Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/090204

Efficient B-Spline Series Method for Solving Fractional Fokker-Planck Equation

Mohammed Abuomar[1,2], *Muhammed Ibrahem Syam*[2,*] *and Amirah Azmi*[1]

[1] School of Mathematical Sciences, College of Science, Universiti Sains Malaysia, Penang, Malaysia
[2] Department of Mathematical Sciences, College of Science, UAE University, Al-Ain, UAE

Received: 3 Mar. 2022, Revised: 15 May 2022, Accepted: 12 Jul. 2022
Published online: 1 Apr. 2023

Abstract: In this paper, an analytical method based on the B-spline method is used to study the fractional Fokker Planck equation. In this study, the B-Spline series method is derived, and its effectiveness in handling this problem is demonstrated. To illustrate the effectiveness of the suggested method, two examples are given. Since the maximum error in our approximation is only about 10^{-15}, the suggested method is clearly very accurate as we can see from the fact that the approximate solution is very close to the exact solution. This study demonstrates the proposed approach's potential and its applicability to additional physical models.

Keywords: Fractional Fokker-Planck equation, B-Spline method, Caputo derivative.

1 Introduction

September 30th, 1695 marks the born of fractional calculus with reason to a profound problem raised by a French mathematician called L'Hospital in a letter to a German mathematician named Leibniz. The adorable response of Leibniz as recorded to that profound problem laminated a major inspiration for all generations of scientists and is incessant to motivate the minds of modern researchers. Fractional calculus has maintained the attention of best-level mathematicians for last three centuries, and over the last decade it has been used to tackle the dynamics of complex systems from different fields of science and engineering. The most important applications of fractional calculus in science and engineering are fluid mechanics, quantum mechanics, electromagnetics, viscoelasticity, electricity, electrochemistry, biological population models, optics, signals processing, and ecological systems [1]. A lot of prominent scientists reporting results in fractional calculus during 18th and 19th centuries included Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and others, see [2,3,4,5,6,7,8,9,10,11]. This part introduces a few definitions of fractional calculus for $\eta \in (0,1]$.

Definition 1. *The Riemann–Liouville fractional derivative is defined as*

$$D^{\eta}h(y) = \frac{1}{\Gamma(1-\eta)}\frac{d}{dy}\int_0^y (y-s)^{-\eta}h(s)ds,$$

where $y > 0$ *and* $\eta > 0$.

Definition 2. *The Caputo fractional derivative of y is defined as*

$$D^{\eta}h(y) = \frac{1}{\Gamma(1-\eta)}\int_0^y (y-s)^{-\eta}h'(s)ds,$$

where $y > 0$ *and* $\eta > 0$.

* Corresponding author e-mail: m.syam@uaeu.ac.ae

Some of the properties of a Caputo derivative include:

1. $D^{\eta}c = 0$, where c is any constant.
2. $D^{\eta}y^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\eta+\gamma+1)}y^{\gamma-\eta}$, where $\gamma \geq \eta$.

There is a long history of studies on the theory of Fokker–Planck equation, dating back to Einstein, Langevin, Fokker, and Planck, among many others [11,12,13]. Because of impressive advancements in analysis and computation, plus a wide range of available applications, the theory of the Fokker-Planck equation is exceedingly rich in content. A wide range of scientific phenomena can be described by it, including the relationship between random force and fluctuations, and nonlinearity in pattern formation. The Fokker–Planck equation can be used to describe many physical, chemical, biological, and economic systems. Numerous algorithms have been explored for numerical solutions of the Fokker–Planck equation in modern years, see [14,15]. The Fokker–Planck equation has received considerable theoretical attention in recent years. Each of the proposed methods has its advantages and limitations, as might be expected. A number of authors have used path integral methods [16,17,18]. A numerical evaluation of the Onsager-Machlup-functions using a mathematical formalism provided by Wehner and Wolfer, [19], has been presented. Eigenvalue expansion is applicable to a wide class of Fokker–Planck operators [20,21,22,23]. The eigenvalue of Fokker–Planck equation can be solved very accurately by using various spectral methods and pseudospectral methods. When the Fokker–Planck equation deals with a Lorentz gas system, the complete set of eigenvalues and eigenfunctions determine the dynamics completely.

In this paper, we discuss the following Fractional Fokker Planck equation (FFPE)

$$D_z^{\eta}\Omega(y,z) + D_y^{\eta}\Omega(y,z) - D_y^{2\eta}\Omega(y,z) = 0, \tag{1}$$

$$\Omega(y,0) = h(y) \tag{2}$$

where $y,z > 0$ and $\eta \in (0,1]$ using the proposed method to increase the accuracy of the approximate solution. We select this method since it is promising method and gives us accurate approximation. In Section 2, some preliminaries which will be used in this paper will be presented. In section 3, we present the method of solution and in section 4, we show some numerical results to show the efficiency of the B-spline method . Finally, we present some conclusions in Section 5.

