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Modified Split-Step Theta Milstein Methods for M-Dimensional Stochastic Differential Equation With Respect To Poisson-Driven Jump

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Abstract: Recently, split-step techniques have been integrated with a Milstein scheme to improve the fundamental analysis of numerical solutions of stochastic differential equations (SDEs). Unfortunately, we note that stability conditions of these methods have restrictions on parameters and step-size to preserve mean-square stability and A-stability of SDEs. We construct new general modified spit-step theta Milstein (MSSTM) methods for using on multi-dimensional SDEs in order to overcome these restrictions. We investigate that the numerical methods are mean-square (MS) stable with no restrictions on parameters for all step-size $h > 0$ when $\theta \in [1/2, 1]$ and it is proved that the methods with $\theta \geq 1/2$ are stochastically A-stable. Furthermore, there is a gap in discussing the split-step Milstein type methods for SDEs with Jump in the literature. Here, we extend the new general methods for SDEs with jump called compensated MSSTM (CMSSTM) methods. The unconditional MS-stability results of CMSSTM methods are proved for SDEs with Poisson-driven jump. Finally, several examples are given to show the effectiveness of the proposed method in approximation of one and two dimensional SDEs compared to some existing methods.

Keywords: Stochastic differential equations, Poisson-driven jump, Spit-step theta Milstein, Convergence, Stability, M-Dimensional

1 Introduction

The m-dimensional Itô stochastic differential equations (SDEs) are considered of the form [18]

$$\begin{aligned} dZ(t) &= f(t,Z(t))dt + \sum_{j=1}^{m} g_j(t,Z(t))dW_j(t), \\ Z(t_0) &= Z_0, \end{aligned} \tag{1}$$

where $f(t,Z(t))$ is the drift coefficient, $g_j(t,Z(t))$ are the diffusion coefficients and $t \in [t_0,T]$. The process $W_j = \{W_j(t) : t \geq 0\}$, $j = 1,\ldots,m$ represents independent Wiener processes on a filtered probability space $(\Omega,\mathscr{F},P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ under the usual conditions.

The SDEs are widely used to simulate many phenomena in biology, financial engineering, neural network and wireless communications [4, 6, 23, 30, 37]. Since it's not easy to find the analytical solutions to SDEs, the interest in numerical solutions has been increased.

The well-known Euler-Maruyama (EM) method for SDEs (1) was presented with convergence order 0.5. In order to improve the properties of numerical methods based on EM scheme, the split-step technique has been provided by Higham et al. [13]. They derived the split-step backward Euler (SSBE) method for (1) with a single noise channel ($m = 1$). Based on Higham's work, many split-step methods were provided, for example the split-step theta (SST) methods which generalize the SSBE method when $\theta = 1$ [17]. Although some numerical methods based on the Euler-scheme are general mean-square stable (MS-stable) and A-stable, these methods converge with order 0.5. To improve the convergence properties of numerical methods, the Milstein scheme was presented with strong convergence order 1.0, by the additional term of the Itô-Taylor

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expansion, as follows

$$\begin{aligned} y_{n+1} &= y_n + hf(t_n,y_n) + \sum_{j=1}^{m} g_j(t_n,y_n)\Delta W_j(t_n) \\ &+ \frac{1}{2}\sum_{j=1}^{m} L^j g_j(t_n,y_n)\left[(\Delta W_j(t_n))^2 - h\right] \\ &+ \sum_{i,j=1,i\neq j}^{m} L^i g_j(t_n,y_n) I_{(i,j)}^{(t_n,t_{n+1})}, \end{aligned} \tag{2}$$

where

$$I_{(i,j)}^{(s,t)} = \int_s^t \int_s^u \mathrm{d}W_i(v)\mathrm{d}W_j(u) \quad \text{and} \quad L^j = \sum_{k=1}^{d} g_k^j \frac{\partial}{\partial y_k},$$

The fundamental properties contain convergence of numerical methods based on a Milstein scheme for SDEs (1) with single and multi noise were discussed (See, [5, 7–9, 24, 27, 30–35, 38]). Stability properties are one of the important tools measure the numerical methods quality. To get insight into stability behavior of the numerical methods for SDEs (1), the scalar equation with a single noise channel ($m = 1$) has been used to compare the stability region of the proposed method with that of existing numerical methods

$$\begin{aligned} \mathrm{d}Z(t) &= aZ(t)\mathrm{d}t + bZ(t)\mathrm{d}W(t), \qquad t > 0, \\ Z(t_0) &= Z_0, \end{aligned} \tag{3}$$

