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A Hybrid Technique for Space-Time Fractional Parabolic Differential Equations

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Abstract: The main objective of the present article is to present an accurate efficient technique for approximating the solutions of the diffusion equation of fractional space-time Lévy-Feller type. The suggested method depends on spectral collocation algorithm and implicit non standard finite difference method. The fractional space-time Lévy-Feller diffusion equations are acquired by updating the classical diffusion equations such that the time derivative of the first order will be the fractional Caputo operator and the second-order space derivative will modify to be the Riesz-Feller derivative. The utilized spectral method uses the well known Legendre orthogonal polynomials and the Gauss-Lobatto Chebyshev collocation points. The method depends basically on conversion these kinds of fractional differential equations into a system of algebraic equations which may be solved easily using appropriate technique. The numerical outcomes are presented in the form of tables and graphs to emphasize the reliability of the introduced technique to approximate the solutions of the fractional space-time-Lévy-Feller diffusion equations.

Keywords: Fractional space-time Lévy-Feller diffusion equations, Caputo fractional derivative, Riesz-Feller fractional derivative, Legendre polynomials, collocation method, non standard finite difference method.

1 Introduction, Motivation and Preliminaries

The field of fractional calculus has obtained an importance in recent years. Although, the long history of calculus with fractional integrals and derivatives in mathematics, a enormous number of real life applications of this framework has appeared fundamentally through the latest decades. A lot of phenomena in viscoelasticity, fluid mechanics, system control, physics, chemistry, hydrology, finance, and many other fields of science can be introduced more accurately by fractional models ([1], [2], [3], [4], [5], [6], [7]) and [8]. These fractional models, described using fractional derivatives and fractional integrals, which are more appropriate than the traditional integer-order derivatives and integrals to describe the memorial and hereditary properties of several processes and materials [9], [10] and [11]. In last years, the fractional differential equations have been used a lot and with increasing attention, depending on their applications in many fields of engineering and science[12], [13].

Fractional derivatives in physics are used to describe anomalous diffusion equations, where particles moved following Lévy stable motion [14], their spread is not the classical Brownian motion [5]. Some of anomalous diffusion applications are: reaction-diffusion equations, kinetic equations of the diffusion, Fokker-Planck equations, and diffusion advection equations. These equations provided by fractional derivatives are taken into consideration considered to be very helpful models to describe the transportation of many systems like the dynamical systems [15]. In many papers, the authors analyzed the behavior of time and/or space fractional derivatives anomalous diffusion equations (see,[14], [16], [17], [18], [19], [20]).

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Spectral methods are highly accurate numerical techniques when used to find the the numerical approximation solutions of linear and nonlinear fractional differential equations. These approaches are used in scientific computing and applied mathematics (see, e.g. [21], [22], [23]). The approximation solution is expressed as a finite linear combination of a chosen set of orthogonal basis functions. There are three kinds of spectral methods: collocation, Galerkin, and Tau. The Tau spectral method is similar to the Galerkin spectral method, except the basis function does not need to satisfy boundary conditions. The collocation spectral method usually uses a set of grid points called collocation points, so it is similar to a finite difference method. Among the three famous types of spectral technique, the collocation spectral method has been increasingly common to approximate the solution of differential equations which have derivatives with respect to the time, so we will focus on collocation methods in this work. It is important to choose the collocation points to guarantee the efficiency and convergence of the spectral collocation technique ([22] [24], [25]).

It is well known that the analysis of the spectral technique with Legendre basis is much simpler than the analysis of the spectral technique with Chebyshev basis, because in the L_2 space the Legendre polynomials are reciprocally orthogonal in terms of the L_2 inner product. The main disadvantage of the Legendre collocation method is that the quadrature points of the Gauss-kind can not be found in explicit form. Furthermore, the values of these polynomials of higher order is not rigorous because there is an error, thi serror resulted from the rounding. Moreover, we can take the strength of the Legendre and Chebyshev polynomials by establishing what was named spectral Chebyshev-Legendre (SCL) procedure. This technique hires the zeros of Chebyshev polynomials and avoiding the error of the round off related by finding the Legendre's net nodes [26].

The fundamental solution of the following standard diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = d(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + s(x,t), \quad t \geq 0, \quad x \in \mathbb{R}, \tag{1}$$

is the function of the probability density with the variance is proportional to time for the normal distribution. Replacing the first order time derivative by the Caputo fractional operator and the space derivative of second-order by the Riesz-Feller operator of order α and skewness θ, ($|\theta| \leq min\{\alpha, 2-\alpha\}$), ([18], [19], [27], [28]), then the classical diffusion model will be the time-space fractional diffusion model:

$${}_0^cD_t^\beta u(x,t) = d(x,t)D_\theta^\alpha u(x,t) + s(x,t), \quad t \geq 0, \quad x \in \mathbb{R}. \tag{2}$$

equipped with the essential solution to be as the Lévy probability distribution.

Our aim in this articl is concerned with appling of the spectral collocation method compined with implicit non standard finite difference approach to find the approximation solution of Eq.(2) in $\Omega : a \leq x \leq b$, a one dimensional domain, with the following initial condition:

$$u(x,0) = f(x), \tag{3}$$

and Dirichlet boundary conditions:

$$u(a,t) = 0, \; u(b,t) = 0, \tag{4}$$

With a view to highlighting the rigor of the introduced approach, we present some numerical examples. In reality, we motivated to interested to utilize the spectral method to approximate the solution of fractional space-time Lévy-Feller diffusion equation because no work used this technique in the literature.