2 Preliminaries

In this section, we present the concept fractional B-splines (FBS) and their properties. Let us define the fractional truncated power functions (FTPFs) and the fractional B-spline.

Definition 3. *Let $\eta > 0$ be a real number, then the FTPF is given as*

$$y_+^{\eta} = \begin{cases} y^{\eta}, & y \geq 0 \\ 0, & y < 0 \end{cases}$$

and the FBS is given by

$$S_{\eta}(y) = \frac{1}{\Gamma(\eta+1)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\eta+2)}{k!\Gamma(\eta-k+2)}(y-k)_+^{\eta}. \tag{3}$$

It is wroth to mention that when $\eta > 0$ is integer, the sum in definition (2.1) will be finite sum and its degree is η. Also, its compact support, which is defined by $\{y \in \Re : y_+^{\eta} > 0\}$, is $[0,\eta+1]$ and it is always positive. However, when η is non-integer, then the sum is infinite and does not have compact support but it decays very fast as y becomes large. Also, it has both positive and negative values but the negative values decay very fast as η becomes large. To understand these functions, we plot the graphs of $S_{\eta}(y)$ for specific choices of η, $\eta = \frac{k}{4}, k = 0:8$. We notice that the compact support of $S_{\eta}(y)$ is $[0,6]$ when η is integer and decay to zero when η is non-integer number. Also, it is positive when η is integer and has positive sign and negative sign when η is non-integer. Using the Caputo derivative and its power rule, one can get

$$D^{\nu}y_+^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\nu+1)}y_+^{\eta-\nu}, 0 < \nu \leq \eta$$

which yields that

$$D^{\nu}S_{\eta}(y) = \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(\eta+2)}{k!\Gamma(\eta-k+2)\Gamma(\eta-\nu+1)}(y-k)_+^{\eta-\nu}. \tag{4}$$

To understand the behavior of the fractional derivatives of $S_5(y)$, $S_{\frac{26}{5}}$, and $S_{\frac{27}{5}}$ for $\nu = \frac{1}{5}, \frac{3}{5}, 1$, we sketch Figures (2)-(4), respectively.

We note from Figures (2)-(4) that the derivatives of $S_{\eta}(y)$ decay to zero as y.

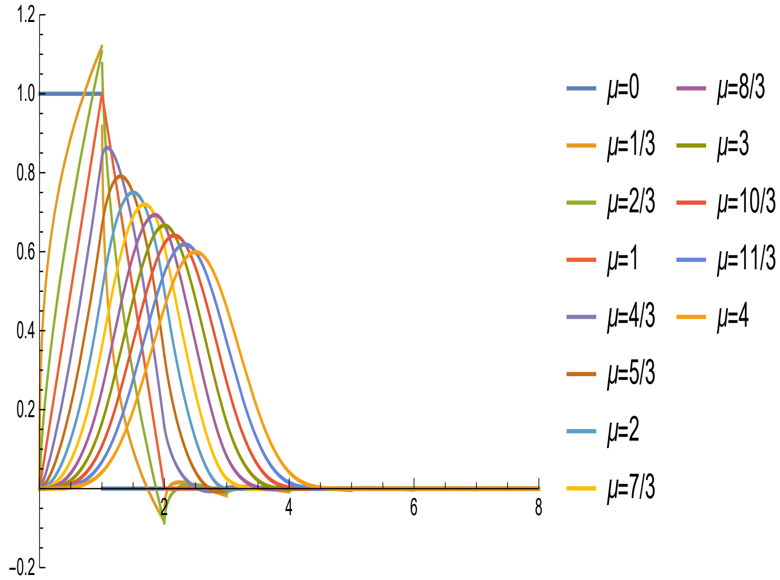

Fig. 1: $S_\eta(y)$ for $\eta = \frac{k}{4}, k = 0:8$.