where $a, b \in \mathbb{R}$. Saito et al. [25] proposed the MS-stability concept of numerical methods for SDEs (3). Though Euler-type methods always possess bigger stability regions than their Milstein type counterparts [1], the Milstein scheme has been interest because of higher convergence order. While considering fully implicit methods for SDEs, the problem appears in the implicit stochastic terms as the unboundedness of the diffusion term, leads to instability. To overcome the drawback of the fully implicit methods, the idea to combine semi-implicit method with implicit ones was presented in [2]. Based on this technique, Higham [12] discussed the benefits of semi-implicit Milstein scheme with $\theta > 1$ in terms of the stability for (3). In addition, Omar et al. [22] provided the composite Milstein methods for SDEs. Recently, the split-step technique has presented by Higham [13], who can be considered a pioneer of driving split-step numerical methods for SDEs. With respect to using split-step techniques, split-step Milstein type methods can be classified in two families: the split-step Milstein methods, and modified split-step Milstein methods. According to this classification the split-step Milstein type methods have been discussed (for example, [27, 31–34]). In addition, based on Split-Step Theta (SST) methods which have advantages in flexibility and stability [17], Zong et al. [38] introduced the split-step theta-Milstein (SSTM) methods. The MS-stability of SSTM methods with $\theta \in [0,1]$ was discussed. In our previous work [7], we derived the stability functions of the drifting split-step theta Mistien (DSSTM) method, then proved that the methods with $\theta \geq \frac{3}{2}$ are stochastically A-stable. On the other side, the modified split-step Milstein type methods were introduced based on collecting all deterministic terms in the first splitting step of the method using the fact that $I_{(j,j)} = \frac{1}{2}[(\Delta W_n)^2 - h]$ in (2). The modified split-step Milstein type methods have been presented for SDEs (See, [24, 32, 34, 35]). *Despite using different techniques in order to improve stability properties and stability regions of Milstein type methods, we can see that the existing numerical methods based on Milstein scheme still have some restrictions on the parameters and step-size to be A-stable.*

As the first contributions of this work, we are interested in improving the existing results in [7, 38] which discussed DSSTM methods for SDEs to remove the existing restrictions on the parameters and step-size. Following, Wang et al. [34], we construct the new general methods; the modified spit-step theta Milstein (MSSTM) methods with strong convergence order 1.0 for solving SDEs (1). The proposed methods are derived based on the split-step technique by combining the SST method with Milstein scheme. Then, the MS-stability is investigated for linear SDEs (3). We prove that the proposed methods MSSTM can share the MS-stability of the exact solution for all step-size with no restrictions on parameters.

Stochastic differential equations with jump (SDEJ) used to model real-world phenomena in different fields. One of the important tools to measure the quality of numerical methods to approximate SDEJ, is stability. Based on the Euler scheme with convergence order 0.5, the MS-stability analysis of the implicit Euler method for linear SDEs with Poisson-driven jumps was discussed [15]. Wang and Gan [36] investigated MS-stability of the compensated stochastic theta methods as an extension of the deterministic A-stability property for a linear test equation with $\theta \in [\frac{1}{2}, 1]$. Using the split-step techniques, Higham and Kloeden [14] discussed the MS-stability of the split-step backward Euler (SSBE) method and the compensated SSBE. Furthermore, compensated split-step theta methods have been discussed for SDEJ in [17, 29]. Based on the Milstein scheme with convergence order 1.0, Hu et al. [16] discussed the Milstein method to approximate the solution of a linear SDEs with Poisson-driven jumps. Furthermore, they showed that the Milstein methods can reproduce the stochastically asymptotical stability for the SDEJ. However, we need more work to remove the restrictions on the parameters and step-size of the existing numerical methods based on Milstein scheme to be A-stable for SDEJ.

As the main contributions of this work, we extend the MSSTM methods for SDEs with Poisson jump (SDEwPJ) which has been called the compensated modified spit-step theta Milstein (CMSSTM) methods. Get insight into the stability behavior of the numerical methods, the

MS-stability is investigated for linear SDEwPJ, as follows: (a) For $\theta \in [\frac{1}{2}, 1]$, the CMSSTM methods are general mean-square (GMS) stable (i.e. the CMSSTM methods can share the MS-stability of the exact solution for all step-size with no restrict on parameters), (b) If $\theta \in [0, \frac{1}{2})$, the CMSSTM methods are MS-stable for all step-size with restrictions on parameters. In the last part of this work, stability regions are discussed to explain that the proposed methods are A-stable. In addition, several examples are given to show the effectiveness of the proposed method in approximation of one and two dimensional SDEs compared to some existing methods.

The paper is organized, as follows: In Section 2, we present some necessary notations and preliminaries. In Section 3, the MSSTM methods are derived and the strong convergence order 1.0 is proved. The MS-stability properties of the MSSTM and CMSSTM methods are considered for linear SDEs and SDEwPJ, respectively. Furthermore, stability regions are discussed to explain that the proposed methods are A-stable in Section 4. In Section 5, numerical results for the convergence and stability are given in one and two dimensional SDEs to demonstrate the properties of proposed methods compared with existing methods. Section 6 is dedicated to Conclusion .

2 Notations and preliminaries

Throughout this paper, we use the following notations. Let $(\Omega, \mathscr{F}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$, which satisfies the usual conditions, i.e. the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ is right-continuous and each $\{\mathscr{F}_t\}, t \geq 0$, contains all P-null sets in $\mathscr{F}$. Let $W(t)$, $t \geq 0$, be $\mathscr{F}_t$-adapted and independent of $\mathscr{F}_0$. $|\cdot|$ is the Euclidean norm in $\mathbb{R}^m$. $a \bigvee b$ presents $\max(a,b)$ and $a \bigwedge b$ presents $\min(a,b)$. Moreover, we assume Z_0 be $\mathscr{F}_0$-measurable and $E(Z_0)^2 < \infty$. Let $f, g : \mathbb{R}^m \to \mathbb{R}^m$ be Borel measurable functions. Let us consider the m-dimensional SDEs (1). For the convenience of readers to accomplish the existence and convergence results, we impose some assumptions on f and g as in [34] in this section.