The construction of this article is as the following: In the next section, several important mathematical tools and definitions of the fractional calculus which are needed for the rest sections are presented and we state some pertinent characteristics of Jacobi and Legendre polynomials. In section 3, numerical scheme for time fractional Lévy-Feller diffusion equation was built. The resulting system of algebraic equations is solved numerically and the solution of the studied problem is introduced. After this section, some numerical results are presented in some figures and tables to clarify the technique and to show the applicability and the efficiency of the introduced technique. Some conclusions are given in last section.

2 Preliminaries and Definitions

Definition 2.1 The Caputo fractional operator with order α, $\alpha \in \mathbb{R}^+$, is introduced by (Caputo, 1967) as the following:

$$({}_0^cD_t^\alpha f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(x)}{(t-x)^{1-n+\alpha}}dx, \quad t > 0, \tag{5}$$

where $n = [\alpha] + 1,$ and the function $f(x) \in C^n[0, \infty[$.

It is very easy to prove that the fractional derivative of any constant using the Caputo definition sense is zero, and

$$ {}_0^cD_t^\alpha e^{vt} = v^n t^{n-\alpha} E_{1,n-\alpha+1}(vt) \tag{6}$$

We notice that if $\alpha \in \mathbb{N}$ then the Caputo fractional derivative is identical with the standard differential operator. In addition, the Caputo's fractional differential operator is a linear operator; i.e.

$$ {}_0^cD_t^\alpha(\lambda f(t) + \gamma g(t)) = \lambda\, {}_0^cD_t^\alpha f(t) + \gamma\, {}_0^cD_t^\alpha g(t).$$

This property is similar to the integer-order derivatives.

Definition 2.2 For $1 < \alpha < 2$ the Riesz-Feller fractional operator D_θ^α with $|\theta| \le min\{\alpha, 2-\alpha\}$, is formulated as the following (see e.g. [14], [29], [18], [19]):

$$D_\theta^\alpha f(x) = -(c_+ D_+^\alpha + c_- D_-^\alpha) f(x), \tag{7}$$

such that $c_\pm$ are the coefficients with the following form:

$$c_+ = c_+(\alpha,\theta) = \frac{sin((\alpha-\theta)\pi/2)}{sin(\alpha\pi)}, \quad c_- = c_-(\alpha,\theta) = \frac{sin((\alpha+\theta)\pi/2)}{sin(\alpha\pi)}, \tag{8}$$

and

$$(D_+^\alpha f)(x) = (\frac{d}{dx})^n (I_+^{n-\alpha} f)(x), \quad (D_-^\alpha f)(x) = (-\frac{d}{dx})^n (I_-^{n-\alpha} f)(x), \tag{9}$$

are defined by the left-side Riemann-Liouville fractional operator and the right-side Riemann-Liouville fractional operator, respectively, for $x \in \mathbb{R}$ and α, positive number $n-1 < \alpha \le n$, $n = 1, 2$. In expressions (9) the fractional integrals $I_\pm^{n-\alpha}$ are the left-side Weyl integrals and right-side of Weyl fractional integrals, that written in the following form:

$$(I_+^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi, \quad (I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{+\infty} \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi. \tag{10}$$

For $\alpha = 2$ $(\theta = 0)$, $D_\theta^\alpha f(x) = \frac{d^2 f(x)}{dx^2}$.

Depending on the definition (2), we notice that Riesz-Feller fractional operator is written as a composition of both left side Riemann-Liouville fractional operator and right side of Riemann-Liouville fractional operator, this combination is linear, therefore:

$$D_\theta^\alpha(\beta h(x) + \gamma g(x)) = \beta D_\theta^\alpha h(x) + \gamma D_\theta^\alpha g(x).$$

Now, we introduce a few properties of the Legender polynomials.

Definition 2.3 On the interval $[-1,\ 1]$ the Legendre polynomials are defined by the following three-terms returning formula:

$$L_{k+1}(x) = \frac{(2k+1)x}{k+1} L_k(x) - \frac{k}{k+1} L_{k-1}(x), \quad k = 1,\ 2,\ \ldots,$$

such that

$$L_0(x) = 1 \text{ and } L_1(x) = x.$$

$L_k(x)$, the Legendre polynomials of degree k have the following analytic expression:

$$L_k(x) = \sum_{i=0}^{\lfloor k/2 \rfloor} (-1)^i \frac{(2k-2i)!}{2^k i!(k-i)!(k-2i)!} x^{(k-2i)}.$$

The shifted Legendre polynomials can be used on the interval $[a,b]$ by changing the variable $x = \frac{2\tilde{x}-b-a}{b-a}$ where $\tilde{x} \in [a,b]$. Assume the shifted Legendre polynomials as the following:

$$L_k^*(\tilde{x}) = L_k(\frac{2\tilde{x}-b-a}{b-a}), \quad k = 1,\ 2, \ldots .$$

Such that

$$L_0^*(\tilde{x}) = 1 \ \text{ and } \ L_1^*(\tilde{x}) = \frac{2\tilde{x}-b-a}{b-a},$$

$$L_{k+1}^*(\tilde{x}) = \frac{(2k+1)(2\tilde{x}-b-a)}{(k+1)(b-a)} L_k^*(\tilde{x}) - \frac{k}{k+1} L_{k-1}^*(\tilde{x}), \quad k = 1,\ 2,\ \ldots .$$

It is well known that $L^*_k(a) = (-1)^k$ and $L^*_k(b) = 1$.