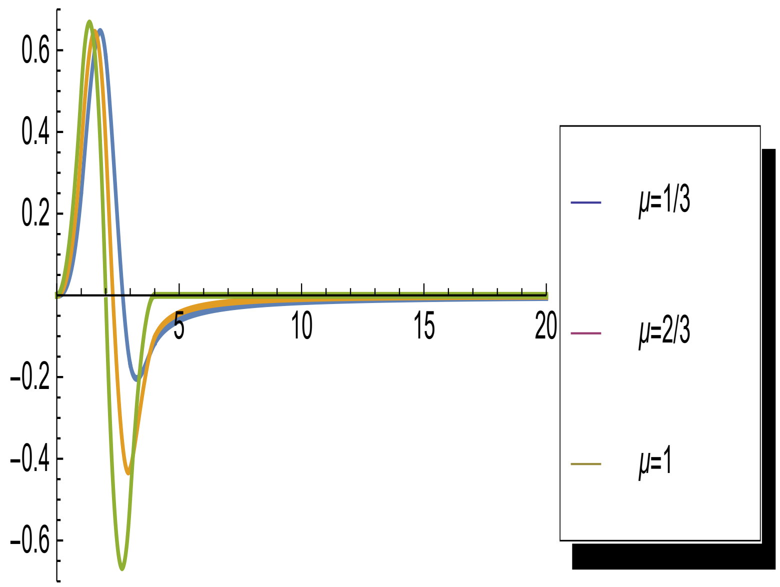

Fig. 2: $D^\nu S_5(y)$ for $\nu = \frac{k}{5}, k = 1,3,5$.

3 Method of solution

In this section, we present the proposed method. Generate a uniform partition to $[0,1]$ as $\Pi = \{k\lambda : 0 \le k \le K\}$ where $\lambda = \frac{1}{K}$. As we notice from Figure (1) that $S_\nu(y)$ does not have compact support but it decays to zero as y becomes large. For this reason, we can assume that the support of $S_\nu(y)$ is $[0,L]$ where $L > 0$ is an integer. Let

$$S_{\eta,\lambda} = \{S_{\eta,k}(y) = S_\eta(\frac{y}{\lambda} - k), k \in \mathbb{I}_\lambda, y \in [0,1]\} \tag{5}$$

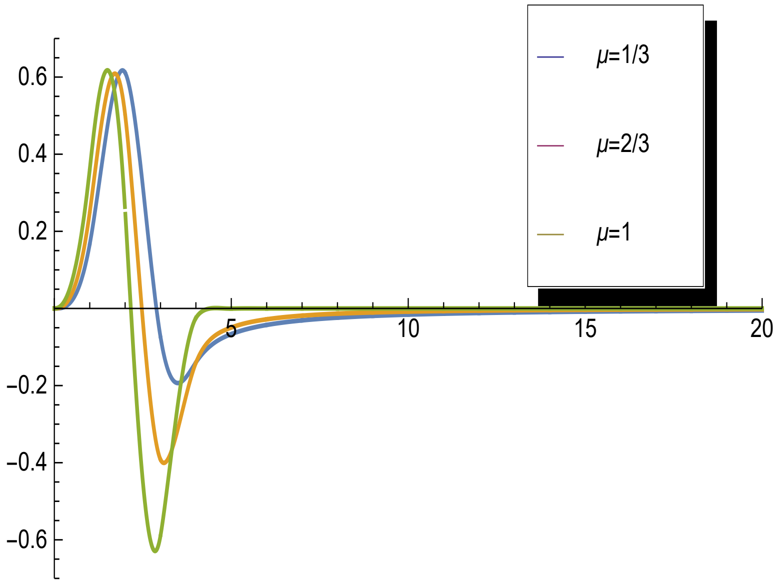

Fig. 3: $D^{\nu}S_{\frac{26}{5}}(y)$ for $\nu = \frac{k}{5}, k = 1,3,5$.

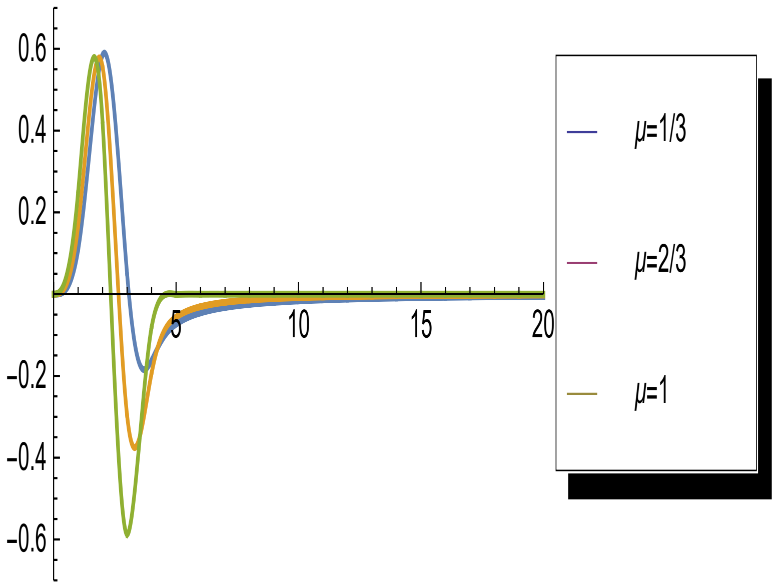

Fig. 4: $D^{\nu}S_{\frac{27}{5}}(y)$ for $\nu = \frac{k}{5}, k = 1,3,5$.

