C_3-p)}{2E}\right\}+1\right) \\ &+\frac{A}{B}p+K_1, \end{aligned} \quad (71)$$

where $p=\alpha x+\beta y+\nu z-\gamma t$, with K_0 and K_1, integration constants.

Exact solutions of (7) using Kudryashov's method

We invoke the Kudryashov method [62] to determine additional exact solutions of (7). We begin by assuming that the solutions to the fourth-order NLNODE (60) can be written in the form:

$$U(p)=\sum_{i=0}^{M}A_iY^i(p), \quad (72)$$

where $Y(p)$ solves the Riccati equation:

$$Y'(p)=Y^2(p)-Y(p), \quad (73)$$

which has an exact solution given by:

$$Y(p)=\frac{1}{1+e^p}. \quad (74)$$

The value of M in (72) can be determined by using the balancing procedure [62] and A_i, $i=0,1,\ldots,M$ are constants which we will determine. We balance the highest order derivative with the nonlinear term, that is, $U''''(p)$ and $U(p)'U''(p)$ respectively. This gives $M=1$. Thus the solution (72) can be written as:

$$U(p)=A_0+A_1Y(p). \quad (75)$$

Substituting (75) into (60) and invoking (73), we obtain:

$$\begin{aligned} -2a\,a_1\,\alpha\,\gamma\,Y^3(p)&+3\,a\,a_1\,\alpha\,\gamma\,Y^2(p) \\ &-a\,a_1\alpha\,\gamma\,Y(p)-2\,b\,a_1\beta\,\gamma\,Y^3(p) \\ &+3\,b\,a_1\beta\,\gamma\,Y^2(p)-b\,a_1\beta\,\gamma\,Y(p) \\ &-2\,c\,a_1\nu\,\gamma\,Y^3(p)+3\,c\,a_1\nu\,\gamma\,Y^2(p) \\ &-c\,a_1\nu\,\gamma\,Y(p)-24\,d\,a_1\beta\,\alpha^3Y^5(p) \\ &+60\,d\,a_1\beta\,\alpha^3Y^4(p)-50\,d\,a_1\beta\,\alpha^3Y^3(p) \\ &+15\,d\,a_1\beta\,\alpha^3Y^2(p)-d\,a_1\beta\,\alpha^3Y(p) \\ &-12\,{a_1}^2\alpha^2\beta\,Y^5(p)+30\,{a_1}^2\alpha^2\beta\,Y^4(p) \\ &-24\,{a_1}^2\alpha^2\beta\,Y^3(p)+6\,{a_1}^2\alpha^2\beta\,Y^2(p) \\ &+2\,e\,a_1\alpha^2Y^3(p)-3\,e\,a_1\alpha^2Y^2(p) \\ &+e a_1\alpha^2Y(p)=0. \end{aligned}$$

Equating the coefficients of like powers of $Y(p)$ in equation (76) we obtain the following five algebraic equations in terms of a_0 and a_1:

$$\begin{aligned} &Y^5(p):2\,da_1\beta\,\alpha^3+{a_1}^2\alpha^2\beta=0, \\ &Y^4(p):2\,da_1\beta\,\alpha^3+{a_1}^2\alpha^2\beta=0, \\ &Y^3(p):ea_1\alpha^2-25\,da_1\beta\,\alpha^3 \\ &\qquad -12\,{a_1}^2\alpha^2\beta-\,aa_1\alpha\,\gamma \\ &\qquad -ba_1\beta\,\gamma-ca_1\nu\,\gamma=0, \\ &Y^2(p):5\,da_1\beta\,\alpha^3+2\,{a_1}^2\alpha^2\beta+aa_1\alpha\,\gamma \\ &\qquad -ea_1\alpha^2+ba_1\beta\,\gamma+ca_1\nu\,\gamma=0, \\ &Y^1(p):25\,\gamma^3\lambda\,a_1+16\,\gamma^2\lambda\,{a_1}^2 \\ &\qquad +8\,\gamma\,\lambda^2{a_1}^2-4\,\gamma\,va_1=0. \end{aligned} \quad (76)$$

The solution of these equations is:

$$\begin{aligned} a_0&=a_0, \\ a_1&=\frac{2(a\alpha\,\gamma-\alpha^2e+b\beta\,\gamma+c\gamma\,\nu)}{\alpha^2\beta}, \\ d&=\frac{\alpha^2e-a\alpha\,\gamma-b\beta\,\gamma-c\gamma\,\nu}{\alpha^3\beta}. \end{aligned} \quad (77)$$

Thus, the solution of the (3+1)-D KP-like (7) reads:

$$u(t,x,y,z)=a_0+\frac{2\,a\alpha\,\gamma-2\,\alpha^2e+2\,b\beta\,\gamma+2\,c\gamma\nu}{\alpha^2\beta\,\left(1+\mathrm{e}^{\alpha\,x+\beta\,y-\gamma\,t+\nu\,z}\right)}. \quad (78)$$

3.6 Invariant solutions under symmetry X_5

The characteristic equations of the symmetry X_5 yield the invariants $G(p,q,k)=u(t,x,y,z)$, $p=tz$, $q=cx-az$, $k=bx-ay$. Insertion of this value of u into (7) produces:

$$\begin{aligned} &c\,p\,G_{pp}+cG_p+a\,c^3d\,G_{qqkq}+3\,a\,b\,c^2d\,G_{qkkq} \\ &\quad +3a\,b^2c\,d\,G_{kkkq}+a\,b^3d\,G_{kkkk}+3\,a\,c^2G_kG_{qq} \\ &\quad +9a\,b\,c\,G_kG_{kq}+6a\,b^2G_kG_{kk}+3a\,c^2G_qG_{kq} \\ &\quad +3a\,b\,c\,G_qG_{kk}+c^2e\,G_{qq}+2b\,c\,e\,G_{kq}+b^2eG_{kk}=0. \end{aligned} \quad (79)$$

Equation (79) has five symmetries, namely:

$$\begin{aligned} &\Gamma_1=\frac{\partial}{\partial q}, \qquad \Gamma_2=\frac{\partial}{\partial k}, \\ &\Gamma_3=\frac{\partial}{\partial G}, \qquad \Gamma_4=\ln p\frac{\partial}{\partial G}, \\ &\Gamma_5=3acp\frac{\partial}{\partial p}+(3ack-3abq)\frac{\partial}{\partial k}+(beq-cek)\frac{\partial}{\partial G}. \end{aligned}$$

Characteristic equations of $\Gamma=\Gamma_1+\gamma\Gamma_2$, where γ is a constant yields the invariants $G(p,q,k)=U(g)$, $g=$

$k-\gamma q$, which transforms equation (79) to the NLNODE:

$$(3a\,b\,c^2d\gamma^2+a\,b^3d-3a\,b^2c\,d\gamma-a\,c^3d\gamma^3)U''''\\+(6a\,c^2\gamma^2-12a\,b\,c\gamma+6ab^2)U''U'\\+(c^2e\gamma^2+b^2e-2bce\gamma)U''=0,$$

or

$$\mathcal{E}U'U''+\mathcal{F}U''+\mathcal{Z}U''''=0, \tag{80}$$

where

$$\mathcal{Z}=3abc^2d\gamma^2+ab^3d-3ab^2cd\gamma-ac^3d\gamma^3,\\ \mathcal{E}=6ac^2\gamma^2-12abc\gamma+6ab^2,\\ \mathcal{F}=c^2e\gamma^2+b^2e-2bce\gamma.$$

Solution of (7) by direct integration.

We seek to find the solution of 3D-KPLike (7) by utilizing the NLNODE (80). Integrating (80) with respect to g yields:

$$\frac{\mathcal{E}}{2}U'^2+\mathcal{F}U'+\mathcal{Z}U'''+P_0=0, \tag{81}$$

where P_0 is an arbitrary constant of integration.

Let $U'(g)=\rho(g)$, then equation (81) becomes:

$$\frac{\mathcal{E}}{2}\rho^2+\mathcal{F}\rho+\mathcal{Z}\rho''+P_0=0. \tag{82}$$

Multiplying (82) by $\rho'(g)$ and integrating with respect to g gives:

$$\frac{\mathcal{Z}}{2}\rho'^2+\frac{\mathcal{E}}{6}\rho^3+\frac{\mathcal{F}}{2}\rho^2+P_0\rho+P_1=0, \tag{83}$$

where P_1 is an arbitrary constant.

Then,

$$\rho'^2+\frac{\mathcal{E}}{3\mathcal{Z}}\rho^3+\frac{\mathcal{F}}{\mathcal{Z}}\rho^2+\frac{2}{\mathcal{Z}}P_0\rho+\frac{2}{\mathcal{Z}}P_1=0. \tag{84}$$

Suppose that r_1, r_2 and r_3 are real roots $(r_1>r_2>r_3)$ of a cubic equation:

$$\rho^3+\frac{3\mathcal{F}}{\mathcal{E}}\rho^2+\frac{6}{\mathcal{E}}P_0\rho+\frac{6}{\mathcal{E}}P_1=0. \tag{85}$$

that satisfy the conditions:

$$r_1r_2r_3=-\frac{6}{\mathcal{E}}P_1,\\ r_1r_2+r_1r_3+r_2r_3=\frac{6}{\mathcal{E}}P_0,\\ r_1+r_2+r_3=-\frac{3\mathcal{F}}{\mathcal{E}}.$$

Then equation (84) is written as:

$$\rho'^2=-\frac{\mathcal{E}}{3\mathcal{Z}}(\rho-r_1)(\rho-r_2)(\rho-r_3),$$

and has the solution:

$$\rho(r)=r_2+(r_1-r_2)\,\mathrm{cn}^2\\ \times\left\{\sqrt{\frac{\mathcal{E}(r_1-r_3)}{12\mathcal{Z}}}\,(g-g_0)\,\middle|\,M^2\right\}, \tag{86}$$
$$M^2=\frac{r_1-r_2}{r_1-r_3},$$

where r_0 is a constant and cn is the Jacobi cosine function.