Any square integrable function $y(\tilde{x})$ in $[a,b]$, can be written using of the shifted Legendre polynomials in the following:

$$y(\tilde{x}) = \sum_{i=0}^{\infty} c_i L^*_i(\tilde{x}),$$

such that the coefficients c_i have the following form:

$$c_i = \frac{2i+1}{b-a}\int_a^b y(\tilde{x})L^*_i(\tilde{x})d\tilde{x}, \qquad i = 0,\ 1,\ 2,\ \dots .$$

In numerical studies, the first $(m+1)$ terms of these shifted polynomials are used, therefor, we can write:

$$y_m(\tilde{x}) = \sum_{i=0}^{m} c_i L^*_i(\tilde{x}).$$

The Legendre polynomials are spaciale case of Jacobi polynomials ($J^{\alpha,\beta}_k(x), \alpha, \beta > -1$) when $\alpha = \beta = 0$, i.e., $L_k(x) = J^{0,0}_k(x)$. We recall here some properites of Jacobi polynomials:

$$J^{\alpha,\beta}_{k+1} = (a^{\alpha,\beta}_k x - b^{\alpha,\beta}_k J^{\alpha,\beta}_k(x)) - c^{\alpha,\beta}_k J^{\alpha,\beta}_{k-1}(x), k = 1,\ 2,\ \dots,$$

$$J^{\alpha,\beta}_0(x) = 1, \qquad J^{\alpha,\beta}_1(x) = \frac{(\alpha+\beta+2)x + \alpha - \beta}{2},$$

where

$$a^{\alpha,\beta}_k = \frac{(2k+\alpha+\beta+1)(2k+\alpha+\beta+2)}{2(k+1)(k+\alpha+\beta+1)},$$

$$b^{\alpha,\beta}_k = \frac{(2k+\alpha+\beta+1)(\beta^2-\alpha^2)}{2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},$$

$$c^{\alpha,\beta}_k = \frac{(k+\alpha)(k+\beta)(2k+\alpha+\beta+2)}{(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta)},$$

and

$$J^{\alpha,\beta}_k(-x) = (-1)^k J^{\beta,\alpha}_k(x), \qquad J^{\alpha,\beta}_k(1) = \frac{\Gamma(k+\alpha+1)}{k!\Gamma(\alpha+1)}, \tag{11}$$

The shifted Jacobi polynomials in $[a,b]$ is ${}_*J^{\alpha,\beta}_k(\tilde{x}) = J^{\alpha,\beta}_k(\frac{2\tilde{x}-b-a}{b-a})$.

The following two theorems explain that the derivatives of Legendre and shifted Legendre polynomials using Riemann-Liouville fractional operator can be written in tearms of Jacobi polynomials.

Theorem 2.1

$$_{-1}D^p_x L_k(x) = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}(1+x)^{-p} J^{p,-p}_k(x), \tag{12}$$

$$_xD^p_1 L_k(x) = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}(1-x)^{-p} J^{-p,p}_k(x). \tag{13}$$

For $0 < p < 1$.

*Proof.*see [30]

Theorem 3.2

$$_aD^p_{\tilde{x}} L^*_k(\tilde{x}) = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}(\tilde{x}-a)^{-p}\ {}_*J^{p,-p}_k(\tilde{x}), \tag{14}$$

$$_{\tilde{x}}D^p_b L^*_k(\tilde{x}) = \frac{\Gamma(k+1)}{\Gamma(k-p+1)}(b-\tilde{x})^{-p}\ {}_*J^{-p,p}_k(\tilde{x}). \tag{15}$$

When $\tilde{x} \in [a,b]$ and $0 < p < 1$.

*Proof.*see [30]

3 Numerical scheme for time fractional Lévy-Feller diffusion equation

Let $u(x,t) = 0$ for $x \in \mathbb{R}\backslash[a,b]$ in Eq. (2), (as [18]). In the current section, we treat the time fractional Lévy-Feller diffusion model in the following form

$$\begin{cases} {}_0^cD_t^{\beta} u(x,t) = d(x,t)D_{\theta}^{\alpha}u(x,t) + s(x,t), & t > 0, \;\; a < x < b, \\ u(a,t) = 0, \qquad u(b,t) = 0, & t > 0, \\ u(x,0) = f(x), & a \leqslant x \leqslant b. \end{cases} \tag{16}$$

We approximate $u(x,t)$ as the following:

$$u_m(x,t) = \sum_{i=0}^{m} u_i(t)L_i^*(x), \tag{17}$$

in terms of Legendre polynomials.