Assumption 1*The functions $f(t,x)$ and $g(t,x)$ in SDEs (1) satisfy Lipschitz condition. There exists a constant k_1 such that for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^m$*

$$|f(t,x) - f(t,y)|^2 \vee |g(t,x) - g(t,y)|^2 \leq k_1|x-y|^2, \quad (4)$$

and Linear growth condition. There is a constant k_2

$$|f(t,x)|^2 \vee |g(t,x)|^2 \leq k_2(1+|x|^2). \quad (5)$$

We note that from ([19], Theorem 3.1), therein $f, g \in C^1$ ensure the existence of a unique solution to the SDEs (1) under the Assumption 1. Furthermore, the following assumption for the diffusion function $g(t,x)$ will be used in the following sections.

Assumption 2*The functions $g_j(t,x)$ for all $j = 1, \dots, m$ in (1) also satisfy*

$$\left|L^j g_j(t,x) - L^j g_j(t,y)\right|^2 \leq k_1|x-y|^2, \quad (6)$$

and

$$\left|L^j g_j(t,x)\right|^2 \leq k_2(1+|x|^2), \quad (7)$$

for all $x, y \in \mathbb{R}^d$, $t \in [t_0, T]$, where k_1, k_2 are defined in Assumption 1.

Saito and Mitsui [25] derived a condition that characterizes the MS-stability of linear SDEs (3) as follows

Lemma 1.*If the constants a,b in (3) satisfy*

$$a < -\frac{1}{2}b^2, \quad (8)$$

then the solution of (3) is asymptotically stable in the mean-square sense, that is,

$$\lim_{t\to\infty} E|y(t)|^2 = 0. \quad (9)$$

To be precise, we state the definitions of the numerical stability in the following.

Definition 1. *[19] Under the exact solution stability condition (8), a numerical method is said to be MS-stable, if there exists a $h^* > 0$, such that any application of the method to (3) generates numerical approximation y_n, which satisfy*

$$\lim_{n\to\infty} E|y_n|^2 = 0, \quad (10)$$

for all step-size $h \in (0, h^*)$.

Definition 2. *[19] Under the exact solution stability condition (8), a numerical method is said to be general mean-square (GMS) stable, if any application of the method to (3) generates numerical approximation y_n, which satisfy*

$$\lim_{n\to\infty} E|y_n|^2 = 0, \quad (11)$$

for all step-size $h > 0$.

3 The MSSTM methods

The main idea to derive the DSSTM methods for SDEs (1) with a single noise channel ($m = 1$) was presented by applying the SST methods to the drift part and the Milstein method to the diffusion part (For more details see [7, 8, 38]). The fundamental properties implicate convergence and stability have been discussed for the different types of split-step Milstein methods [5, 7, 8, 11, 27, 31, 32, 35, 38]. Our analysis is motivated by convergence order in multi-dimensional case and GMS-stability and A-stability with no restrictions on the parameters and step-size. In this section, we state the outline to construct the new MSSTM methods for more general systems. Furthermore, the strong convergence order 1.0 of the proposed methods is discussed.

To derive the MSSTM methods for more general systems, the m-dimensional stochastic process $Z=\{Z(t):t\geq 0\}$ with $E(Z_0)^2<\infty$ satisfies the m-dimensional SDEs (1) is considered as

$$\mathrm{d}Z(t)=f(t,Z(t))\mathrm{d}t+\sum_{j=1}^{m} g_j(t,Z(t))\mathrm{d}W_j(t),$$
$$Z(t_0)=Z_0,$$

where $t\in[t_0,T]$, $f(t,Z(t))$ is the drift coefficient, $g_j(t,Z(t))$ are the diffusion coefficients which are m-dimensional function satisfied Lipschitz conditions and linear growth conditions in Assumptions 1 and 2. The process $W_j=\{W_j(t):t\geq 0\}$, $j=1,\ldots,m$ represents independent Wiener processes on a filtered probability space $(\Omega,\mathscr{F},P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ under the usual conditions. For $0\leq s\leq t<\infty$ and $y\in\mathbb{R}^m$, the variable $Z_t^{s,y}$ denotes the value of a solution of (1) at time t which starts in $y\in\mathbb{R}^m$ at time s.

To improve the fundamental analysis (specially the stability) of the drifting split-step backward Milstein (DSSBM) method, Wang et al. [34] derived the modified split-step backward Milstein (MSSBM) methods with convergence order 1.0 for SDEs (1.1) with $m=1$. Following [34], using SST methods on the drift function instead of the backward Euler method in MSSBM methods, we can derive a new general family of MSSTM methods. A MSSTM methods applied to (1) can be written in the general form as follows

$$y_n^*=y_n+h\left[\theta f(t_n,y_n^*)-\frac{1}{2}\sum_{j=1}^{m}L^j g_j(t_n,y_n^*)\right], \tag{12}$$

$$\begin{aligned} y_{n+1}=&\, y_n^*+(1-\theta)hf(t_n,y_n^*)+\sum_{j=1}^{m} g_j(t_n,y_n^*)\Delta W_j(t_n)\\ &+\frac{1}{2}\sum_{j=1}^{m}L^j g_j(t_n,y_n^*)(\Delta W_j(t_n))^2\\ &+\sum_{i,j=1,i\neq j}^{m} L^i g_j(t_n,y_n^*)I_{(i,j)}^{(t_n,t_{n+1})}, \end{aligned} \tag{13}$$

where

$$I_{(i,j)}^{(s,t)}=\int_s^t\int_s^u \mathrm{d}W_i(v)\mathrm{d}W_j(u) \quad and \quad L^j=\sum_{k=1}^{d} g_k^j\frac{\partial}{\partial y_k},$$

and y_n is an approximation to $Z(t_n)$, $\theta\in[0,1]$, with increments $\Delta W_j(t)=W_j(t_{n+1})-W_j(t_n)$ are independent $N(0,h)$-distributed Gaussian random variables, $h=t_{n+1}-t_n, n=0,1,\ldots,N-1$ and $y(0)=y_0$. Moreover, y_n is $\{\mathscr{F}_{t_n}\}$-measurable at the mesh-point t_n.