Thus, by returning to the original variables, we obtain the solution of the 3D-KPLike (7) as:

$$u(t,x,y,z)=\mathcal{F}_0\left[\mathrm{EllipticE}\left\{\mathrm{sn}\left[\mathcal{F}_1(g-g_0),K^2\right],K^2\right\}\right]\\ +\left\{r_2-(r_1-r_2)\frac{1-K^4}{K^4}\right\}\\ \times(g-g_0)+k_1, \tag{87}$$

where

$$\mathcal{F}_0=\sqrt{\frac{12\mathcal{C}(r_1-r_2)^2}{(r_1-r_3)\mathcal{E}K^8}},\ \mathcal{F}_1=\sqrt{\frac{\mathcal{E}(r_1-r_2)}{12\mathcal{C}}},$$

with $g=bx-ay-\gamma(cx-az)$ and k_1 an arbitrary constant.

3.7 Invariant solutions under symmetry X_6

From the symmetry X_6 we get the group invariant solution:

$$F(p,q,k)=u(t,x,y,z)-\frac{bz\ln(bz-cy)}{3c}\\ +\frac{e\,(bz-cy)}{3c}, \tag{88}$$

where $p=x$, $q=z$ and $k=(cy-bz)/t$ and equation (7) transforms to the NLNPADE:

$$a\,k^2F_{kp}+c\,k^2F_{kq}+c\,d\,k\,F_{ppkp}\\ +bqF_{pp}+3ckF_{pp}F_k+3ckF_pF_{kp}=0. \tag{89}$$

Equation (89) has three Lie symmetries:

$$U_1=\frac{\partial}{\partial p}+F^1(q)\frac{\partial}{\partial F},\\ U_2=3c\frac{\partial}{\partial q}-\left\{b\ln k-F^2(q)\right\}\frac{\partial}{\partial F},\\ U_3=(2aq-cp)\frac{\partial}{\partial p}-cq\frac{\partial}{\partial q}+4ck\frac{\partial}{\partial k}\\ +\left\{cF+cF^3(q)\right\}\frac{\partial}{\partial F}, \tag{90}$$

where F^1, F^2 and F^3 are arbitrary functions of q.

However, here we take a special case $F^1 = F^2 = F^3 = 0$, and obtain:

$$Y_1 = \frac{\partial}{\partial p},\ Y_2 = 3c\frac{\partial}{\partial q} - b\ln k\frac{\partial}{\partial F},$$
$$Y_3 = (2aq - cp)\frac{\partial}{\partial p} - cq\frac{\partial}{\partial q} + 4ck\frac{\partial}{\partial k} + cF\frac{\partial}{\partial F}. \tag{91}$$

We use the symmetry Y_2 to perform reductions on (89). This symmetry has invariants:

$$F(p,q,k) + \frac{1}{3c}(bq\ln k) = W(r,j), \quad r = p,\ \ s = k. \tag{92}$$

Substituting (92) into (89) yields NLNPADE:

$$3\,a\,s\,W_{rs} + 3\,c\,d\,W_{rrrs} + 9\,c\,W_{rr}W_s + 9c\,W_r\,W_{rs} - b = 0. \tag{93}$$

The given equation (93) has the following operator as its Lie symmetry:

$$W_1 = \frac{\partial}{\partial r},\ W_2 = 3c\frac{\partial}{\partial s} - ar\frac{\partial}{\partial W},\ W_3 = \frac{\partial}{\partial W}.$$

We use symmetry W_2 to perform reductions on (93).

This symmetry has invariants $W(r,s) + (ars)/(3c) = N(r)$, $r = \xi$ and using them yields the LNORDE:

$$b + 3a\xi N'' + 3aN' = 0, \tag{94}$$

whose solution is

$$N(\xi) = C_1\ln(\xi) - \frac{b}{3a}\xi + C_2, \tag{95}$$

with C_1 and C_2 being arbitrary constants of integration.

Hence, the group-invariant solution of (7) under X_6 is:

$$u(t,x,y,z) = \frac{b}{3c}\{z\ln(bz-cy)\} - \frac{e}{3c}(bz-cy) + C_1\ln(x) - \frac{b}{3a}x + C_2 - \frac{b}{3c}\left\{z\ln\left[\frac{1}{t}(cy-bz)\right]\right\}.$$

3.8 Invariant solutions under symmetry X_7

The use of symmetry X_7 gives the invariants:

$$p = z, \qquad q = (cx - az)/\sqrt{t}c,$$
$$k = \sqrt{t}(cy - bz)/c,$$
$$U(p,q,k) = t^{1/2}u(t,x,y,z).$$

and using these, equation (7) transforms to:

$$2eU_{qq} - cU_p - 2dU_{qqkq} - 6U_{qq}U_k - 6U_qU_{kq} - cqU_{pq} + ckU_{kp} = 0. \tag{96}$$

Equation (96) has the following operators as its Lie symmetries:

$$Q_1 = \frac{\partial}{\partial p}, \qquad Q_2 = \frac{\partial}{\partial U},$$
$$Q_3 = 9p\frac{\partial}{\partial p} + 3q\frac{\partial}{\partial q} + (ek - 3U)\frac{\partial}{\partial U},$$
$$Q_4 = -3q\frac{\partial}{\partial q} + 9k\frac{\partial}{\partial k} + (2ek + 3U)\frac{\partial}{\partial U}.$$

The symmetry Q_3 has the invariants:

$$r = \frac{q}{\sqrt[3]{p}}, \qquad s = k,$$
$$H(r,s) = \left(U(p,q,k) - \frac{1}{3}ek\right)\frac{q}{\sqrt[3]{p}}, \tag{97}$$

which transform the PDE (96) to:

$$c\,H + 3\,c\,r\,H_r - 6\,d\,H_{rrrs} - 18\,H_{rr}\,H_s - 18H_rH_{rs} + c\,r^2H_{rr} - c\,s\,H_s - c\,r\,s\,H_{rs} = 0. \tag{98}$$

The above equation admits the Lie symmetry:

$$G = -r\frac{\partial}{\partial r} + 3s\frac{\partial}{\partial s} + H\frac{\partial}{\partial H},$$

which has invariants:

$$rH(r,s) = F(\xi), \qquad \xi = r^3 s,$$

and so the PDE (98) reduces to the NLNODE:

$$9F(\xi)F''(\xi) - 27\xi^2 d\,F''''(\xi) - 108\xi\,d\,F'''(\xi) + (\xi c - 60d)F''(\xi) + c\,F'(\xi) - 54\,r\,F'(\xi)\,F''(\xi) - 36F(\xi)\,F'(\xi) = 0.$$

The use of symmetry Q_4 provides us with the invariants:

$$r = p, \qquad s = kq^3,$$
$$H(r,s) = \left(U(p,q,k) - \frac{1}{3}ek\right)q,$$

and these invariants convert equation (96) to NLNPADE:

$$9H_{ss}\,H - 27\,d\,s^2H_{ssss} - 54\,sH_{ss}\,H_s - 108\,d\,s - 60\,d\,H_{ss} - 18H_s^2 - cH_{rs} = 0. \tag{99}$$

Equation (99) has two Lie point symmetries, viz.,

$$T_1 = \frac{\partial}{\partial r},\ T_2 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}.$$

The symmetry operator T_2 has invariants:

$$H(r,s) = W(r), \qquad \xi = \frac{s}{r},$$

and these transform the PDE (99) to the NLNODE

$$c\,W'(\xi) - 27\,d\,\xi^2W''''(\xi) + c\,\xi W''(\xi) + 108\xi\,d\,W'''(\xi) - 54\xi W''(\xi)W'(\xi) - 60\,d\,W''(\xi) - 1\,8W'^2(\xi) + 9\,W''(\xi)\,W(\xi) = 0. \tag{100}$$

4 Graphical representation of solutions and discussions

In this section, we present the graphical descriptions of some of the obtained solutions in the previous section. The results comprise various solutions of interest ranging from exponential, trigonometry, and hyperbolic to Jacobi elliptic function solutions. Besides, several algebraic solutions consisting of arbitrary functions were achieved. These arbitrary functions can assume any possible mathematical functions with the result satisfying (7). Therefore, using computer software, we represent a few of the various interesting solutions in this study with some graphical display of solitary waves in the form of three-dimensional (3D), two-dimensional (2D), and density plots.

In the first place, we explore the dynamics of algebraic solution (21) in Figure 1 with functions $f_1(z) = \operatorname{sech}(z)$, $f_2(z) = \sin(z)$, and $G(t) = -\sin(t)$, where $x = 1$, $y = 0$, $-8 \leq t, z \leq 8$. We further have as earlier presented, a fixed value of y with $f_1(z) = \operatorname{sech}(z)$, $f_2(z) = \cos(z)$, $G(t) = -\cos(t)$, where $x = 1.2$, and $-10 \leq t, z \leq 10$, thus, we plot Figure 2. In addition, Figure 3 is presented for the solution of (7) with $f_1(z) = \operatorname{sech}(z)$, $f_2(z) = \cos(z)$, and $G(t) = -\cos(t)$, where $x = 2$, $y = 0$, alongside $-8 \leq t, z \leq 8$. Next, we display solution (55) in Figure 4 where we take $f_1(t) = \operatorname{sech}(t)$, $f_2(t, x) = \sin[ax - (e-3)t]$, $F(z) = -\sin(z)$, in which $a = 1$, $e = 5$, $x = y = 0$, and $-8 \leq t, z \leq 8$. Further to that, in Figure 5, we assign $f_1(t)$ as earlier done and $F(z) = \cos(z)$, $f_2(t, x) = \tanh^2[ax - (e-3)t]$, where $a = 1$, $e = 5$, $x = 0.1$, $y = 0$, and $-8 \leq t, z \leq 8$. We present in Figure 6 and Figure 7 portrayals of solution (55) with the assignment functions and parameter values as in Figure 5 but with different values of x and intervals of t and z, where for Figure 6 function $f_2(t, x)$ is doubled. We notice that by fixing y and other involved constants and varying x in dissimilar intervals of t and z, we obtain diverse notable soliton interactions as demonstrated in the Figures.