By using relations (14, 15) and characteristics of fractional Riemann-Liouville operator, we approximate $D_{\theta}^{\alpha}u(x,t)$ as:

$$D_{\theta}^{\alpha}u_m(x,t) = -\sum_{i=0}^{m} u_i(t)\frac{d}{dx}\Theta_{i,\alpha-1}(x), \qquad for \;\; 1 < \alpha < 2, \tag{18}$$

where

$$\Theta_{i,\alpha}(x) = \frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\left(c_+(x-a)^{-\alpha}\;{}_*J_i^{\alpha,-\alpha}(x) + c_-(b-x)^{-\alpha}\;{}_*J_i^{-\alpha,\alpha}(x)\right).$$

Therefore Eq. (16) take the following forms, depending on equation **??**diff of u 2):

$$\sum_{i=0}^{m} {}_0^cD_t^{\beta} u_i(t)L_i^*(x) = -d(x,t)\sum_{i=0}^{m} u_i(t)\frac{d}{dx}\Theta_{i,\alpha-1}(x) + s(x,t), \tag{19}$$

Now by collocating Eq. (19) at $x_j, \;\; j = 1\;,2\;,...,\; m-1, \;\; (a < x_j < b)$, specific $(m-1)$ points, as follows:

$$\sum_{i=0}^{m} {}_0^cD_t^{\beta} u_i(t)L_i^*(x_j) = -d(x_j,t)\sum_{i=0}^{m} u_i(t)\frac{d}{dx}\Theta_{i,\alpha-1}(x)\big|_{x=x_j} + s(x_j,t), \tag{20}$$

Now, plugging in Eq. (17) in the conditions at the beginning time to find the u_i as constants at $t = 0$ and by plugging the same equation in the conditions at the boundaries we have the following equations:

$$\sum_{i=0}^{m}(-1)^i u_i(t) = 0, \qquad \sum_{i=0}^{m} u_i(t) = 0, \tag{21}$$

Equation (20), and equations (21), built a system of with the unknowns $u_i, \; i = 0,1,...,m, \; (m+1)$ unknowns. this system is a time fractional ordinary differential equations. This system is solved using non standard implicit finite difference method [31] with the fowlloing discritization of the Caputo fractional direvative:

Let us consider N_n to be a non-negative integer number then the mesh points have the following coordinates:

$$for \quad n = 0,\; 1,\; 2,\; ...,\; N_n, \quad we\; have \;\; t_n = n\triangle t,$$

such that

$$h = \frac{t_{final}}{N_n} := \triangle t.$$

We denoted by $x_n, \; y_n$ and z_n at the grid point (t_n) to the approximating values of x, y and z. Caputo fractional derivative oerator will approximate using the nonstandard differences technique depending on the Grünwald-Letnikov operator as the following:

$${}_0^cD_t^{\alpha}x(t)\mid_{t=t_n} = \frac{1}{(\phi(\triangle t))^{\alpha}}(x_{n+1} - \sum_{i=1}^{n+1} w_i x_{n+1-i} - q_{n+1}x_0), \tag{22}$$

where

$$w_i = (-1)^{i-1}\binom{\alpha}{i}, \quad w_1 = \alpha,$$

$$q_i = \frac{i^{-\alpha}}{\Gamma(1-\alpha)}, \quad i = 1,\ 2,\ \ldots,\ n+1.$$

Theorem 3.1 [32] Let $0 < \alpha < 1$, then w_i and q_i, the coefficients in (22), has the following properties

$$0 < w_{i+1} < w_i < \ldots < w_1 = \alpha < 1, \tag{23}$$

$$0 < q_{i+1} < q_i < \ldots < q_1 = \frac{1}{\Gamma(1-\alpha)}. \tag{24}$$

*Proof.*see [32].

The technique is built in the coming steps:

$$\begin{cases} \sum\limits_{i=0}^{m}(-1)^i u_i^n = 0, \\ \sum\limits_{i=0}^{m}\frac{1}{(\phi(\triangle t))^{\beta}}(u_i^{n+1} - \sum\limits_{k=1}^{n+1} w_k u_i^{n+1-k} - q_{n+1}u_i^0)L_{i,j}^* = -d_j^{n+1}\sum\limits_{i=0}^{m} u_i^{n+1}\frac{d}{dx}\Theta_{i,\alpha-1}(x)\big|_{x=x_j} + s_j^{n+1}, \quad j = 1,2,\ldots,m-1, \\ \sum\limits_{i=0}^{m} u_i^n = 0, \end{cases} \tag{25}$$

with the initial conditions:

$$\sum_{i=0}^{m} u_i^0 L_{i,j}^* = f_j, \quad j = 0,1,2,\ldots,m,$$

where $u_i^n = u_i(t_n),\quad L_{i,j}^* = L_i^*(x_j),\quad d_j^n = d(x_j,t_n),\quad s_j^n = s(x_j,t_n)\quad and \quad f_j = f(x_j)$. The above system (25) can be formulated in matrix form as the following:

$$\boldsymbol{A}^{n+1}\boldsymbol{U}^{n+1} = \boldsymbol{B^n}\boldsymbol{U}^n + \boldsymbol{B}^{n-1}\boldsymbol{U}^{n-1} + \ldots + \boldsymbol{B^1}\boldsymbol{U}^1 + \boldsymbol{B^0}\boldsymbol{U}^0 + \boldsymbol{S}^{n+1}, \tag{26}$$

such that:

$$\boldsymbol{U}^n = (u_0^n, u_1^n, \ldots, u_m^n)^T,$$
$$\boldsymbol{S}^n = (0, s_1^n, s_2^n, \ldots, s_{m-1}^n, 0)^T,$$

$$\boldsymbol{A}^{n+1} = \begin{pmatrix} \frac{1}{\phi^\beta}L_{0,0}^* + d_0^{n+1}\frac{d}{dx}\Theta_{0,\alpha-1,0} & \frac{1}{\phi^\beta}L_{1,0}^* + d_0^{n+1}\frac{d}{dx}\Theta_{1,\alpha-1,0} & \cdots & \frac{1}{\phi^\beta}L_{m,0}^* + d_0^{n+1}\frac{d}{dx}\Theta_{m,\alpha-1,0} \\ \frac{1}{\phi^\beta}L_{0,1}^* + d_1^{n+1}\frac{d}{dx}\Theta_{0,\alpha-1,1} & \frac{1}{\phi^\beta}L_{1,1}^* + d_1^{n+1}\frac{d}{dx}\Theta_{1,\alpha-1,1} & \cdots & \frac{1}{\phi^\beta}L_{m,1}^* + d_1^{n+1}\frac{d}{dx}\Theta_{m,\alpha-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\phi^\beta}L_{0,m}^* + d_m^{n+1}\frac{d}{dx}\Theta_{0,\alpha-1,m} & \frac{1}{\phi^\beta}L_{1,m}^* + d_m^{n+1}\frac{d}{dx}\Theta_{1,\alpha-1,m} & \cdots & \frac{1}{\phi^\beta}L_{m,m}^* + d_m^{n+1}\frac{d}{dx}\Theta_{m,\alpha-1,m} \end{pmatrix},$$

$$\boldsymbol{B^n} = \begin{pmatrix} \frac{1}{\phi^\beta}w_1L_{0,0}^* & \frac{1}{\phi^\beta}w_1L_{1,0}^* & \frac{1}{\phi^\beta}w_1L_{2,0}^* & \frac{1}{\phi^\beta}w_1L_{3,0}^* & \cdots & \frac{1}{\phi^\beta}w_1L_{m,0}^* \\ \frac{1}{\phi^\beta}w_1L_{0,1}^* & \frac{1}{\phi^\beta}w_1L_{1,1}^* & \frac{1}{\phi^\beta}w_1L_{2,1}^* & \frac{1}{\phi^\beta}w_1L_{3,1}^* & \cdots & \frac{1}{\phi^\beta}w_1L_{m,1}^* \\ \frac{1}{\phi^\beta}w_1L_{0,2}^* & \frac{1}{\phi^\beta}w_1L_{1,2}^* & \frac{1}{\phi^\beta}w_1L_{2,2}^* & \frac{1}{\phi^\beta}w_1L_{3,2}^* & \cdots & \frac{1}{\phi^\beta}w_1L_{m,2}^* \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\phi^\beta}w_1L_{0,m}^* & \frac{1}{\phi^\beta}w_1L_{1,m}^* & \frac{1}{\phi^\beta}w_1L_{2,m}^* & \frac{1}{\phi^\beta}w_1L_{3,m}^* & \cdots & \frac{1}{\phi^\beta}w_1L_{m,m}^* \end{pmatrix},$$

$$\boldsymbol{B}^{n-1} = \begin{pmatrix} \frac{1}{\phi^\beta}w_2L_{0,0}^* & \frac{1}{\phi^\beta}w_2L_{1,0}^* & \frac{1}{\phi^\beta}w_2L_{2,0}^* & \frac{1}{\phi^\beta}w_2L_{3,0}^* & \cdots & \frac{1}{\phi^\beta}w_2L_{m,0}^* \\ \frac{1}{\phi^\beta}w_2L_{0,1}^* & \frac{1}{\phi^\beta}w_2L_{1,1}^* & \frac{1}{\phi^\beta}w_2L_{2,1}^* & \frac{1}{\phi^\beta}w_2L_{3,1}^* & \cdots & \frac{1}{\phi^\beta}w_1L_{m,1}^* \\ \frac{1}{\phi^\beta}w_2L_{0,2}^* & \frac{1}{\phi^\beta}w_2L_{1,2}^* & \frac{1}{\phi^\beta}w_2L_{2,2}^* & \frac{1}{\phi^\beta}w_2L_{3,2}^* & \cdots & \frac{1}{\phi^\beta}w_2L_{m,2}^* \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\phi^\beta}w_2L_{0,m}^* & \frac{1}{\phi^\beta}w_2L_{1,m}^* & \frac{1}{\phi^\beta}w_2L_{2,m}^* & \frac{1}{\phi^\beta}w_2L_{3,m}^* & \cdots & \frac{1}{\phi^\beta}w_2L_{m,m}^* \end{pmatrix},$$

$$\vdots$$

$$\boldsymbol{B}^0 = \begin{pmatrix} \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{0,0} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{1,0} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{2,0} & \cdots & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{m,0} \\ \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{0,1} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{1,1} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{2,1} & \cdots & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{m,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{0,m} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{1,m} & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{2,m} & \cdots & \frac{1}{\phi^\beta}(w_{n+1}+q^{n+1})L^*_{m,m} \end{pmatrix},$$

By substituting the coefficients u_i, $i = 0,1,2,\ldots,m$, which are computed by solving (26) as well as the Legendre polynomials in (17). Then, the approximation solution of u is obtained.