be a basis for the fractional spline space where $\mathbb{I}_{\lambda} = \{-L, -L+1, ..., K\}$. Because $S_{\eta}(\frac{y}{\lambda} - k) \subseteq [0,1]$, we let $K = L-1$. Hence, our spline space has dimension $2L$. Let us assume that the solution of equation (1.1) by

$$\Omega_{\eta,\lambda}(y,z) = \sum_{k=-L}^{L-1} \omega_k(z) S_{\eta}(\frac{y}{\lambda} - k). \tag{6}$$

We choose the collocation points as follows

$$\Upsilon = \{y_k = k\rho : 1 \leq k \leq 2L\} \tag{7}$$

where $\rho = \frac{1}{2L}$. We implement the collocation method by forcing the approximate solution to satisfy Equation (1.1) at the collocation points. Thus,

$$D_z^\eta \Omega_{\eta,\lambda}(y_k,z) + D_y^\eta \Omega_{\eta,\lambda}(y_k,z) - D_z^{2\eta} \Omega_{\eta,\lambda}(y_k,z) = 0 \tag{8}$$

which gives that

$$\sum_{k=-L}^{L-1} D_z^\eta \omega_k(z) S_{\eta,k}(y_m) + \sum_{k=-L}^{L-1} \omega_k(z) D_y^\eta S_{\eta,k}(y_m) - \sum_{k=-L}^{L-1} \omega_k(z) D_y^{2\eta} S_{\eta,k}(y_m) = 0 \tag{9}$$

for $m = 1,...,2L$. Let

$$h(y) = \sum_{k=-L}^{L-1} h_k S_{\eta,k}(y). \tag{10}$$

Equation (1.2) yields that

$$\Omega_{\eta,\lambda}(y,0) = \sum_{k=-L}^{L-1} \omega_k(0) S_{\eta,k}(y) = \sum_{k=-L}^{L-1} h_k S_{\eta,k}(y). \tag{11}$$

Thus,

$$\omega_k(0) = h_k, k = -L : L-1. \tag{12}$$

Let

$$\omega(z) = \begin{pmatrix} \omega_{-L}(z) \\ \vdots \\ \omega_{L-1}(z) \end{pmatrix}, A = (S_{\eta,k}(y_m))_{-L\leq k\leq L-1, 1\leq m\leq 2L},$$

$$B = (D_y^\eta S_{\eta,k}(y_m) - D_y^{2\eta} S_{\eta,k}(\beta y_m))_{-L\leq k\leq L-1, 1\leq m\leq 2L},$$

$$H = \begin{pmatrix} h_L \\ \vdots \\ h_{L-1} \end{pmatrix}.$$

Equations (3.5) and (3.7) can be written in the matrix form as

$$AD_z^\eta \omega(z) + B\omega(z) = 0, \tag{13}$$

$$\omega(0) = H. \tag{14}$$

Assume that $\omega(z)$ is approximated by

$$\omega(z) = \sum_{j=0}^{\infty} \frac{z^{j\eta}}{\Gamma(1+j\eta)} \omega_j. \tag{15}$$

Then, the s^{th} truncation series is given by

$$\omega_s(z) = \sum_{j=0}^{s} \frac{z^{j\eta}}{\Gamma(1+j\eta)} \omega_j. \tag{16}$$

Because $\omega(0) = H$, then

$$\omega_s(z) = H + \sum_{j=1}^{s} \frac{z^{j\eta}}{\Gamma(1+j\eta)} \omega_j. \tag{17}$$

To find the vectors $\omega_j, j = 1 : s$, we solve the iterative equation

$$D^{(j-1)\eta} Res\omega_s(0) = 0 \tag{18}$$

where

$$Res\omega_s(z) = AD_z^\eta \omega_s(z) + B\omega_s(z) = 0$$

$$= BH + \sum_{j=1}^{s} \frac{z^{(j-1)\eta}}{\Gamma(1+(j-1)\eta)} A\omega_j + \sum_{j=1}^{s} \frac{z^{j\eta}}{\Gamma(1+j\eta)} B\omega_j = 0.$$

Hence,

$$A\omega_j + B\omega_{j-1} = 0, j = 1,2,...,s$$
$$\omega_0 = H$$

which can be simplified as

$$\omega_j = (-1)^j (A^{-1}B)^j H = 0, j = 0,1,2,...,s. \quad (19)$$

We summarize the proposed method in the following algorithm.

Algorithm 3.1

Step 1. Use Equation (3.15) to find $\omega_j(z)$ for $j = 0,1,...,s$.
Step 2. Use Equation (3.13) to find the approximate solution of Problem (3.9)-(3.10).
Step 3. Use Equation (3.2) to find the approximate solution $\Omega(y,z)$ of Problem (1.1)-(1.2).
Step 4. Stop.