Now, we depict Jacobi elliptic solution (67) in Figure 8 with parameter values $\alpha = 0.3$, $\beta = 0.5$, $\nu = 5$, $\gamma = -0.2$, $\theta_1 = 90$, $\theta_2 = 50.05$, $\theta_3 = 0.04$, where variables $y = 1.4$, $z = 2$, and $-5 \leq t, x \leq 5$, whereas in Figure 9, same value allocations are utilized but with varying interval of t and x. Finally, we represent elliptic integral function solution (87) in Figure 10, by using dissimilar values $\alpha = -1.1$, $\beta = 1.1$, $\nu = 0.7$, $\gamma = 0.4$, $\mathcal{C} = 70$, $\mathcal{E} = 10$, $k_1 = 1$, $r_1 = 90$, $r_2 = 40.05$, $r_3 = 0.05$, where variables $t = 0.01$, $z = 0.02$, and $-5 \leq x, y \leq 5$. Moreover, we further exhibit the dynamics of (87) in Figure 11 by assigning $\alpha = -1.1$, $\beta = 1.1$, $\nu = 0.7$, $\gamma = 0.4$, $\mathcal{C} = 70$, $\mathcal{E} = 10$, $k_1 = 1$, $r_1 = 90$, $r_2 = 40.05$, $r_3 = 0.05$, with variables $t = 0.01$, $z = 0.02$, and $-10 \leq x, y \leq 10$. It is observed that the elliptic function solutions exhibit various periodic waves at various values of parameters and dissimilar intervals.

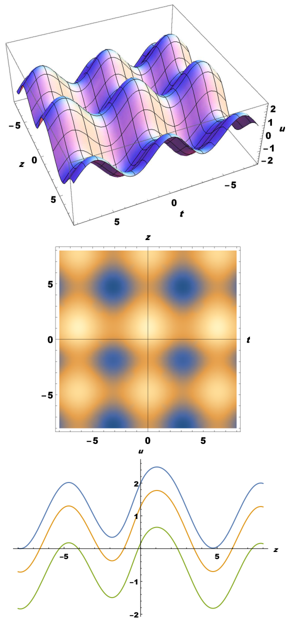

Fig. 1: Wave profile representing algebraic solution (21) at $x = 1$ and $y = 0$.

5 Conservation laws of (7)

We devote this section to secure conservation laws related to the 3D-KPLike (7) via Ibragimov's theorem [63–65]. Using the salient information provided in [64], we have the following theorem:

Theorem 51 *The adjoint equation of 3D-KPLike (7) is expressed as:*

$$\begin{aligned} \mathcal{G}^* \equiv\ & av_{tx} + bv_{ty} + cv_{tz} - dv_{xxxy} + v_{xx}(e - 3u_y) \\ & - 6v_x u_{xy} - 3u_x v_{xy} = 0, \end{aligned} \tag{101}$$

and the formal Lagrangian given as:

$$\begin{aligned} \mathcal{L} = v\mathcal{G} \equiv\ & v(au_{tx} + bu_{ty} + cu_{tz} - du_{xxxy} \\ & + eu_{xx} - 3u_x u_{xy} - 3u_{xx}u_y), \end{aligned} \tag{102}$$

with

$$\begin{aligned} \mathcal{G} =\ & au_{tx} + bu_{ty} + cu_{tz} - du_{xxxy} + eu_{xx} \\ & - 3u_x u_{xy} - 3u_{xx}u_y. \end{aligned} \tag{103}$$

Therefore, using the earlier outlined information, we have the conserved vectors associated with Lie symmetries obtained in (12), as:

$$T_1^t = evu_{xx} - 3vu_x u_{xy} - 3vu_y u_{xx} - dvu_{xxxy}$$

$$+\frac{1}{2}cv_z u_t + \frac{1}{2}bv_y u_t + \frac{1}{2}av_x u_t$$

$$+\frac{1}{2}cvu_{tz}+\frac{1}{2}bvu_{ty}+\frac{1}{2}avu_{tx},$$

$$T_1^x=ev_xu_t-\frac{3}{2}v_yu_xu_t-3u_yv_xu_t-\frac{3}{2}vu_{xy}u_t$$

$$-\frac{3}{4}dv_{xxy}u_t+\frac{1}{2}av_tu_t+\frac{3}{2}vu_xu_{ty}+\frac{1}{4}dv_{xx}u_{ty}$$

$$-evu_{tx}+3vu_yu_{tx}+\frac{1}{2}dv_{xy}u_{tx}-\frac{1}{2}dv_xu_{txy}$$

$$-\frac{1}{4}dv_yu_{txx}+\frac{3}{4}dvu_{txxy}-\frac{1}{2}avu_{tt},$$

$$T_1^z=\frac{1}{2}cu_tv_t-\frac{1}{2}cvu_{tt};$$

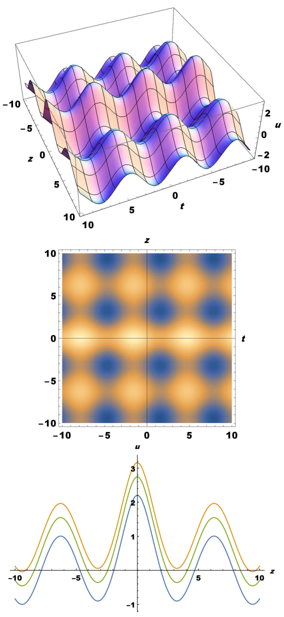

Fig. 2: Wave profile representing algebraic solution (21) at $x=1.2$ and $y=0$.

$$T_1^y=\frac{3}{2}vu_{xx}u_t-\frac{3}{2}u_xv_xu_t-\frac{1}{4}dv_{xxx}u_t+\frac{1}{2}bv_tu_t$$

$$+\frac{3}{2}vu_xu_{tx}+\frac{1}{4}dv_{xx}u_{tx}-\frac{1}{4}dv_xu_{txx}$$

$$+\frac{1}{4}dvu_{txxx}-\frac{1}{2}bvu_{tt},$$

$$T_2^t=\frac{1}{2}au_xv_x-\frac{1}{2}au_{xx}v-\frac{1}{2}bu_{xy}v-\frac{1}{2}cu_{xz}v$$

$$+\frac{1}{2}bu_xv_y+\frac{1}{2}cu_xv_z,$$

$$T_2^x=\frac{1}{2}au_{tx}v+bu_{ty}v+cu_{tz}v-\frac{1}{4}du_{xxxy}v$$

$$-3u_xu_{xy}v+\frac{1}{2}av_tu_x-\frac{3}{4}du_xv_{xxy}$$

$$+\frac{1}{2}du_{xx}v_{xy}+\frac{1}{4}dv_{xx}u_{xy}-\frac{1}{2}dv_xu_{xxy}$$

$$-\frac{1}{4}du_{xxx}v_y+eu_xv_x-\frac{3}{2}u_x^2v_y-3u_xu_yv_x,$$

$$T_2^y=\frac{1}{4}du_{xxxx}v-\frac{1}{2}bu_{tx}v+3u_{xx}u_xv$$

$$+\frac{1}{2}bv_tu_x-\frac{1}{4}du_xv_{xxx}+\frac{1}{4}du_{xx}v_{xx}$$

$$-\frac{1}{4}du_{xxx}v_x-\frac{3}{2}u_x^2v_x,$$

$$T_2^z=\frac{1}{2}cv_tu_x-\frac{1}{-}2cu_{tx}v;$$

$$T_3^t=\frac{1}{2}cv_zu_y+\frac{1}{2}bv_yu_y+\frac{1}{2}av_xu_y\frac{1}{2}cvu_{yz}$$

$$-\frac{1}{2}bvu_{yy}-\frac{1}{2}avu_{xy},$$

$$T_3^x=\frac{3}{2}vu_{xy}u_y-3v_xu_y^2-\frac{3}{2}v_yu_xu_y+ev_xu_y$$

$$-\frac{3}{4}dv_{xxy}u_y+\frac{1}{2}av_tu_y+\frac{3}{2}vu_{yy}u_x-evu_{xy}$$

$$+\frac{1}{2}du_{xy}v_{xy}-\frac{1}{2}dv_xu_{xyy}+\frac{1}{4}du_{yy}v_{xx}$$

$$-\frac{1}{4}dv_yu_{xxy}+\frac{3}{4}dvu_{xxyy}-\frac{1}{2}avu_{ty},$$

$$T_3^y = evu_{xx} - \frac{3}{2}u_y u_x v_x - \frac{1}{4}du_{xxy}v_x - \frac{3}{2}vu_x u_{xy}$$