4 Numerical examples

For demonstrating the accuracy and the effectiveness of the introduced approach, two numerical problems are offered, in details. We will use the $(\phi(\triangle t))^\beta = 1 - e^{-\triangle t}$

Example 1.[18]. In a bounded domain, the following space-time fractional Lévy-Feller fractional diffusion equation was considered :

$$\begin{cases} {}^c_0D^\beta_t u(x,t) = D^\alpha_\theta u(x,t), \quad t>0, \ 0<x<2\pi, \\ u(0,t)=0, \qquad u(\pi,t)=0, \quad t>0, \\ u(x,0)=sin(x), \qquad 0 \leqslant x \leqslant 2\pi. \end{cases} \tag{27}$$

Figure (1) shows the behavior of the obtained numerical solutions by the introduced technique when $m = 5$ and $T_{final} = 2$ with different values of α and β for example 1.

Figure (2), with $\alpha = 1.8$ and $T_{final} = 1$, shows the numerical outcomes by the introduced technique for Ex.(1) with $m = 5$ for different values of β.

*Example 2.*In the following, the space-time fractional Lévy-Feller fractional diffusion equation with force term was considered :

$$\begin{cases} {}^c_0D^\beta_t u(x,t) = d(x,t)D^\alpha_\theta u(x,t) + s(x,t), \quad t>0, \ 0<x<1, \\ u(0,t)=0, \qquad u(1,t)=0, \quad t>0, \\ u(x,0)=x(1-x), \qquad 0 \leqslant x \leqslant 1, \end{cases} \tag{28}$$

where

$$d(x,t) = \Gamma(3-\alpha)x,$$

$$S(x,t) = x(x-1)t^{1-\beta}E_{1,2-\beta}(-t) + x\Big\{\frac{(2-\alpha)}{sin(\alpha\pi)}\Big(sin(\frac{(\alpha-\theta)\pi}{2})x^{1-\alpha} + sin(\frac{(\alpha+\theta)\pi}{2})(1-x)^{1-\alpha}\Big)$$
$$-\frac{2}{sin(\alpha\pi)}\Big(sin(\frac{(\alpha-\theta)\pi}{2})x^{2-\alpha} + sin(\frac{(\alpha+\theta)\pi}{2})(1-x)^{2-\alpha}\Big)\Big\}e^{-t}.$$

This equation, when $1 < \alpha \leq 2$, has the following exact solution:

$$u(x,t) = x(1-x)e^{-t}.$$

The absolute error, the difference between the approximate solutions and their corresponding exact values, is found using the following difference $E(x,t) = |u_{exact}(x,t) - u_{approx.}(x,t)|$. Also,