Remark. Let $m=1$, the MSSTM methods (12-13) with parameter ($\theta=1$) reduce the MSSBM methods which derived in [34], while the MSSTM methods (12-13) with parameter ($\theta=0$) are the modified forward Milstein (MSSFM) methods which derived in [26].

In order to state the convergence theorem for the general MSSTM methods (12-13) for SDEs (1), we recall the following theorem concerning the strong convergence order (See, [20, 21]).

Theorem 1. *Assume that for one-step discrete time approximation y_n, the local mean error and mean-square error for all $N=1,2,\ldots$, and $n=1,2,\ldots,N-1$ satisfy the estimates*

$$\left|E\left[y_{n+1}-Z_{t_{n+1}}^{t_n,y_n}|\mathscr{F}_{t_n}\right]\right|\leq K\left(1+|y_n|^2\right)^{1/2}h^{p_1}, \tag{14}$$

$$\left(E\left[\left|y_{n+1}-Z_{t_{n+1}}^{t_n,y_n}\right|^2|\mathscr{F}_{t_n}\right]\right)^{1/2}\leq K\left(1+|y_n|^2\right)^{1/2}h^{p_2}, \tag{15}$$

with $p_2\geq\frac{1}{2}$ and $p_1\geq p_2+\frac{1}{2}$. Then

$$\left(E\left[\left|y_k-Z_{t_k}^{0,Z_0}\right|^2|\mathscr{F}_0\right]\right)^{1/2}\leq K\left(1+|Z_0|^2\right)^{1/2}h^{p_2-1/2}, \tag{16}$$

holds for each $k=0,1,2,\ldots,N$. Here K is independent on h but dependent on the length of the time interval $T-t_0$.

The following theorem provides the strong convergence order 1.0 of the MSSTM methods (12-13) for SDEs (1).

Theorem 2. *Under Assumptions 1 and 2, the approximation solution y_n of the MSSTM methods (12-13) converges with strong order 1.0, that is for all $k=0,1,\ldots,N$, $0<\theta\leq 1$ and $0<h<1/(\theta\sqrt{k_1})$*

$$\left(E\left[\left|y_k-Z_{t_k}\right|^2|\mathscr{F}_0\right]\right)^{1/2}=K\left(1+|Z_0|^2\right)^{1/2}h, \tag{17}$$

where K is independent on h.

Proof. First, in order to show (14) holds for the MSSTM methods (12-13) with $p_1=2$, using (12-13) we have

$$\begin{aligned} y_{k+1}=&\, y_k+hf(t_k,y_k^*)+\sum_{j=1}^{m} g_j(t_k,y_k^*)\Delta W_j(t_k)\\ &+\frac{1}{2}\sum_{j=1}^{m}L^j g_j(t_k,y_k^*)\left[(\Delta W_j(t_k))^2-h\right]\\ &+\sum_{i,j=1,i\neq j}^{m} L^i g_j(t_k,y_k^*)I_{(i,j)}^{(t_k,t_{k+1})}, \end{aligned} \tag{18}$$

and let y_k^M be the approximation of the Milstein method, i.e.

$$\begin{aligned} y_{k+1}^M=&\, y_k^M+hf(t_k,y_k^M)+\sum_{j=1}^{m} g_j(t_k,y_k^M)\Delta W_j(t_k)\\ &+\frac{1}{2}\sum_{j=1}^{m}L^j g_j(t_k,y_k^M)\left[(\Delta W_j(t_k))^2-h\right]\\ &+\sum_{i,j=1,i\neq j}^{m} L^i g_j(t_k,y_k^M)I_{(i,j)}^{(t_k,t_{k+1})}, \end{aligned} \tag{19}$$

$k = 0,1,\dots,N-1$ is introduced and referring to [34] with $n = 0,1,\dots,N-1$, the local errors are given by

$$\left|E\left[y^M_{n+1} - Z^{t_n,y_n}_{t_{n+1}}|\mathscr{F}_{t_n}\right]\right| = O(h^2), \tag{20}$$

$$\left(E\left[\left|y_{n+1} - Z^{t_n,y_n}_{t_{n+1}}\right|^2|\mathscr{F}_{t_n}\right]\right)^{1/2} = O(h^{3/2}). \tag{21}$$

Consider the local mean error with $y_n = Z_{t_n}$ i.e.