4 Numerical results

In this section, we give two examples to illustrate the efficiency of the proposed method.

Example 1: Consider the following problem [22]

$$D_z^\eta \Omega(y,z) + D_y^\eta \Omega(y,z) - D_z^{2\eta} \Omega(y,z) = 0$$

with

$$\Omega(y,0) = \frac{-y^{3\eta}}{\Gamma(3\eta+1)}$$

where $\Omega(y,z)$ is the external potential and $D_z^\eta \Omega(y,z)$ and $D_z^{2\eta} \Omega(y,z)$ represent the negative external forces in the system. The exact solution is

$$\Omega(y,z) = \frac{-y^\eta z^\eta}{(\Gamma(\eta+1))^2} + \frac{2z^{2\eta}}{\Gamma(2\eta+1)} + \frac{y^{2\eta}z^\eta - y^\eta z^{2\eta}}{\Gamma(2\eta+1)\Gamma(\eta+1)} + \frac{z^{3\eta} - y^{3\eta}}{\Gamma(3\eta+1)}.$$

The maximum absolute errors in approximate solution for $L = s = 6$ are reported in Tables 1 for $\eta = \frac{1}{k}$ for $k = 1,2,...,10$ where

$$error_{max} = max\{|\Omega(y,z) - \Omega_{\eta,\frac{1}{5}}(y,z)| : y,z \in \{0,0.1,...,1\}\}.$$

Table 1: Error in Example 1

η	$error_{max}$
$\frac{1}{10}$	0
$\frac{2}{10}$	$1.2E-15$
$\frac{3}{10}$	$1.3E-15$
$\frac{4}{10}$	$1.5E-15$
$\frac{5}{10}$	$1.7E-15$
$\frac{6}{10}$	$1.9E-15$
$\frac{7}{10}$	$2.1E-15$
$\frac{8}{10}$	$2.2E-15$
$\frac{9}{10}$	$2.4E-15$
1	$2.7E-15$

The graphs of the exact and approximate solutions for $L = s = 6$ and $\eta = \frac{1}{3}, \frac{2}{3}, 1$ are presented in Figures 5-7.

Example 2: Consider the following problem [22]

$$D_z^\eta \Omega(y,z) + D_y^\eta \Omega(y,z) - D_z^{2\eta} \Omega(y,z) = 0$$

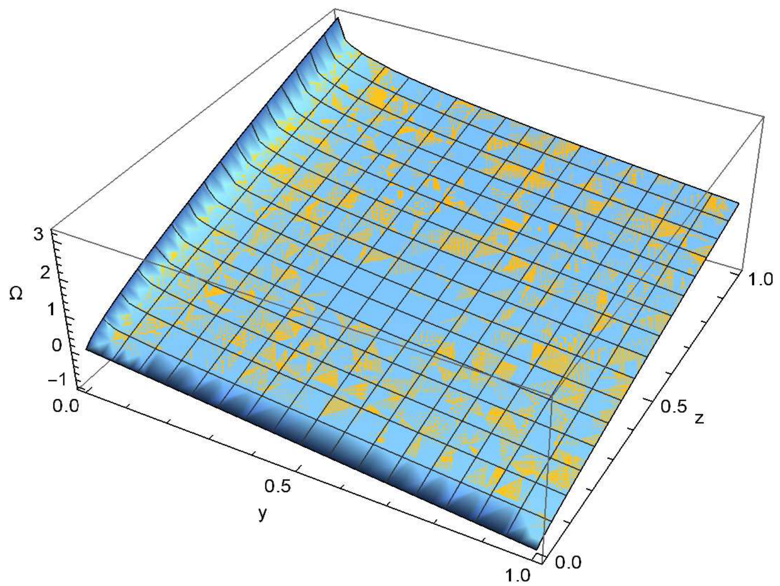

Fig. 5: Exact and approximate solutions for $\eta = \frac{1}{3}$ in Example 1.