$$M = max\{E(x,t): \ a \leq x \leq b, \ 0 \leq t \leq T_{final}\}.$$

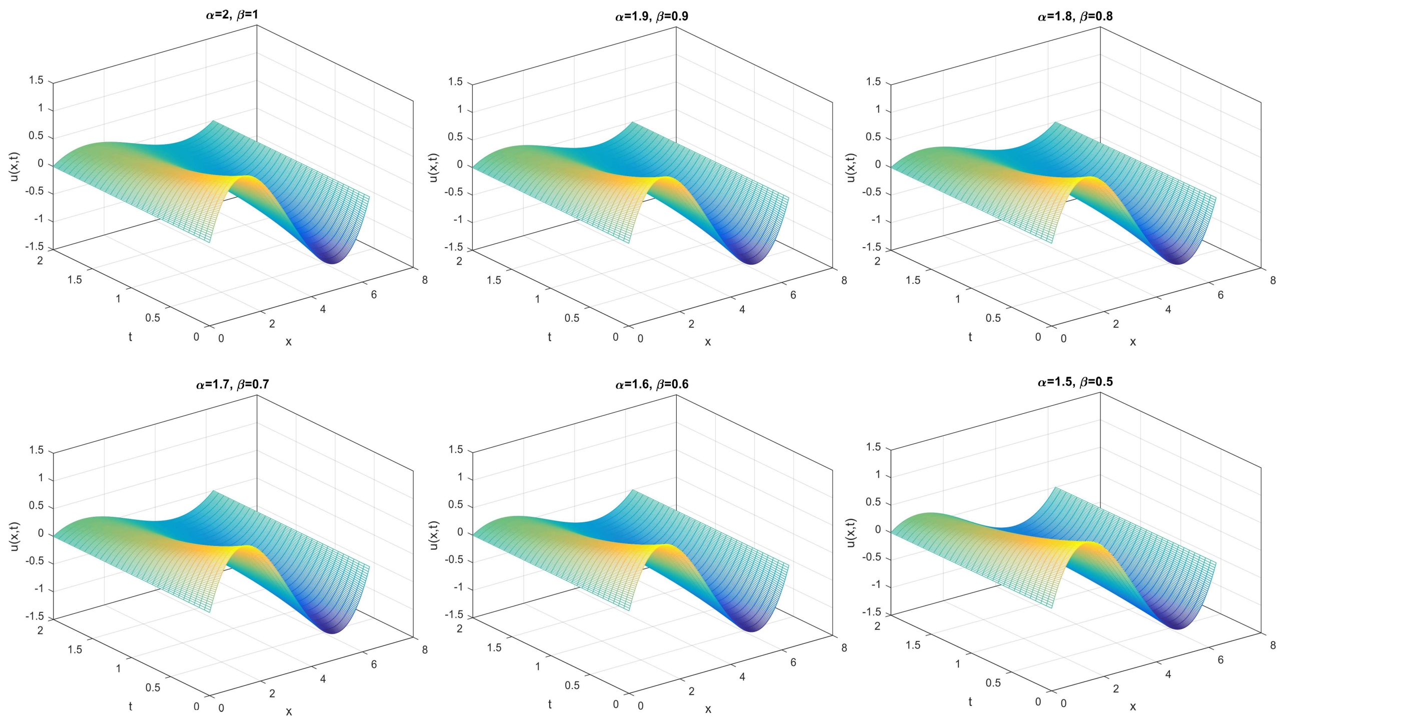

Fig. 1: The approximation solutions by the proposed method for example (1).

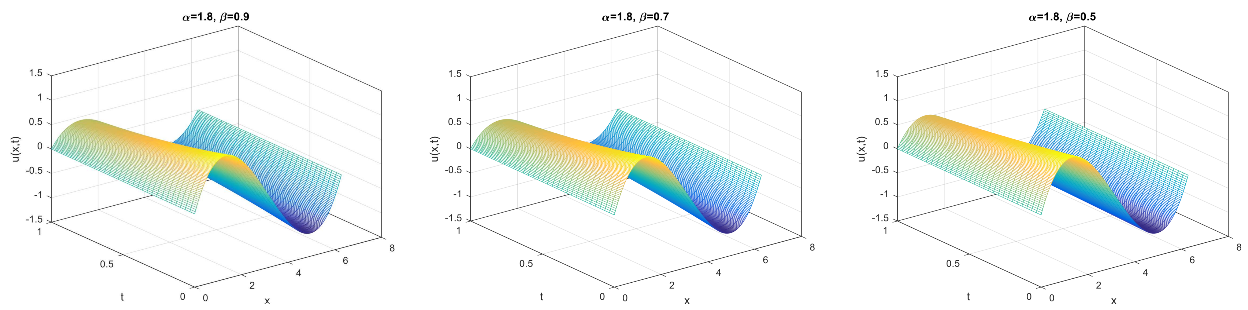

Fig. 2: The approximation solutions using the proposed method for example (1) when $\alpha = 1.8$.

is the maximum absolute errors.

In order to show the accuracy of our method to solve the proposed problem, we stated in table (1), for $T_{final} = 1\ and\ \triangle t = 0.05$, the errors for problem (2), where $\alpha = 1.8,\ \theta = -0.1$ and $\beta = 0.9$ for different values of m.

In table (2), for $T_{final} = 1\ and\ \triangle t = 0.001$, for example (2), we found M for different values of $\alpha,\ \beta$ and m.

In table (3), for $T_{final} = 1\ and\ \triangle t = 0.001$, for problem (2), we found M when $\alpha = 1.5$ for different values of $\theta,\ \beta$ and m.

In figure (3) we see the exact solution for Ex.(2) when $T_{final} = 1.5$, the approoximatin solution and the errors of using our method when $m = 4$ and $\triangle t = 0.001$.

Table 1: The errors calculated by the proposed method for example (2) at $t = 1$

x	$m = 3$, CPU time=273 s	$m = 5$, CPU time=397 s	$m = 7$, CPU time=677 s
0	0	0	0
0.1	4.2567e-04	2.1538e-04	1.1189e-04
0.2	3.5419e-04	3.1581e-04	2.9803e-04
0.3	4.5278e-04	4.2638e-04	3.1417e-04
0.4	2.9573e-03	2.8456e-04	2.7764e-04
0.5	4.2611e-03	3.5231e-03	4.3241e-05
0.6	5.6725e-03	5.1263e-04	9.6739e-05
0.7	3.4189e-03	3.2092e-04	2.5721e-04
0.8	2.5897e-04	2.4175e-04	2.0126e-04
0.9	1.7342e-04	1.3489e-04	1.0372e-04
1	0	0	0

Table 2: The maximum errors calculated by the proposed method for example (2)

	$m = 2$	$m = 4$	$m = 6$	$m = 8$
CPU time	290 s	478 s	765 s	968 s
$\alpha = 2,\ \beta = 1$	1.0864e-03	2.1156e-04	1.0021e-04	7.0498e-05
$\alpha = 1.9,\ \beta = 0.9$	8.6749e-03	3.6281e-04	2.0163e-04	8.1542e-05
$\alpha = 1.8,\ \beta = 0.8$	7.8534e-03	4.6237e-04	1.2310e-04	9.0853e-05
$\alpha = 1.7,\ \beta = 0.7$	9.8945e-03	2.5876e-03	2.0231e-04	4.1749e-05
$\alpha = 1.6,\ \beta = 0.6$	5.9578e-03	4.7284e-04	2.2671e-04	7.0953e-05
$\alpha = 1.5,\ \beta = 0.5$	6.7843e-03	3.8569e-04	3.0066e-04	6.2673e-05