$$-\frac{3}{2}vu_y u_{xx} + \frac{1}{4}du_{xy}v_{xx} - \frac{1}{4}du_y v_{xxx}$$

$$-\frac{3}{4}dvu_{xxxy} + \frac{1}{2}bu_y v_t + cvu_{tz}$$

$$+\frac{1}{2}bvu_{ty} + avu_{tx},$$

$$T_3^z = \frac{1}{2}cu_y v_t - \frac{1}{2}cvu_{ty},$$

$$T_4^t = \frac{1}{2}cu_z v_z - \frac{1}{2}cvu_{zz} + \frac{1}{2}bu_z v_y$$

$$-\frac{1}{2}bvu_{yz} + \frac{1}{2}au_z v_x - \frac{1}{2}avu_{xz},$$

$$T_4^x = \frac{3}{2}vu_{yz}u_x - \frac{3}{2}u_z v_y u_x + eu_z v_x$$

$$-3u_z u_y v_x - evu_{xz} + 3vu_y u_{xz}$$

$$-\frac{3}{2}vu_z u_{xy} + \frac{1}{2}du_{xz}v_{xy} - \frac{1}{2}dv_x u_{xyz}$$

$$+\frac{1}{4}du_{yz}v_{xx} - \frac{1}{4}dv_y u_{xxz} - \frac{3}{4}du_z v_{xxy}$$

$$+\frac{3}{4}dvu_{xxyz} + \frac{1}{2}au_z v_t - \frac{1}{2}avu_{tz},$$

$$T_4^y = \frac{3}{2}vu_x u_{xz} - \frac{3}{2}u_z u_x v_x - \frac{1}{4}du_{xxz}v_x$$

$$+\frac{3}{2}vu_z u_{xx} + \frac{1}{4}du_{xz}v_{xx} - \frac{1}{4}du_z v_{xxx}$$

$$+\frac{1}{4}dvu_{xxxz} + \frac{1}{2}bu_z v_t - \frac{1}{2}bvu_{tz},$$

$$T_4^z = evu_{xx} - 3vu_x u_{xy} - 3vu_y u_{xx} - dvu_{xxxy}$$

$$+\frac{1}{2}cu_z v_t + \frac{1}{2}cvu_{tz} + bvu_{ty} + avu_{tx};$$

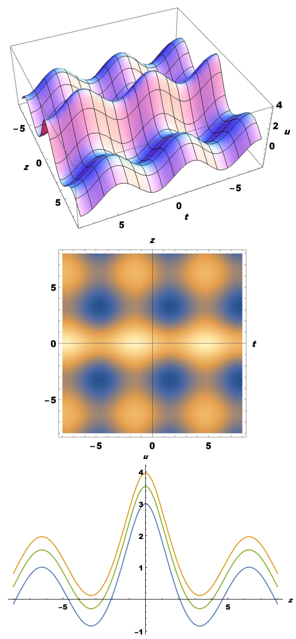

Fig. 3: Wave profile representing algebraic solution (21) at $x = 2$ and $y = 0$.

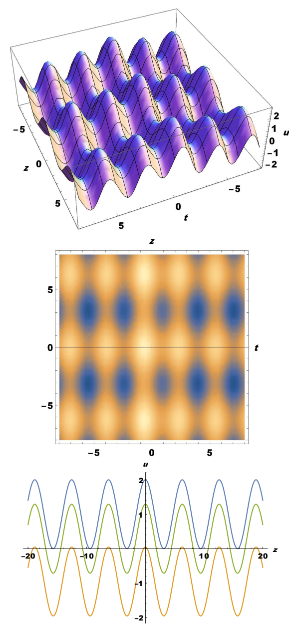

Fig. 4: Wave profile representing algebraic solution (55) at $x = 0$ and $y = 0$.

$$T_5^t = \frac{1}{2}zvu_{xx}a^2 - \frac{1}{2}zu_xv_xa^2 + \frac{1}{2}cvu_xa - \frac{1}{2}czv_zu_xa$$
$$-\frac{1}{2}bzv_yu_xa - \frac{1}{2}czu_zv_xa - \frac{1}{2}bzu_yv_xa + czvu_{xz}a$$
$$+bzvu_{xy}a + \frac{1}{2}ctv_xu_ta + \frac{1}{2}ctvu_{tx}a + \frac{1}{2}c^2vu_z$$
$$-\frac{1}{2}c^2zu_zv_z + \frac{1}{2}c^2zvu_{zz} + \frac{1}{2}bcvu_y - \frac{1}{2}bczv_zu_y$$
$$-\frac{1}{2}bczu_zv_y - \frac{1}{2}b^2zu_yv_y + bczvu_{yz} + \frac{1}{2}b^2zvu_{yy}$$
$$-3ctvu_xu_{xy} + cetvu_{xx} - 3ctvu_yu_{xx}$$
$$-cdtvu_{xxxy} + \frac{1}{2}c^2tv_zu_t + \frac{1}{2}bctv_yu_t + \frac{1}{2}c^2tvu_{tz}$$
$$+\frac{1}{2}bctvu_{ty},$$

$$T_5^x = \frac{3}{2}zv_yu_x^2a - \frac{1}{2}zu_xv_ta^2 - \frac{1}{2}zvu_{tx}a^2$$
$$-ezu_xv_xa + 3zu_yu_xv_xa + 3zvu_xu_{xy}a$$
$$-\frac{1}{2}dzv_{xy}u_{xx}a - \frac{1}{4}dzu_{xy}v_{xx}a + \frac{1}{2}dzv_xu_{xxy}a$$
$$+\frac{3}{4}dzu_xv_{xxy}a + \frac{1}{4}dzv_yu_{xxx}a + \frac{1}{4}dzvu_{xxxy}a$$
$$-\frac{1}{2}cvu_ta - \frac{1}{2}czu_zv_ta - \frac{1}{2}bzu_yv_ta$$
$$+\frac{1}{2}ctu_tv_ta - \frac{1}{2}czvu_{tz}a - \frac{1}{2}bzvu_{ty}a$$
$$-\frac{1}{2}ctvu_{tt}a + \frac{3}{2}czu_zv_yu_x + \frac{3}{2}bzu_yv_yu_x$$
$$-\frac{3}{2}czvu_{yz}u_x - \frac{3}{2}bzvu_{yy}u_x + 3bzu_y^2v_x$$
$$-cezu_zv_x - bezu_yv_x + 3czu_zu_yv_x + cezvu_{xz}$$
$$-3czvu_yu_{xz} + bezvu_{xy} + \frac{3}{2}czvu_zu_{xy}$$
$$-\frac{3}{2}bzvu_yu_{xy} - \frac{1}{2}cdzu_{xz}v_{xy} - \frac{1}{2}bdzu_{xy}v_{xy}$$
$$+\frac{1}{2}cdzv_xu_{xyz} + \frac{1}{2}bdzv_xu_{xyy} - \frac{1}{4}cdzu_{yz}v_{xx}$$
$$-\frac{1}{4}bdzu_{yy}v_{xx} + \frac{1}{4}cdzv_yu_{xxz} + \frac{1}{4}bdzv_yu_{xxy}$$
$$+\frac{3}{4}cdzu_zv_{xxy} + \frac{3}{4}bdzu_yv_{xxy} - \frac{3}{4}cdzvu_{xxyz}$$
$$-\frac{3}{4}bdzvu_{xxyy} - \frac{3}{2}ctv_yu_xu_t + cetv_xu_t$$
$$-3ctu_yv_xu_t - \frac{3}{2}ctvu_{xy}u_t - \frac{3}{4}cdtv_{xxy}u_t$$
$$+\frac{3}{2}ctvu_xu_{ty} + \frac{1}{4}cdtv_{xx}u_{ty} - cetvu_{tx}$$
$$+3ctvu_yu_{tx} + \frac{1}{2}cdtv_{xy}u_{tx} - \frac{1}{2}cdtv_xu_{txy}$$
$$-\frac{1}{4}cdtv_yu_{txx} + \frac{3}{4}cdtvu_{txxy},$$

$$T_5^z = \frac{1}{2}tu_tv_tc^2 - \frac{1}{2}vu_tc^2 - \frac{1}{2}zu_zv_tc^2$$
$$-\frac{1}{2}zvu_{tz}c^2 - \frac{1}{2}tvu_{tt}c^2 + 3zvu_xu_{xy}c$$
$$-ezvu_{xx}c + 3zvu_yu_{xx}c + dzvu_{xxxy}c$$
$$-\frac{1}{2}bzu_yv_tc - \frac{1}{2}azu_xv_tc$$
$$-\frac{1}{2}bzvu_{ty}c - \frac{1}{2}azvu_{tx}c;$$

$$T_6^t = \frac{3}{2}zvu_{yy}b^2 - \frac{3}{2}zu_yv_yb^2 + \frac{1}{2}cevb$$
$$-\frac{3}{2}czv_zu_yb - \frac{1}{2}ceyv_yb + \frac{3}{2}cyu_yv_yb$$

$$+\frac{3}{2}czvu_{yz}b-\frac{3}{2}cyvu_{yy}b-\frac{3}{2}azu_{y}v_{x}b$$

$$+\frac{3}{2}azvu_{xy}b+\frac{3}{2}ctv_{y}u_{t}b+\frac{3}{2}ctvu_{ty}b$$

$$-\frac{1}{2}c^{2}eyv_{z}+\frac{3}{2}c^{2}yv_{z}u_{y}-\frac{3}{2}c^{2}yvu_{yz}$$

$$-\frac{1}{2}aceyv_{x}+\frac{3}{2}acyu_{y}v_{x}-\frac{3}{2}acyvu_{xy}$$

$$-9ctvu_{x}u_{xy}+3cetvu_{xx}-9ctvu_{y}u_{xx}$$

$$-3cdtvu_{xxxy}+\frac{3}{2}c^{2}tv_{z}u_{t}+\frac{3}{2}actv_{x}u_{t}$$

$$+\frac{3}{2}c^{2}tvu_{tz}+\frac{3}{2}actvu_{tx},$$

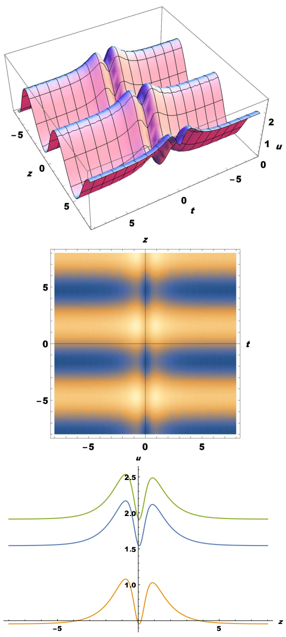

Fig. 5: Wave profile representing algebraic solution (55) at $x=0.1$ and $y=0$.