$$\begin{aligned} H_1 &= \left|E\left[y_{n+1} - Z^{t_n,y_n}_{t_{n+1}}|\mathscr{F}_{t_n}\right]\right| \\ &= \left|E\left[y^M_{n+1} - Z^{t_n,y_n}_{t_{n+1}}|\mathscr{F}_{t_n}\right] + E\left[y_{n+1} - y^M_{n+1}|\mathscr{F}_{t_n}\right]\right| \\ &\le K(1+|y_n|^2)^{1/2}h^2 + H_2, \end{aligned} \tag{22}$$

from (18) and (20), H_2 is defined by

$$\begin{aligned} H_2 &= \left|E\left[y_{n+1} - y^M_{n+1}|\mathscr{F}_{t_n}\right]\right| \\ &= |E[h(f(t_n,y^*_n) - f(t_n,y_n)) \\ &+ \sum_{j=1}^m (g_j(t_n,y^*_n) - g_j(t_n,y_n))\,\Delta W_j(t_n) \\ &+ \frac{1}{2}\sum_{j=1}^m \left(L^j g_j(t_n,y^*_n) - L^j g_j(t_n,y_n)\right)(\Delta W_j(t_n))^2 \\ &- \frac{1}{2}h\sum_{j=1}^m \left(L^j g_j(t_n,y^*_n) - L^j g_j(t_n,y_n)\right) \\ &+ \sum_{i,j=1,i\neq j}^m \left(L^i g_j(t_n,y^*_n) - L^i g_j(t_n,y_n)\right) I^{(t_n,t_{n+1})}_{(i,j)}|\mathscr{F}_{t_n}]|. \end{aligned} \tag{23}$$

In view of the independence of y_n, y^*_n, the symmetric property of $\Delta W_j(t_n), j = 1,\dots,m$ in those expressions involving this zero-mean Gaussian variable, the property of Itô integrals $E(I_{(i,j)}) = 0$, and Assumptions 1, 2, we obtain

$$H_2 \le \sqrt{k_1}h\,|E[y^*_n - y_n|\mathscr{F}_{t_n}]|\,. \tag{24}$$

On the other hand

$$\begin{aligned} |E[y^*_n - y_n|\mathscr{F}_{t_n}]| &= |E[\theta h f(t_n,y^*_n) \\ &- \frac{1}{2}h\sum_{j=1}^m L^j g_j(t_n,y^*_n)|\mathscr{F}_{t_n}]| \\ &\le \frac{(\theta+\frac{1}{2})h\sqrt{k_2}}{(1-\theta h\sqrt{k_1})}\left(1+|y_n|^2\right)^{1/2}. \end{aligned} \tag{25}$$

Substituting (25) in (24), we obtain

$$\begin{aligned} H_2 &\le \frac{(\theta+\frac{1}{2})\sqrt{k_1k_2}}{(1-\theta h\sqrt{k_1})}\left(1+|y_n|^2\right)^{1/2}h^2 \\ &\le K\left(1+|y_n|^2\right)^{1/2}h^2, \end{aligned} \tag{26}$$

which yields

$$H_1 = \left|E\left[y_{n+1} - Z^{t_n,y_n}_{t_{n+1}}|\mathscr{F}_{t_n}\right]\right| = O(h^2). \tag{27}$$

From (Kloeden and Platen [18], Lemma 5.7.2), it is known that $E\left[(\Delta W_j(t_n))^2|\mathscr{F}_{t_n}\right] \le O(h)$ and $E\left[I^2_{(i,j)}|\mathscr{F}_{t_n}\right] \le O(h^2)$. Similarly, the local mean-square error of the MSSTM methods for $n = 0,1,\dots,N-1$ by standard argument

$$\begin{aligned} H_3 &= \left(E\left[\left|y_{n+1} - Z^{t_n,y_n}_{t_{n+1}}\right|^2|\mathscr{F}_{t_n}\right]\right)^{1/2} \\ &\le \left(E\left[\left|y^M_{n+1} - Z^{t_n,y_n}_{t_{n+1}}\right|^2|\mathscr{F}_{t_n}\right]\right)^{1/2} \\ &+ \left(E\left[\left|y_{n+1} - y^M_{n+1}\right|^2|\mathscr{F}_{t_n}\right]\right)^{1/2} \\ &\le K\left(1+|y_n|^2\right)^{1/2}h^{3/2}. \end{aligned} \tag{28}$$

for $1-\theta h\sqrt{k_1} > 0$. With respect to (21) and Theorem 1, the proof is completed.

4 Mean-square Stability

4.1 Stability of MSSTM methods for SDEs

In this section, the Itô scalar linear SDEs (3) are concerned as

$$\begin{aligned} \mathrm{d}Z(t) &= aZ(t)\mathrm{d}t + bZ(t)\mathrm{d}W(t), \qquad t > 0, \\ Z(t_0) &= Z_0, \end{aligned}$$

to discuss stability properties of the MSSTM methods in order to compare the stability region of the proposed method with that of existing numerical methods. We prove that the MSSTM methods (12-13) with $\theta \in [\frac{1}{2},1]$ are GMS-stable for (3) (i.e. the numerical methods can share the MS-stability of the exact solution with no restrictions on parameter and step-size).

The MSSTM methods (12-13) for linear SDEs (3) have the form (3)

$$y^*_n = y_n + h\theta a y^*_n - \frac{1}{2}hb^2y^*_n, \tag{29}$$

$$y_{n+1} = y^*_n + (1-\theta)hay^*_n + by^*_n\Delta W_n + \frac{1}{2}b^2y^*_n(\Delta W_n)^2. \tag{30}$$

The following theorem gives the main results of MSSTM methods.