Table 3: The maximum errors calculated by the proposed method when $\alpha = 1.5$ for example (2)

	$m = 3$	$m = 5$	$m = 6$	$m = 9$
CPU time	362 s	531 s	849 s	991 s
$\theta = 0,\ \beta = 1$	5.6924e-03	6.8724e-04	4.0245e-04	9.1456e-05
$\theta = 0.1,\ \beta = 0.9$	6.0723e-03	1.9531e-03	1.7429e-04	7.9741e-05
$\theta = 0.2,\ \beta = 0.8$	3.6623e-03	6.7289e-03	3.0963e-04	6.7309e-05
$\theta = 0.3,\ \beta = 0.7$	1.3681e-03	5.7290e-04	9.0189e-04	3.5793e-06
$\theta = 0.4,\ \beta = 0.6$	9.0553e-03	7.1689e-04	8.0527e-04	3.5709e-05
$\theta = 0.5,\ \beta = 0.5$	8.0127e-03	1.0762e-03	1.0931e-04	2.0742e-05

5 Conclusion

An efficient numerical method is introduced to approximate the solutions of the time fractional Lévy-Feller diffusion equation. The fractional time derivatives were given in the Caputo sense. The Chebyshev-Legendre collocation approch combination with the non standard implicit finite difference method is implemented for constructing the proposed technique. This technique creates an algebraic system, the unknowns of this system are the coefficients of the spectral expansion. The high accurate approximation results of fractional time Lévy-Feller equations which are achieved by using some terms of the shifted Legendre polynomial expansion is the essential advantage of this technique. The proposed technique can approximate accurately the solutions of any space time fractional parabolic differential equation. Two numerical examples are introduced to show the availability of the proposed method to find the approximation numerical solutions of all kinds of the space-time fractional differential equations. We introduced some comparisons between the obtained numerical solution of the proposed problem with its exact solution to confirm the accuracy and validity of our schema.

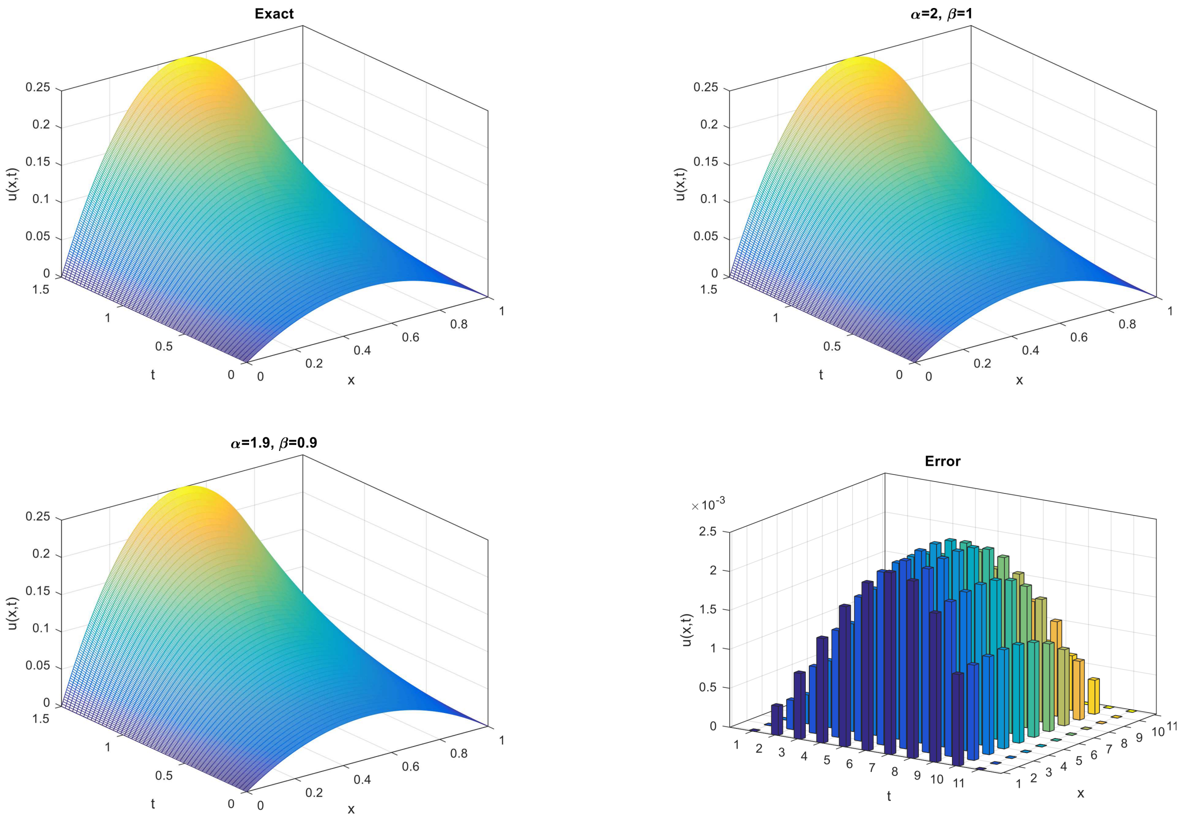

Fig. 3: The exact, numerical solution using the proposed method and the error for example (2).

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