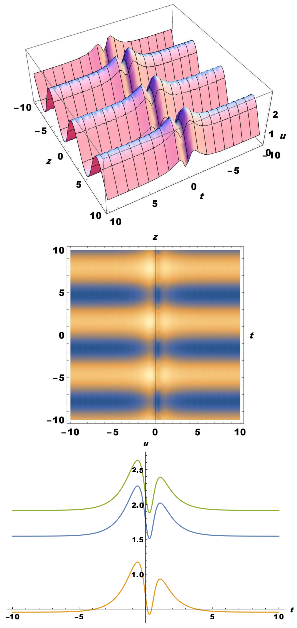

Fig. 6: Wave profile representing algebraic solution (55) at $x=0.5$ and $y=0$.

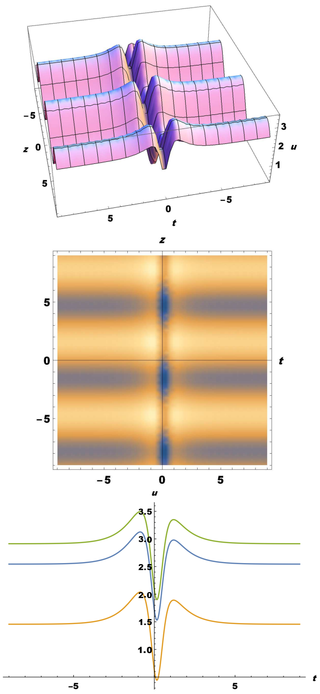

Fig. 7: Wave profile representing algebraic solution (55) at $x=0.3$ and $y=0$.

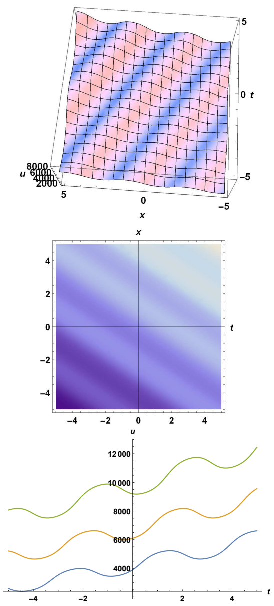

Fig. 8: Wave profile representing elliptic function (67) at $y = 1.4$ and $z = 2$.

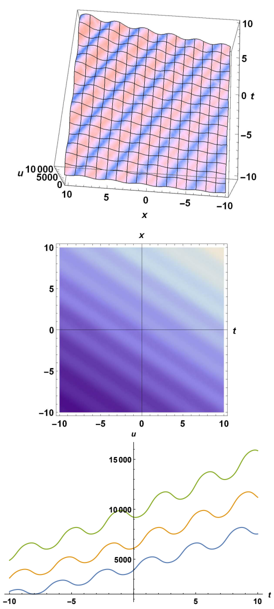

Fig. 9: Wave profile representing elliptic function (67) at $y = 1.4$ and $z = 2$.

$$T_5^y = \frac{3}{2}zu_yu_xv_xb - \frac{1}{2}zu_yv_tb^2 - \frac{1}{2}zvu_{ty}b^2$$

$$+\frac{3}{2}zvu_xu_{xy}b - ezvu_{xx}b + \frac{3}{2}zvu_yu_{xx}b$$

$$-\frac{1}{4}dzu_{xy}v_{xx}b + \frac{1}{4}dzv_xu_{xxy}b + \frac{1}{4}dzu_yv_{xxx}b$$

$$+\frac{3}{4}dzvu_{xxxy}b - \frac{1}{2}cvu_tb - \frac{1}{2}czu_zv_tb - \frac{1}{2}azu_xv_tb$$

$$+\frac{1}{2}ctu_tv_tb - \frac{1}{2}czvu_{tz}b - \frac{1}{2}azvu_{tx}b - \frac{1}{2}ctvu_{tt}b$$

$$+\frac{3}{2}azu_x^2v_x + \frac{3}{2}czu_zu_xv_x - \frac{3}{2}czvu_xu_{xz} - \frac{3}{2}czvu_zu_{xx}$$

$$-3azvu_xu_{xx} - \frac{1}{4}cdzu_{xz}v_{xx} - \frac{1}{4}adzu_{xx}v_{xx}$$

$$+\frac{1}{4}cdzv_xu_{xxz} + \frac{1}{4}adzv_xu_{xxx} + \frac{1}{4}cdzu_zv_{xxx}$$

$$+\frac{1}{4}adzu_xv_{xxx} - \frac{1}{4}cdzvu_{xxxz} - \frac{1}{4}adzvu_{xxxx}$$

$$-\frac{3}{2}ctu_xv_xu_t + \frac{3}{2}ctvu_{xx}u_t - \frac{1}{4}cdtv_{xxx}u_t + \frac{1}{4}cdtvu_{txxx}$$

$$+\frac{3}{2}ctvu_xu_{tx} + \frac{1}{4}cdtv_{xx}u_{tx} - \frac{1}{4}cdtv_xu_{txx},$$

$$T_6^z = \frac{3}{2}yu_yv_tc^2 - \frac{3}{2}vu_tc^2$$

$$-\frac{1}{2}eyv_tc^2 + \frac{3}{2}tu_tv_tc^2$$

$$-\frac{3}{2}yvu_{ty}c^2 - \frac{3}{2}tvu_{tt}c^2$$

$$-\frac{3}{2}bzu_yv_tc + \frac{3}{2}bzvu_{ty}c;$$

$$T_6^x = \frac{3}{2}cyv_yu_xe - cyv_xe^2 - \frac{3}{2}cvu_xe + 6cyu_yv_xe$$

$$-3bzu_yv_xe - \frac{3}{2}cyvu_{xy}e + 3bzvu_{xy}e - \frac{1}{4}cdv_{xx}e$$

$$+\frac{3}{4}cdyv_{xxy}e + 3ctv_xu_te - \frac{1}{2}acyv_te - 3ctvu_{tx}e$$

$$+\frac{9}{2}cvu_yu_x-\frac{9}{2}cyu_yv_yu_x+\frac{9}{2}bzu_yv_yu_x$$

$$+\frac{9}{2}cyvu_{yy}u_x-\frac{9}{2}bzvu_{yy}u_x-9cyu_y^2v_x+9bzu_y^2v_x$$

$$+\frac{9}{2}cyvu_yu_{xy}-\frac{9}{2}bzvu_yu_{xy}-\frac{3}{2}cdv_xu_{xy}$$

$$+\frac{3}{2}cdyu_{xy}v_{xy}-\frac{3}{2}bdzu_{xy}v_{xy}+\frac{3}{2}bdzv_xu_{xyy}$$

$$+\frac{3}{4}cdu_yv_{xx}+\frac{3}{4}cdyu_{yy}v_{xx}-\frac{3}{2}cdyv_xu_{xyy}$$

$$-\frac{3}{4}bdzu_{yy}v_{xx}+\frac{9}{4}cdvu_{xxy}-\frac{3}{4}cdyv_yu_{xxy}$$

$$+\frac{3}{4}bdzv_yu_{xxy}-\frac{9}{4}cdyu_yv_{xxy}+\frac{9}{4}bdzu_yv_{xxy}$$

$$+\frac{9}{4}cdyvu_{xxyy}-\frac{9}{4}bdzvu_{xxyy}-\frac{3}{2}acvu_t-\frac{9}{2}ctv_yu_xu_t$$

$$-9ctu_yv_xu_t-\frac{9}{2}ctvu_{xy}u_t-\frac{9}{4}cdtv_{xxy}u_t+\frac{3}{2}acyu_yv_t$$

$$-\frac{3}{2}abzu_yv_t+\frac{3}{2}actu_tv_t-\frac{3}{2}acyvu_{ty}+\frac{3}{2}abzvu_{ty}$$

$$+\frac{9}{2}ctvu_xu_{ty}+\frac{3}{4}cdtv_{xx}u_{ty}+9ctvu_yu_{tx}+\frac{3}{2}cdtv_{xy}u_{tx}$$

$$-\frac{3}{2}cdtv_xu_{txy}-\frac{3}{4}cdtv_yu_{txx}+\frac{9}{4}cdtvu_{txxy}-\frac{3}{2}actvu_{tt},$$

$$T_6^y=\frac{9}{2}zvu_xu_{xy}b-\frac{3}{2}zu_yv_tb^2-\frac{3}{2}zvu_{ty}b^2$$

$$+\frac{9}{2}zu_yu_xv_xb-3ezvu_{xx}b+\frac{9}{2}zvu_yu_{xx}b$$

$$-\frac{3}{4}dzu_{xy}v_{xx}b+\frac{3}{4}dzv_xu_{xxy}b+\frac{3}{4}dzu_yv_{xxx}b$$

$$+\frac{9}{4}dzvu_{xxxy}b-\frac{3}{2}cvu_tb-\frac{1}{2}ceyv_tb$$

$$+\frac{3}{2}cyu_yv_tb+\frac{3}{2}ctu_tv_tb-3czvu_{tz}b$$

$$+\frac{3}{2}cyvu_{ty}b-3azvu_{tx}b-\frac{3}{2}ctvu_{tt}b+\frac{3}{2}ceyu_xv_x$$

$$-\frac{9}{2}cyu_yu_xv_x-\frac{9}{2}cyvu_xu_{xy}+\frac{3}{2}ceyvu_{xx}$$

$$-\frac{9}{2}cyvu_yu_{xx}+\frac{3}{4}cdyu_{xy}v_{xx}-\frac{3}{4}cdyv_xu_{xxy}$$

$$+\frac{1}{4}cdeyv_{xxx}-\frac{3}{4}cdyu_yv_{xxx}-\frac{9}{4}cdyvu_{xxxy}$$

$$-\frac{9}{2}cyvu_yu_{xx}+\frac{3}{4}cdyu_{xy}v_{xx}-\frac{3}{4}cdyv_xu_{xxy}$$

$$-\frac{9}{2}ctu_xv_xu_t+\frac{9}{2}ctvu_{xx}u_t-\frac{3}{4}cdtv_{xxx}u_t$$

$$+3c^2yvu_{tz}+3acyvu_{tx}+\frac{9}{2}ctvu_xu_{tx}$$

$$+\frac{3}{4}cdtv_{xx}u_{tx}-\frac{3}{4}cdtv_xu_{txx}+\frac{3}{4}cdtvu_{txxx},$$