Theorem 3. *Suppose that MS-stability condition (8) holds, we have*

1. *If $\theta \in [\frac{1}{2},1]$, then the MSSTM methods are GMS-stable for all $h > 0$.*
2. *If $\theta \in [0,\frac{1}{2})$ and $a^2 \le \frac{b^2(a+\frac{1}{2}b^2)}{(2\theta-1)}$, then the MSSTM methods are MS-stable for all $h > 0$.*
3. *If $\theta \in [0,\frac{1}{2})$ and $a^2 > \frac{b^2(a+\frac{1}{2}b^2)}{(2\theta-1)}$, then the MSSTM methods are MS-stable for all $h \in (0,\tilde{h}(a,b,\theta))$,*

$$\tilde{h} < \frac{-(2a+b^2)}{(1-2\theta)a^2 + b^2(a+\frac{1}{2}b^2)}.$$

Proof. Note that y_n is $\mathscr{F}_{t_n}$-measurable at the mesh point t_n, we easily know from (12-13) that y_n^* is also $\mathscr{F}_{t_n}$-measurable at related meash-point, ΔW_n is independent of $\mathscr{F}_{t_n}$. From $E[\triangle W_n]=0$, $E[|\triangle W_n|^2]=h$, $E[\triangle W_n^3]=0$ and $E[|\triangle W_n|^4]=3h^2$, it is easy to see from (29-30) that

$$E(y_{n+1})^2 = P(a,b,\theta,h)E(y_n)^2, \tag{31}$$

where

$$P(a,b,\theta,h) = \frac{1}{(1-\theta ha+\frac{1}{2}hb^2)^2}((1+(1-\theta)ha)^2 + 2b^2h + \frac{3}{4}b^4h^2 + (1-\theta)h^2ab^2). \tag{32}$$

Hence $E(y_{n+1})^2 \to 0$,when $(n\to\infty)$ if and only if

$$\left((1-2\theta)a^2 + b^2(a+\frac{1}{2}b^2)\right)h + (2a+b^2) < 0. \tag{33}$$

With respect to condition (8), we know that $2a+b^2<0$ and the following results can be obtained.

Case I: If $\theta\in[\frac{1}{2},1]$, then the MSSTM methods are GMS-stable for all $h>0$.

Case II: If $\theta\in[0,\frac{1}{2})$, then $(1-2\theta)a^2>0$. Hence,

$$(1-2\theta)a^2 + b^2\left(a+\frac{1}{2}b^2\right) \tag{34}$$

If $(1-2\theta)a^2+b^2(a+\frac{1}{2}b^2)\le 0$, then $a^2 \le \frac{b^2(a+\frac{1}{2}b^2)}{2\theta-1}$. Hence, we can see that (33) holds. The MSSTM methods are MS-stable for all $h>0$.

If $(1-2\theta)a^2+b^2(a+\frac{1}{2}b^2) > 0$, then $a^2 > \frac{b^2(a+\frac{1}{2}b^2)}{2\theta-1}$. Hence, we can see that the MSSTM methods are MS-stable for $h\in(0,\tilde{h}(a,b,\theta))$, then (33) holds. The MSSTM methods are MS-stable for $h\in(0,\tilde{h}(a,b,\theta))$.

The proof is complete.

4.2 Stability of CMSSTM methods for SDEwPJ

The stability results of the MSSTM methods are derived for SDEs with Poisson-driven jump (SDEwPJ). In this work, the compensated MSSTM (CMSSTM) methods are constructed for SDEwPJ. The unconditional MS-stability results of CMSSTM methods are discussed. The SDEwPJ are considered, as follows:

$$\mathrm{d}Z(t) = f(Z(t^-))\mathrm{d}t + g(Z(t^-))\mathrm{d}W(t) + z(Z(t^-))\mathrm{d}N(t), \qquad t>0, \tag{35}$$

where $Z(0^-)=Z_0$ with $Z(t^-)$ denotes $\lim_{s\to t^-}Z(s)$, f,g and $W(t)$ are defined similarly, $z:\mathbb{R}^m\to\mathbb{R}^m$ is a given function and scalar Poisson process $N(t)$ with intensity $\lambda>0$. The SDEwPJ are considered the generalization of both deterministic and random parts. Note that, the SDEwPJ (35) have a unique solution on $[0,\infty)$, if f,g, and z satisfy the local Lipschitz condition and the linear growth condition (see, [10, 28]).

We first define the MSSTM methods for SDEwPJ (35), as follows:

$$y_n^* = y_n + h[\theta f(y_n^*) - \frac{1}{2}\acute{g}(y_n^*)g(y_n^*)], \tag{36}$$

$$y_{n+1} = y_n^* + (1-\theta)hf(y_n^*) + g(y_n^*)\Delta W_n + \frac{1}{2}\acute{g}(y_n^*)g(y_n^*)(\Delta W_n)^2 + z(y_n^*)\Delta N_n, \tag{37}$$

where y_n is approximation to $Z(t_n)$ with constant step-size h such that $t=nh$, $\theta\in[0,1]$, with increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0,h)$-distributed Gaussian random variables; $\Delta N_n := N(t_{n+1}) - N(t_n)$ are independent Poisson distributed random variables with distribution $N(\lambda h,\lambda h)$ and $y(0)=y(0^-)$.