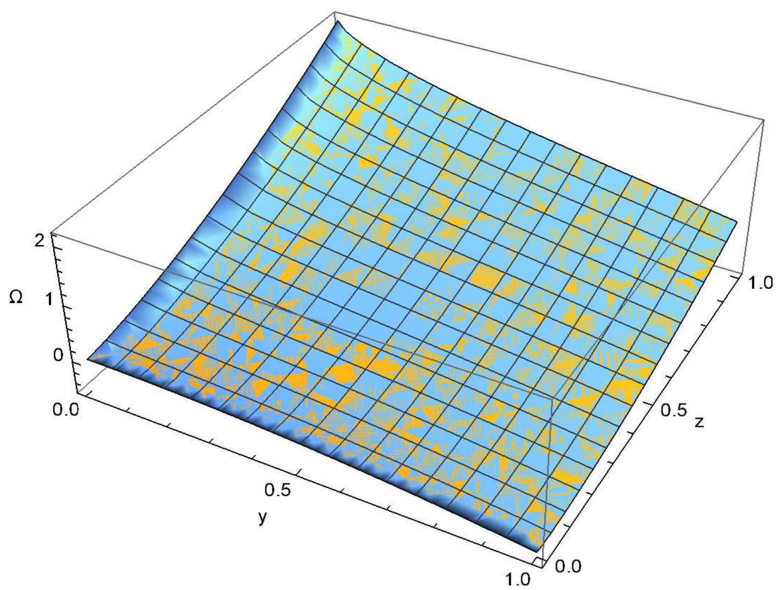

Fig. 6: Exact and approximate solutions for $\eta = \frac{2}{3}$ in Example 1.

with

$$\Omega(y,0) = \frac{y^{2\eta}}{\Gamma(2\eta+1)}$$

where $\Omega(y,z)$ is the external potential and $D_z^{\eta}\Omega(y,z)$ and $D_z^{2\eta}\Omega(y,z)$ represent the negative external forces in the system. The exact solution is

$$\Omega(y,z) = \frac{-y^{\eta}z^{\eta}}{(\Gamma(\eta+1))^2} + \frac{z^{2\eta}}{\Gamma(\eta+1)} + \frac{y^{2\eta}+z^{2\eta}}{\Gamma(2\eta+1)}.$$

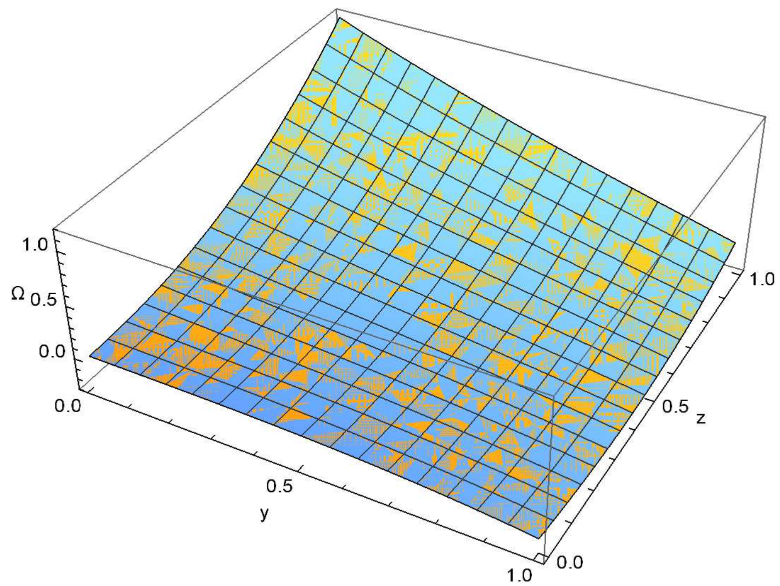

Fig. 7: Exact and approximate solutions for $\eta = 1$ in Example 1.

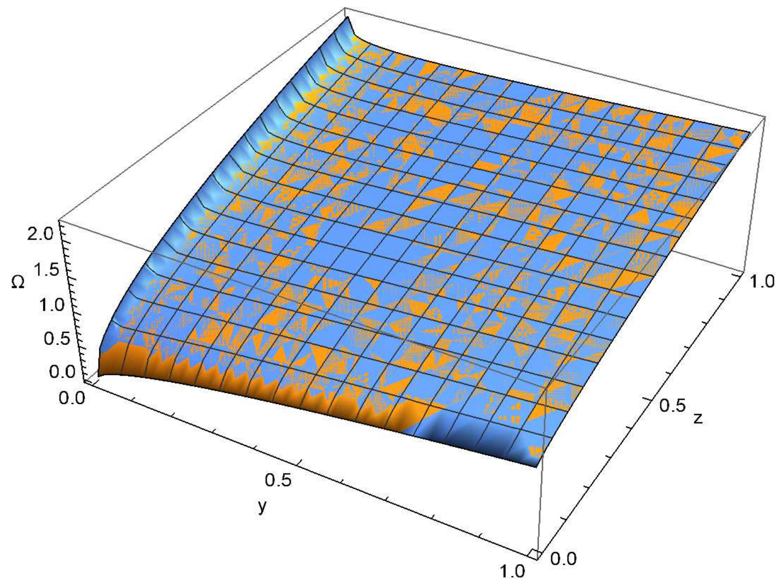

Fig. 8: Exact and approximate solutions for $\eta = \frac{1}{3}$ in Example 2.