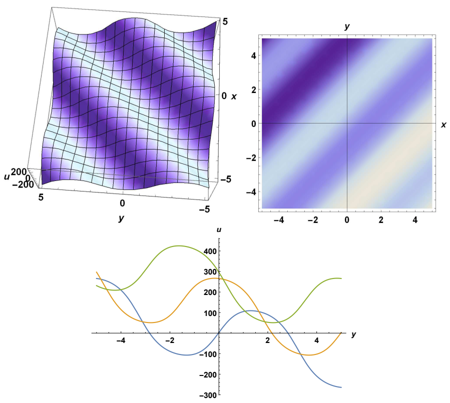

Fig. 10: Wave profile representing elliptic function (87) at $t=0.01$ and $z=0.02$.

$$T_7^t=\frac{1}{2}zvu_{xx}a^2-\frac{1}{2}zu_xv_xa^2-\frac{1}{2}cvu_xa$$

$$-\frac{1}{2}czv_zu_xa-\frac{1}{2}bzv_yu_xa+\frac{1}{2}cuv_xa$$

$$-\frac{1}{2}cyu_yv_xa+\frac{1}{2}bzu_yv_xa+\frac{1}{2}cxu_xv_xa$$

$$+\frac{1}{2}czvu_{xz}a+\frac{1}{2}cyvu_{xy}a-\frac{1}{2}cxvu_{xx}a$$

$$+ctv_xu_ta+ctvu_{tx}a-\frac{1}{2}c^2vu_z$$

$$+\frac{1}{2}c^2uv_z-\frac{1}{2}bcvu_y-\frac{1}{2}c^2yv_zu_y$$

$$+\frac{1}{2}bczv_zu_y+\frac{1}{2}bcuv_y-\frac{1}{2}bcyu_yv_y$$

$$+\frac{1}{2}b^2zu_yv_y+\frac{1}{2}c^2yvu_{yz}-\frac{1}{2}bczvu_{yz}$$

$$+\frac{1}{2}bcyvu_{yy}-\frac{1}{2}b^2zvu_{yy}+\frac{1}{2}c^2xv_zu_x$$

$$+\frac{1}{2}bcxv_yu_x-\frac{1}{2}c^2xvu_{xz}-\frac{1}{2}bcxvu_{xy}$$

$$-6ctvu_xu_{xy}+2cetvu_{xx}-6ctvu_yu_{xx}$$

$$-2cdtvu_{xxxy}+c^2tv_zu_t+bctv_yu_t$$

$$+c^2tvu_{tz}+bctvu_{ty},$$

$$+ctu_tv_ta-czvu_{tz}a+\frac{1}{2}cyvu_{ty}a$$

$$-\frac{3}{2}bzvu_{ty}a+\frac{1}{2}cxvu_{tx}a-ctvu_{tt}a$$

$$-\frac{3}{2}cxv_yu_x^2-2cevu_x+6cvu_yu_x$$

$$-\frac{3}{2}cuv_yu_x+\frac{3}{2}cyu_yv_yu_x-\frac{3}{2}bzu_yv_yu_x$$

$$-\frac{3}{2}cyvu_{yy}u_x+\frac{3}{2}bzvu_{yy}u_x+3cyu_y^2v_x$$

$$-3bzu_y^2v_x+ceuv_x-ceyu_yv_x$$

$$+bezu_yv_x-3cuu_yv_x+cexu_xv_x$$

$$-3cxu_yu_xv_x+ceyvu_{xy}-bezvu_{xy}$$

$$-\frac{3}{2}cuvu_{xy}-\frac{3}{2}cyvu_yu_{xy}+\frac{3}{2}bzvu_yu_{xy}$$

$$-3cxvu_xu_{xy}-\frac{1}{2}cdv_xu_{xy}+cdu_xv_{xy}$$

$$-\frac{1}{2}cdyu_{xy}v_{xy}+\frac{1}{2}bdzu_{xy}v_{xy}+\frac{1}{2}cdyv_xu_{xyy}$$

$$-\frac{1}{2}bdzv_xu_{xyy}-\frac{3}{4}cdv_yu_{xx}+\frac{1}{2}cdxv_{xy}u_{xx}$$

$$-\frac{1}{4}cdyu_{yy}v_{xx}+\frac{1}{4}bdzu_{yy}v_{xx}+\frac{1}{4}cdxu_{xy}v_{xx}$$

$$+\frac{3}{2}cdvu_{xxy}+\frac{1}{4}cdyv_yu_{xxy}-\frac{1}{4}bdzv_yu_{xxy}$$

$$-\frac{1}{2}cdxv_xu_{xxy}-\frac{3}{4}cduv_{xxy}+\frac{3}{4}cdyu_yv_{xxy}$$

$$-\frac{3}{4}bdzu_yv_{xxy}-\frac{3}{4}cdxu_xv_{xxy}-\frac{3}{4}cdyvu_{xxyy}$$

$$+\frac{3}{4}bdzvu_{xxyy}-\frac{1}{4}cdxv_yu_{xxx}-\frac{1}{4}cdxvu_{xxxy}$$

$$T_7^x=\frac{3}{2}zv_yu_x^2a-\frac{1}{2}zu_xv_ta^2-\frac{1}{2}zvu_{tx}a^2$$

$$-ezu_xv_xa+3zu_yu_xv_xa$$

$$+3zvu_xu_{xy}a-\frac{1}{2}dzv_{xy}u_{xx}a-\frac{1}{4}dzu_{xy}v_{xx}a$$

$$+\frac{1}{2}dzv_xu_{xxy}a+\frac{3}{4}dzu_xv_{xxy}a+\frac{1}{4}dzv_yu_{xxx}a$$

$$+\frac{1}{4}dzvu_{xxxy}a-\frac{3}{2}cvu_ta+\frac{1}{2}cuv_ta$$

$$-\frac{1}{2}cyu_yv_ta+\frac{1}{2}bzu_yv_ta+\frac{1}{2}cxu_xv_ta$$

$$-3ctv_yu_xu_t + 2cetv_xu_t - 6ctu_yv_xu_t$$

$$-3ctvu_{xy}u_t - \frac{3}{2}cdtv_{xxy}u_t + c^2xvu_{tz}$$

$$+bcxvu_{ty} + 3ctvu_xu_{ty} + \frac{1}{2}cdtv_{xx}u_{ty}$$

$$-2cetvu_{tx} + 6ctvu_yu_{tx} + cdtv_{xy}u_{tx}$$

$$-cdtv_xu_{txy} - \frac{1}{2}cdtv_yu_{txx} + \frac{3}{2}cdtvu_{txxy},$$

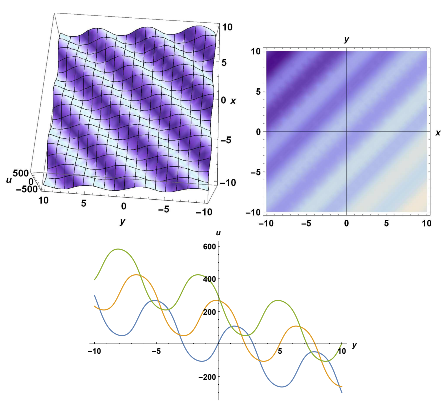

Fig. 11: Wave profile representing elliptic function (87) at $t = 0.01$ and $z = 0.02$.