Noting that the compensated Poisson process

$$\tilde{N}(t) := N(t) - \lambda t, \tag{38}$$

which is a martingale. Defining

$$f_\lambda := f(x) + \lambda z(x), \tag{39}$$

we can rewrite the jump-diffusion system (35) as a compensated SDEs with poisson jump in the form

$$\mathrm{d}X(t) = f_\lambda(X(t^-))\mathrm{d}t + g(X(t^-))\mathrm{d}W(t) + z(X(t^-))\mathrm{d}\tilde{N}(t). \tag{40}$$

We note that f_λ also satisfies the uniform Lipschitz condition and linear growth condition with larger constants

$$K_\lambda = 2(\lambda+1)^2k_1, \qquad L_\lambda = 2(\lambda+1)^2k_2 \tag{41}$$

Using the compensated Poisson process and following [14], we can define the CMSSTM methods for SDEwPJ (40), as follows:

$$y_n^* = y_n + h[\theta f_\lambda(y_n^*) - \frac{1}{2}\acute{g}(y_n^*)g(y_n^*)], \tag{42}$$

$$y_{n+1} = y_n^* + (1-\theta)hf_\lambda(y_n^*) + g(y_n^*)\Delta W_n + \frac{1}{2}\acute{g}(y_n^*)g(y_n^*)(\Delta W_n)^2 + z(y_n^*)\Delta\tilde{N}_n, \tag{43}$$

where $\Delta\tilde{N}_n := \tilde{N}(t_{n+1}) - \tilde{N}(t_n)$.

To study the stability properties of CMSSTM methods, we consider a linear test equation with scalar coefficients

$$\mathrm{d}X(t) = aX(t^-)\mathrm{d}t + bX(t^-)\mathrm{d}W(t) + cX(t^-)\mathrm{d}\tilde{N}(t), \tag{44}$$

where $a,b,c\in\mathbb{R}$. Hence, the MS-stability of the zero solution to equation (44) was given in [15]

$$\lim_{t\to\infty} E|X(t)|^2 = 0 \quad \Leftrightarrow \quad 2a+b^2+\lambda c(c+2)<0. \tag{45}$$

Applying the CMSSTM methods (42-43) to Equation (44), we have

$$y_n^* = y_n + h[\theta(a+\lambda c) - \frac{1}{2}b^2 y]y_n^*, \quad (46)$$

$$y_{n+1} = y_n^* + (1-\theta)h(a+\lambda c)y_n^* + by_n^*\Delta W_n + \frac{1}{2}b^2 y_n^*(\Delta W_n)^2 + cy_n^*\Delta\tilde{N}_n, \quad (47)$$

The MS-stability results of the CMSSTM methods (46-47) for (44) are given in the following theorem.

Theorem 4.*Suppose that MS-stability condition (45) hold, we have*

1. *If $\theta \in [\frac{1}{2}, 1]$, then the CMSSTM methods are GMS-stable for all $h > 0$.*
2. *If $\theta \in [0, \frac{1}{2})$ and $(a+\lambda c)^2 \leq \frac{b^2(2(a+\lambda c)+b^2)}{2(2\theta-1)}$, then the CMSSTM methods are MS-stable for all $h > 0$.*
3. *If $\theta \in [0, \frac{1}{2})$ and $(a+\lambda c)^2 > \frac{b^2(2(a+\lambda c)+b^2)}{2(2\theta-1)}$, then the CMSSTM methods are MS-stable for all $h \in (0, \tilde{h}(a,b,c,\lambda,\theta))$.*

$$\tilde{h} < \frac{-(2(a+\lambda c)+b^2+\lambda c^2)}{(1-2\theta)(a+\lambda c)^2 + \frac{1}{2}b^2(2(a+\lambda c)+b^2}.$$

*Proof.*From (46 - 47), we have

$$y_n^* = \frac{1}{1-\theta h(a+\lambda c)+\frac{1}{2}b^2 h}y_n,$$

$$y_{n+1} = [1+(1-\theta)h(a+\lambda c)+b\Delta W_n + \frac{1}{2}b^2(\Delta W_n)^2 + c\Delta\tilde{N}_n]y_n^*.$$

Note that y_n is $\mathscr{F}_{t_n}$-measurable at the mesh point t_n, easily know from (42-43) that y_n^* is also $\mathscr{F}_{t_n}$-measurable at related meash-point. Increments $\Delta W_n, \Delta\tilde{N}_n$ are used mutually independent. $E[\Delta W_n] = E[\Delta\tilde{N}_n] = 0$, $E[|\Delta W_n|^2] = h$, $E[|\Delta\tilde{N}_n|^2] = \lambda h$, $E[(\triangle W_n)^2 - h] = 0$ and $E[|\triangle W_n|^2 - h]^2 = 2h^2$. Squaring both sides of the above equations and taking mathematical expectation, we get

$$E|y_n^*|^2 = \frac{1}{(1-\theta h(a+\lambda c)+\frac{1}{2}b^2 h)^2}E|y_n|^2, \quad (48)$$

$$E|y_{n+1}|^2 = [(1+(1-\theta)h(a+\lambda c))^2 + 2b^2 h + \frac{3}{4}b^4h^2 + (1-\theta)(a+\lambda c)b^2h^2 + \lambda c^2 h]E|y_n^*|^2. \quad (49)$$

Substituting (48) into (49), we obtain

$$E|y_{n+1}|^2 = P(a,b,c,\theta,\lambda,h)E|y_n|^2, \quad (50)$$

where

$$P(a,b,c,\theta,\lambda,h) = (1+(1-\theta)h(a+\lambda c))^2 + 2b^2h + \frac{3}{4}b^4h^2 + (1-\theta)(a+\lambda c)b^2h^2 + \lambda c^2 h \,/(1-\theta h(a+\lambda c)+\frac{1}{2}b^2 h)^2. \quad (51)$$