The maximum absolute errors in approximate solution for $L = s = 6$ are reported in Tables 2 for $\eta = \frac{1}{k}$ for $k = 1,2,...,10$ where

$$error_{max} = max\{|\Omega(y,z) - \Omega_{\eta,\frac{3}{4}}(y,z)| : y,z \in \{0,0.1,...,1\}\}.$$

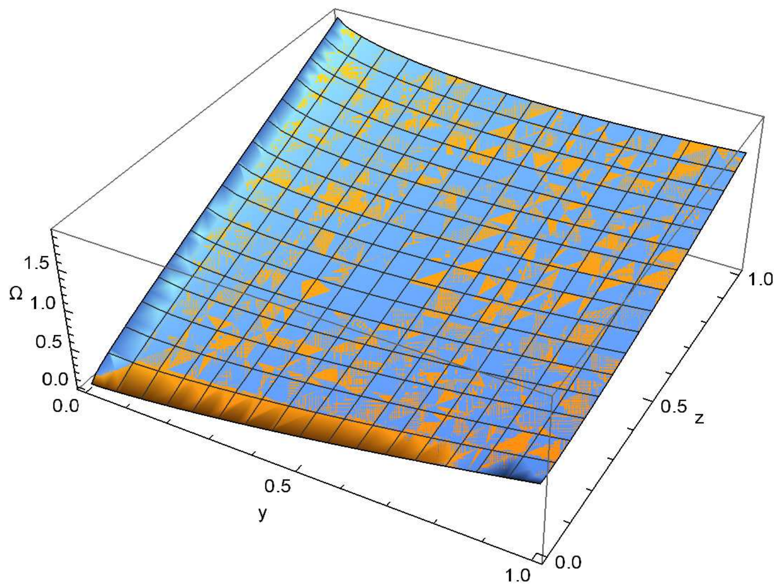

Fig. 9: Exact and approximate solutions for $\eta = \frac{2}{3}$ in Example 2.

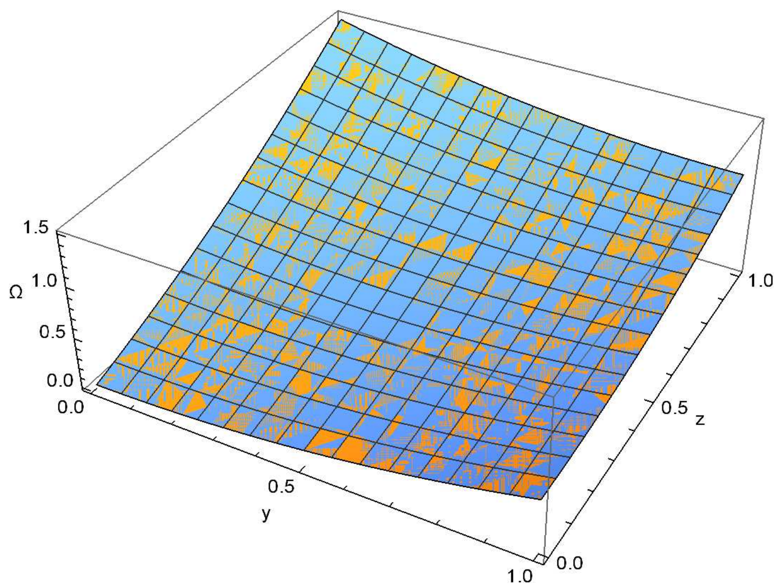

Fig. 10: Exact and approximate solutions for $\eta = 1$ in Example 2.

Table 2: Error in Example 2

η	$error_{max}$
$\frac{1}{10}$	0
$\frac{2}{10}$	$1.1E-16$
$\frac{3}{10}$	$1.2E-15$
$\frac{4}{10}$	$1.4E-15$
$\frac{5}{10}$	$1.5E-15$
$\frac{6}{10}$	$1.7E-15$
$\frac{7}{10}$	$1.8E-15$
$\frac{8}{10}$	$2.0E-15$
$\frac{9}{10}$	$2.2E-15$
1	$2.3E-15$

The graphs of the exact and approximate solutions for $L = s = 6$ and $\eta = \frac{1}{3}, \frac{2}{3}, 1$ are presented in Figures 8-10.

5 Conclusion

The B-Spline method is used to solve the following FFPE

$$D_z^{\eta}\Omega(y,z) + D_y^{\eta}\Omega(y,z) - D_y^{2\eta}\Omega(y,z) = 0, \tag{20}$$

$$\Omega(y,0) = h(y) \tag{21}$$

where $y,z > 0$ and $\eta \in (0,1]$. Theoretical and numerical results are given. Two numerical examples are studied to illustrate the efficiency of the B-Spline series method. Tables (1)-(2) shows that the approximate solution is very close to the exact solution. In addition, Figures (5)-(10) give strong evidences of the efficiency of the proposed method. For the future work, we will use the proposed method to study more applications in physics and engineering.