$$\begin{aligned}
T_7^y = &\frac{1}{2}zu_yv_tb^2 + \frac{1}{2}zvu_{ty}b^2 - \frac{3}{2}zu_yu_xv_xb \\
&- \frac{3}{2}zvu_xu_{xy}b + ezvu_{xx}b - \frac{3}{2}zvu_yu_{xx}b \\
&+ \frac{1}{4}dzu_{xy}v_{xx}b - \frac{1}{4}dzv_xu_{xxy}b - \frac{1}{4}dzu_yv_{xxx}b \\
&- \frac{3}{4}dzvu_{xxxy}b - \frac{3}{2}cvu_tb + \frac{1}{2}cuv_tb \\
&- \frac{1}{2}cyu_yv_tb + \frac{1}{2}cxu_xv_tb - \frac{1}{2}azu_xv_tb \\
&+ ctu_tv_tb + czvu_{tz}b - \frac{1}{2}cyvu_{ty}b \\
&- \frac{1}{2}cxvu_{tx}b + \frac{3}{2}azvu_{tx}b - ctvu_{tt}b \\
&+ 3cvu_x^2 - \frac{3}{2}cxu_x^2v_x + \frac{3}{2}azu_x^2v_x \\
&- \frac{3}{2}cuu_xv_x + \frac{3}{2}cyu_yu_xv_x + \frac{3}{2}cyvu_xu_{xy} \\
&- ceyvu_{xx} + \frac{3}{2}cuvu_{xx} + \frac{3}{2}cyvu_yu_{xx} \\
&+ 3cxvu_xu_{xx} - 3azvu_xu_{xx} - \frac{3}{4}cdv_xu_{xx} \\
&+ \frac{1}{2}cdu_xv_{xx} - \frac{1}{4}cdyu_{xy}v_{xx} + \frac{1}{4}cdxu_{xx}v_{xx} \\
&- \frac{1}{4}adzu_{xx}v_{xx} + \frac{1}{4}cdyv_xu_{xxy} + cdvu_{xxx} \\
&- \frac{1}{4}cdxv_xu_{xxx} + \frac{1}{4}adzv_xu_{xxx} - \frac{1}{4}cduv_{xxx} \\
&+ \frac{1}{4}cdyu_yv_{xxx} - \frac{1}{4}cdxu_xv_{xxx} + \frac{1}{4}adzu_xv_{xxx} \\
&+ \frac{3}{4}cdyvu_{xxxy} + \frac{1}{4}cdxvu_{xxxx} - \frac{1}{4}adzvu_{xxxx} \\
&- 3ctu_xv_xu_t + 3ctvu_{xx}u_t - \frac{1}{2}cdtv_{xxx}u_t \\
&- c^2yvu_{tz} - acyvu_{tx} + 3ctvu_xu_{tx} \\
&+ \frac{1}{2}cdtv_{xx}u_{tx} - \frac{1}{2}cdtv_xu_{txx} + \frac{1}{2}cdtvu_{txxx},
\end{aligned}$$

$$\begin{aligned}
T_7^z = &\frac{1}{2}uv_tc^2 - \frac{3}{2}vu_tc^2 - \frac{1}{2}yu_yv_tc^2 \\
&+ \frac{1}{2}xu_xv_tc^2 + tu_tv_tc^2 + \frac{1}{2}yvu_{ty}c^2 \\
&- \frac{1}{2}xvu_{tx}c^2 - tvu_{tt}c^2 + \frac{1}{2}bzu_yv_tc \\
&- \frac{1}{2}azu_xv_tc - \frac{1}{2}bzvu_{ty}c + \frac{1}{2}azvu_{tx}c;
\end{aligned}$$

$$T_G^t = -\frac{1}{2}aG(t)v_x - \frac{1}{2}bG(t)v_y - \frac{1}{2}cG(t)v_z,$$

$$T_G^x = \frac{3}{2}G(t)u_{xy}v + \frac{1}{2}aG'(t)v - \frac{1}{2}aG(t)v_t + \frac{3}{4}dG(t)v_{xxy} - eG(t)v_x + \frac{3}{2}G(t)u_x v_y + 3G(t)u_y v_x,$$

$$T_G^y = \frac{1}{2}bG'(t)v - \frac{3}{2}G(t)u_{xx}v - \frac{1}{2}bG(t)v_t + \frac{1}{4}dG(t)v_{xxx} + \frac{3}{2}G(t)u_x v_x,$$

$$T_G^z = \frac{1}{2}cG'(t)v - \frac{1}{2}cG(t)v_t,$$

$$T_F^t = \frac{1}{2}cF'(z)v - \frac{1}{2}aF(z)v_x - \frac{1}{2}bF(z)v_y - \frac{1}{2}cF(z)v_z,$$

$$T_F^x = \frac{3}{2}F(z)u_{xy}v - \frac{1}{2}aF(z)v_t + \frac{3}{4}dF(z)v_{xxy} - eF(z)v_x + \frac{3}{2}F(z)u_x v_y + 3F(z)u_y v_x,$$

$$T_F^y = \frac{3}{2}F(z)u_x v_x - \frac{3}{2}F(z)u_{xx}v - \frac{1}{2}bF(z)v_t + \frac{1}{4}dF(z)v_{xxx}$$

$$T_F^z = -\frac{1}{2}cF(z)v_t.$$

Remark 51 *We observe that by invoking Ibragimov's theorem, we obtained nine conservation laws of the 3D-KPLike (7) which contain new variable v and arbitrary functions $F(z)$ and $G(t)$. These conservation laws are not the same, and there is the availability of functions that attest to the fact that a nonlinear differential equation can possess infinitely many conservation laws. In addition, some of them represent conserved quantities such as energy and momentum.*

6 Conclusion

In this work, an investigation of a three-dimensional fourth-order nonlinear Kadomtsev-Petviashvili-like equation (7) was carried out. There are numerous disciplines in which this equation can be used. We performed symmetry analysis on the model and obtained point symmetries given in (12). In order to execute symmetry reductions and create exact solutions, we first reduced the equation using the obtained Lie point symmetries. As a result, diverse group-invariant solutions were obtained. Besides, using the direct integration technique along with Kudryashov's approach, more solutions to (7) were found. Solutions of interest secured include logarithmic, exponential, and hyperbolic functions, as well as elliptic integral functions. In addition, various algebraic function solutions of interest were found. Moreover, the solutions secured were depicted with various diagrammatic representations by making an adequate choice of parameter values. Lastly, Ibragimov's approach was utilized to construct conservation laws for this model. These conservation laws represented conserved quantities that included energy and momentum.

Acknowledgment

The authors thank the North-West University, Mafikeng campus for its continued support.

References

[1] X.X. Du, B. Tian, Q.X. Qu, Y.Q. Yuan, and X.H. Zhao, Lie group analysis, solitons, self-adjointness and conservation laws of the modified Zakharov-Kuznetsov equation in an electron-positron-ion magnetoplasma, *Chaos Solitons Fract.*, **134**, 109709, 2020.

[2] C.R. Zhang, B. Tian, Q.X. Qu, L. Liu, and H.Y. Tian, Vector bright solitons and their interactions of the couple Fokas-Lenells system in a birefringent optical fiber, *Z. Angew. Math. Phys.*, **71**, 1-19, 2020.

[3] X.Y. Gao, Y.J. Guo, and W.R. Shan, Water-wave symbolic computation for the Earth, Enceladus and Titan: The higher-order Boussinesq-Burgers system, auto-and non-auto-Bäcklund transformations, *Appl. Math. Lett.*, **104**, 106170, 2020.

[4] U. Al Khawaja, H. Eleuch, and H. Bahloulid, Analytical analysis of soliton propagation in microcavity wires, *Results Phys.*, **12**, 471–474, 2019.

[5] A.M. Wazwaz, Exact soliton and kink solutions for new (3+1)-dimensional nonlinear modified equations of wave propagation, *Open Eng.*, **7**, 169-174, 2017.

[6] O.D. Adeyemo, and C.M. Khalique, Lie group theory, stability analysis with dispersion property, new soliton solutions and conserved quantities of 3D generalized nonlinear wave equation in liquid containing gas bubbles with applications in mechanics of fluids, biomedical sciences and cell biology, *Commun. Nonlinear Sci. Numer. Simul.*, **123**, 107261, 2023.

[7] M.J. Ablowitz, and P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, UK, 1991.

[8] O.D. Adeyemo, Applications of cnoidal and snoidal wave solutions via an optimal system of subalgebras for a generalized extended (2+1)-D quantum Zakharov-Kuznetsov equation with power-law nonlinearity in oceanography