References

[1] M. Syam, On approximate solutions of Fokker Planck equation by the method of modified residual power series method, *J. Fract. Calc. App.* **10**(2), 225-232 (2019).

[2] T. Omay and D. Baleanu, Fractional unit-root tests allowing for a fractional frequency flexible Fourier form trend: predictability of Covid-19, *Adv. Differ. Equ.* **2021**(1), 167 (2021).

[3] A. Alshomrani, M. Ullah and D. Baleanu, Caputo SIR model for COVID-19 under optimized fractional order, *Adv. Differ. Equ.* **2021**(1), 185 (2021).

[4] S. Ahmad, R. Ullah and D. Baleanu, Mathematical analysis of tuberculosis control model using nonsingular kernel type Caputo derivative, *Adv. Differ. Equ.* **2021**(1), 26 (2021).

[5] T. Abdeljawad and D. Baleanu, Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel, *Chaos Solit. Fract.* **102**, 106-110 (2017).

[6] M. Abdelhakem, D. Mahmoud, D. Baleanu and M. El-kady, Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problem, *Adv. Differ. Equ.* **2021**(1), 110 (2021).

[7] M. Syam and M. Al-Refai, First order delay equations implicit hybrid method, *Alexandria Engn. J.* **(59)**(4), 2677-2681 (2020).

[8] B. Kashkari and M. Syam, A numerical approach for investigating a special class of fractional Riccati equation, *Res. Phys.* **17**, 103080, (2020).

[9] A. K. Alomari, M. Syam, N. R. Anakira, A. F. Jameel, Homotopy Sumudu transform method for solving applications in physics, *Res. Phys.* **18**, 103265 (2020).

[10] A. Coronel-Escamilla and J. Gómez-Aguilar, A novel predictor–corrector scheme for solving variable-order fractional delay differential equations involving operators with Mittag-Leffler kernel, *Disc. Cont. Dyn. Syst. Ser. S* **13**(3), 561-574 (2020).

[11] B. Kashkari and M. Syam, Fractional-order Legendre operational matrix of fractional integration for solving the Riccati equation with fractional order,*Appl. Math. Comput.* **290**, 281-291 (2016).

[12] S. Shahmorad, M. H. Ostadzad and D. Baleanu, A Tau-like numerical method for solving fractional delay integro-differential equations, *Appl. Numer. Math.* **151**, 322-336 (2020).

[13] H. Risken, *The Fokker–Planck equation: methods of solution and application*, Springer, Berlin, 1984.

[14] W. G. Hoover, D. J. Evans, R. B. Hickman, A. J. C. Ladd, W. T. Ashurst and B. Moran, Lennard-jones triple-point bulk and shear viscosities. Green-Kubo theory, Hamiltonian mechanics, and nonequilibrium molecular dynamics, *Phys. Rev. A* **22**(4), 1690-1697 (1980).

[15] B.J. Edwards and M. Dressler, A reversible problem in non-equilibrium thermodynamics: Hamiltonian evolution equations for non-equilibrium molecular dynamics simulations, *J. Non-Newton. Fluid Mech.* **96**(1–2), 163-175 (2001).
[16] G. M. Leonenko and T. N. Phillips Numerical approximation of high-dimensional Fokker-Planck equations with polynomial coefficients, *J. Comp. Appl. Math.* **273**, 296-312 (2015).
[17] B. D. Shizgal, Pseudospectral method of solution of the Schrödinger equation with non classical polynomials; the Morse and Pöschl-Teller (SUSY) potentials, *Comp. Theor. Chem.* **1084**, 51-58 (2016).
[18] B. Attili and M. Syam, Efficient shooting method for solving two point boundary value problems, *Chaos Solit. Fract.* **35**(5), 895–903 (2008).
[19] M. Syam, The modified Broyden-variational method for solving nonlinear elliptic differential equations, *Chaos Solit. Fract.* **32**(2), 392–404 (2007).
[20] Q. Zhang, X. Wang and Z. Sun, The pointwise estimates of a conservative difference scheme for Burgers' equation, *Numer. Meth. Part. Differ. Equ.* **10.1002**, 22494 (2020).
[21] L. N. Ji, C.Z. Qu and S.F. Shen, Conditional Lie-Bäcklund symmetry of evolution system and application for reaction-diffusion system, *Stud. Appl. Math.* **133**, 118-149 (2014).
[22] M. Brics, J. Kaupuzs and R. Mahnke How to solve the Fokker-Planck equation treating mixed eigenvalue spectrum?, *Cond. Matt. Phys.* **16**, 1-13 (2013).
[23] G. S. V. Krishna Mohan et al., An efficient design of finite impulse response — Fractional-order differentiator using shuffled frog leaping algorithm heuristic, *Int. J. Wavel., Multires. Inf. Pro.*, (2019).