and ocean engineering, *J. Ocean Eng. Sci.*, (2022), https://doi.org/10.1016/j.joes.2022.04.012,
[9] M.J. Mardanov, Y.A. Sharifov, Y.S. Gasimov, and C. Cattani, Non-linear first-order differential boundary problems with multipoint and integral conditions, *Fractal Fract.*, **5**(1), 15, 2021.
[10] H. Marghny Mohammed, Alaa Abdel-Razeq, DNA-based steganography using genetic algorithm, *Inf. Sci. Lett.* **9** (2020) 205-210: doi:10.18576/isl/090307
[11] H.M. Baskonus, and W. Gao, Investigation of optical solitons to the nonlinear complex Kundu-Eckhaus and Zakharov-Kuznetsov-Benjamin-Bona-Mahony equations in conformable, *Opt. Quantum Electron*, **54**, 388, 2022.
[12] A.A. Mahmud, T. Tanriverdi, K.A. Muhamad, Exact traveling wave solutions for (2+1)-dimensional Konopelchenko-Dubrovsky equation by using the hyperbolic trigonometric functions methods, *International Journal of Mathematics and Computer in Engineering*, **1**, 11-24, 2023.
[13] B. Gasmi, A. Ciancio, A. Moussa, L. Alhakim, and Y. Mati, New analytical solutions and modulation instability analysis for the nonlinear (1+1)-dimensional Phi-four model, *International Journal of Mathematics and Computer in Engineering*, **1**, 79-90, 2023.
[14] A.R. Adem, C.M. Khalique, and M, Molati, Group classification, symmetry reductions and exact solutions of a Generalized Korteweg-de Vries-Burgers Equation, *Appl. Math. Inf. Sci.*, **9**, 501-506, 2015.
[15] M. Saleh A. A. Hassan, and A.A. Altwaty, Optical Solitons of The Extended Gerdjikov-Ivanov Equation in DWDM System by Extended Simplest Equation Method, *Appl. Math. Inf. Sci.*, **14**(5), 901-907, 2020.
[16] O.D. Adeyemo, and C.M. Khalique, Symmetry solutions and conserved quantities of an extended (1+3)-dimensional Kadomtsev-Petviashvili-like Equation, *Appl. Math. Inf. Sci.*, **15**, 649-660, 2021.
[17] L. Zhang, and C.M. Khalique, Classification and bifurcation of a class of second-order ODEs and its application to nonlinear PDEs, *Discrete and Continuous dynamical systems Series S*, **11**, 777-790,2018.
[18] J. Weiss, M. Tabor, and G. Carnevale, The Painlevé property and a partial differential equations with an essential singularity, *Phys. Lett. A*, **109**, 205-208, 1985.
[19] C. Chun, and R. Sakthivel, Homotopy perturbation technique for solving two point boundary value problems-comparison with other methods, *Comput. Phys. Commun.*, **181**, 1021-1024, 2010.
[20] A.M. Wazwaz, Traveling wave solution to (2+1)-dimensional nonlinear evolution equations, *J. Nat. Sci. Math.*, **1**, 1-13, 2007.
[21] M.T. Darvishi, and M. Najafi, A modification of extended homoclinic test approach to solve the (3+1)-dimensional potential-YTSF equation, *Chin. Phys. Lett.*, **28**, 040202, 2011.
[22] A.H. Salas, and C.A. Gomez, Application of the Cole-Hopf transformation for finding exact solutions to several forms of the seventh-order KdV equation, *Math. Probl. Eng.*, **2010**, 194329, 2010.
[23] A.M. Wazwaz, Partial Differential Equations: Methods and Applications, CRC Press, Boca Raton, Florida, USA, 2002.
[24] C.H. Gu, Soliton Theory and Its Application, Zhejiang Science and Technology Press, Zhejiang, China, 1990.
[25] L.V. Ovsiannikov, Group Analysis of Differential Equations, Academic Press, New York, USA, 1982.
[26] P.J. Olver, Applications of Lie Groups to Differential Equations, second ed., Springer-Verlag, Berlin, Germany, 1993.
[27] Y. Zhou, M. Wang, and Y. Wang, Periodic wave solutions to a coupled KdV equations with variable coefficients, *Phys. Lett. A*, **308**, 31-36, 2003.
[28] X. Zeng, and D.S. Wang, A generalized extended rational expansion method and its application to (1+1)-dimensional dispersive long wave equation, *Appl. Math. Comput.*, **212**, 296-304, 2009.
[29] A.M. Wazwaz, Partial Differential Equations and Solitary Waves Theory, Springer-Verlag, Berlin 2009.
[30] R. Hirota, The Direct Method in Soliton Theory, Cambridge University Press, Cambridge, UK, 2004.
[31] V.B. Matveev, and M.A. Salle, Darboux Transformations and Solitons, Springer, New York, USA, 1991.
[32] A.M. Wazwaz, The tanh method for generalized forms of nonlinear heat conduction and Burgers-Fisher equations, *Appl. Math. Comput.*, **169**, 321-338, 2005.
[33] M. Wang, X. Li, and J. Zhang, The $(G'/G)-$expansion method and travelling wave solutions for linear evolution equations in mathematical physics, *Phys. Lett. A*, **24**, 1257-1268, 2005.
[34] Al-Qrinawi, M. S., El-Agez, T. M., Abdel-Latif, M. S., Taya, S. A., (20121) Capacitance-voltage measurements of hetero-layer OLEDs treated by an electric field and thermal annealing, Int. J. Thin Film Sci. Tech. Vol. 10, No. 3 pp. 217-226: http://dx.doi.org/10.18576/ijtfst/100311
[35] M.S. Osman, One-soliton shaping and inelastic collision between double solitons in the fifth-order variable-coefficient Sawada-Kotera equation, *Nonlinear Dynam.*, **96**, 1491-1496, 2019.
[36] J.H. He, and X.H. Wu, Exp-function method for nonlinear wave equations, *Chaos Solitons Fract.*, **30**, 700-708, 2006.
[37] L. Li, C. Duan, and F. Yu, An improved Hirota bilinear method and new application for a nonlocal integrable complex modified Korteweg-de Vries (mKdV) equation, *Phys. Lett. A*, **383**, 1578-1582, 2019.
[38] J. Yu, D. Wang, and Y. Sun, S, Wu, Modified method of simplest equation for obtaining exact solutions of the Zakharov-Kuznetsov equation, the modified Zakharov-Kuznetsov equation, and their generalized forms, *Nonlinear Dyn.*, **85**, 2449-2465, 2016.
[39] Elhadary, A. A., El-Zein, A., Talaat, M., El-Aragi, G., El-Amawy, A., (2021) Studying The Effect of The Dielectric Barrier Discharge Non- thermal Plasma on Colon Cancer Cell line, Int. J. Thin Film Sci. Tech. 10 (2021), pp. 161-168
[40] N.A. Kudryashov, Simplest equation method to look for exact solutions of nonlinear differential equations, *Chaos Solitons Fract.*, **24**, 1217-1231, 2005
[41] N.A. Kudryashov, and N.B. Loguinova, Extended simplest equation method for nonlinear differential equations, *Appl. Math. Comput.*, **205**, 396-402, 2008.
[42] M. Date, M. Jimbo, M. Kashiwara, and T. Miwa, Operator apporach of the Kadomtsev-Petviashvili equation - Transformation groups for soliton equations III, *JPSJ.*, **50**, 3806-3812, 1981.
[43] Shapaan, M., (2016) DC Conductivity Thermal Stability and Crystallization Kinetics of the Semiconducting 30P2O5

(50-x)V2O5 xB2O3 20Fe2O3 Oxide Glasses. Int. J. of Thin Film Sci. Tech. Vol. 5, No. 3, pp. 143-153 http://dx.doi.org/10.18576/ijtfst/050301
[44] W.X. Ma, and E. Fan, Linear superposition principle applying to Hirota bilinear equations, *Comput. Math. Appl.*, **61**, 950-959, 2011.
[45] A.M. Wazwaz, Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation, *Commun. Nonlinear Sci. Numer. Simul.*, **17**, 491-495, 2012.
[46] W.X. Ma, Lump solutions to the Kadomtsev-Petviashvili equation, *Phys., Lett., A*, **379**, 1975-1978, 2015.
[47] R. Uthayakumar, A. Gowrisankar, Generalized Fractal Dimensions in Image Thresholding Technique *Inf. Sci. Lett.*, **3**, 3, 125-134, 2014.
[48] I. Amer Basim Shaalan, Nadir Fadhil Habubi, Sami Salman Chiad, Ziad Abdulahad Toma, New Design of Hairpin-Koch Fractal Filter for Suppression of Spurious Band *Int. J. Thin Film Sci. Tech.*, **2**, 3, 217-221, 2013.
[49] A.M. Wazwaz, Completely integrable coupled KdV and coupled KP systems, *Commun. Nonlinear Sci. Numer. Simul.*, **15**, 2828-2835, 2010.
[50] M. Kashiwara, and T. Miwa, The τ function of the Kadomtsev-Petviashvili equation, *Proc. Japan Acad.*, **57**, 342-347, 1981.
[51] A. Peckan, The Hirota Direct Method (Master thesis), Bilkent University, 2005.
[52] A.M. Wazawaz, Extended KP equations and extended system of KP equations: multiple-soliton solutions, *Can. J. Phys.*, **89**, 739-743, 2011.
[53] S. Manukure, Y. Zhou, and W.X. Ma, Lump solutions to a (2+1)-dimensional extended KP equation, *Comput. Math. with Appl.*, **75**, 2414-2419, 2018.
[54] X. Lü, W.X. Ma, Y. Zhou, and C.M. Khalique, Rational solutions to an extended Kadomtsev-Petviashvili-like equation with symbolic computation, *Comput. Math. Appl.*, **71**, (2016) 1560-1567.
[55] O.D. Adeyemo, and C.M. Khalique, Symmetry solutions and conserved quantities of an extended (1+3)-dimensional Kadomtsev-Petviashvili-like equation, *Appl. Math. Inf. Sci.*, **15**, (2021) 649-660.
[56] O.D. Adeyemo, and C.M. Khalique, Dynamics of soliton waves of group-invariant solutions through optimal system of an extended KP-like equation in higher dimensions with applications in medical sciences and mathematical physics, *J. Geom. Phys.*, **177**, 104502, 2022.
[57] M.T. Darvishi, M. Najafi, S. Arbabi, and L. Kavitha, Exact propagating multi-anti-kink soliton solutions of a (3+1)-dimensional B-type Kadomtsev-Petviashvili equation, *Nonlinear Dyn.*, **83**, (2016) 1453-1462.
[58] D.F. Lawden, Elliptic Functions and Applications, Vol 8, Springer-Verlag, New-York, NY, USA, 1989.
[59] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 7th edn. Academic Press, New York, (2007).
[60] J. Billingham, A.C. King, Wave motion, Cambridge University Press, Cambridge, 2000.
[61] N.A. Kudryashov, On new travelling wave solutions of the KdV and the KdV-Burgers equations, Commun. *Nonlinear Sci. Numer. Simulat.*, **14**, 1891-1900, 2009.
[62] N.A. Kudryashov, One method for finding exact solutions of nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simulat., 17 (2012) 2248–2253
[63] N.H. Ibragimov, Integrating factors, adjoint equations and Lagrangians, J. Math. Anal. Appl. 318 (2006) 742–757.
[64] N.H. Ibragimov, A new conservation theorem, J. Math. Anal. Appl., 333 (2007) 311–328.
[65] C.M. Khalique, O.D. Adeyemo, Closed-form solutions and conserved vectors of a generalized (3+1)-dimensional breaking soliton equation of engineering and nonlinear science, Mathematics, 8 (2020) 1692.