By recursive calculation, we conclude that $E|y_{n+1}|^2 \to 0, (if \quad n \to \infty)$ if and only if

$$P(a,b,c,\theta,\lambda,h) < 1,$$

which is equivalent to

$$\left[(1-2\theta)(a+\lambda c)^2 + \frac{1}{2}b^2\left(2(a+\lambda c)+b^2\right)\right]h + 2(a+\lambda c)+b^2+\lambda c^2 < 0. \quad (52)$$

With respect to condition (45), we know that $2(a+\lambda c)+b^2+\lambda c^2 < 0$, and the following results can be obtained.

Case I: If $\theta \in [\frac{1}{2}, 1]$, then the CMSSTM methods are GMS-stable for all $h > 0$.

Case II: Let $\theta \in [0, \frac{1}{2})$. Then the CMSSTM methods are MS-stable for all $h > 0$, if $(1-2\theta)(a+\lambda c)^2 + \frac{1}{2}b^2\left(2(a+\lambda c)+b^2\right) \leq 0$ and the methods are MS-stable for all $h \in (0, \tilde{h}(a,b,c,\lambda,\theta))$, if $(1-2\theta)(a+\lambda c)^2 + \frac{1}{2}b^2\left(2(a+\lambda c)+b^2\right) > 0$.

The proof is completed.

4.3 Discussion for A-stability

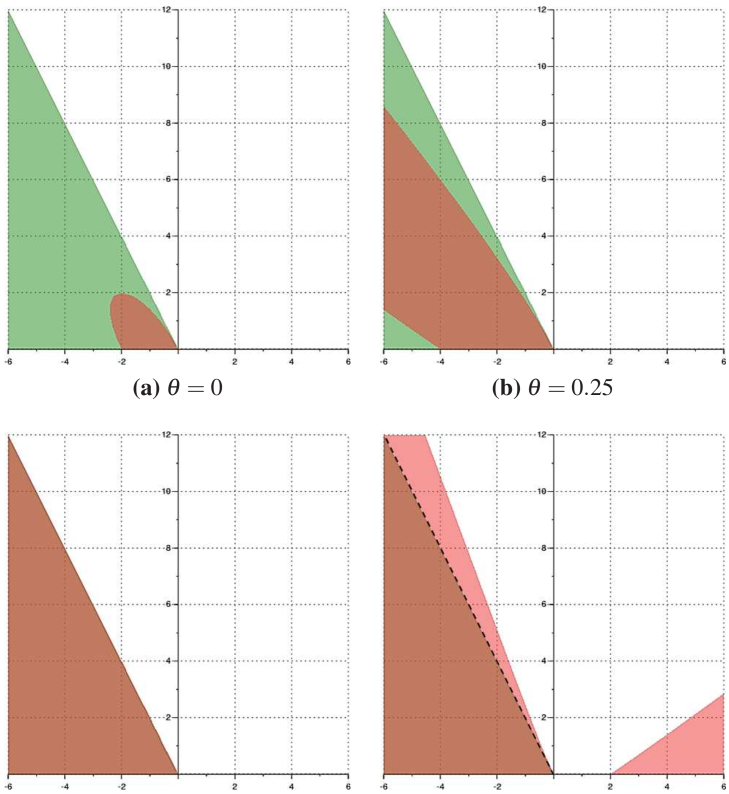
In this section, we aim to compare the stability region of the proposed method with that of existing numerical methods. In addition, the stability regions of the numerical methods are compared with that of the test problem and existing numerical methods to show the efficiency of the proposed method. Also, A-stability approach is investigated. For simplicity and without loss of generalization, we will consider the stability condition (33) of MSSTM methods (29-30) for linear SDEs (3). Note that, it is easy to get the similar results for CMSSTM methods (46-47) to SDEwPJ (44).

The MS-stability regions are the areas under the plotted curves and symmetric about the $x-y$ plane with $x = ah$ and $y = b^2h$. In this case, the stability conditions (MS-stability) (8), (33) for the test problem, MSSTM methods become

$$x + \frac{1}{2}y < 0 \qquad (Problem), \quad (53)$$

$$(1-2\theta)x^2 + \frac{1}{2}y^2 + xy + 2x + y < 0 \qquad (MSSTM), \quad (54)$$

The stability regions of the SSTM and test problem are plotted, as follows: Let $R_{SDE} = \{x, y \in \mathbb{R} : y \geq 0$ and $x + \frac{1}{2}y < 0\}$ denote the MS-stability region of the SDEs (3) and $R_{MSSTM}(\theta) = \{x, y \in \mathbb{R} : y \geq 0$ and (54) hold$\}$ be the MS-stability regions of MSSTM methods. Figure 1 illustrates how the $R_{MSSTM}(\theta)$ varies with θ. The green shading marks the regions R_{SDE} and the red shading superimposes $R_{MSSTM}(\theta)$ for $\theta = 0, 0.25, 0.5, 1.0$. Figure 1 confirms our results.



(a) $\theta = 0$ **(b)** $\theta = 0.25$

(c) $\theta = 0.5$ **(d)** $\theta = 1.0$

Fig. 1: Real MS-stability regions for test problem (green) and MSSTM methods (red)

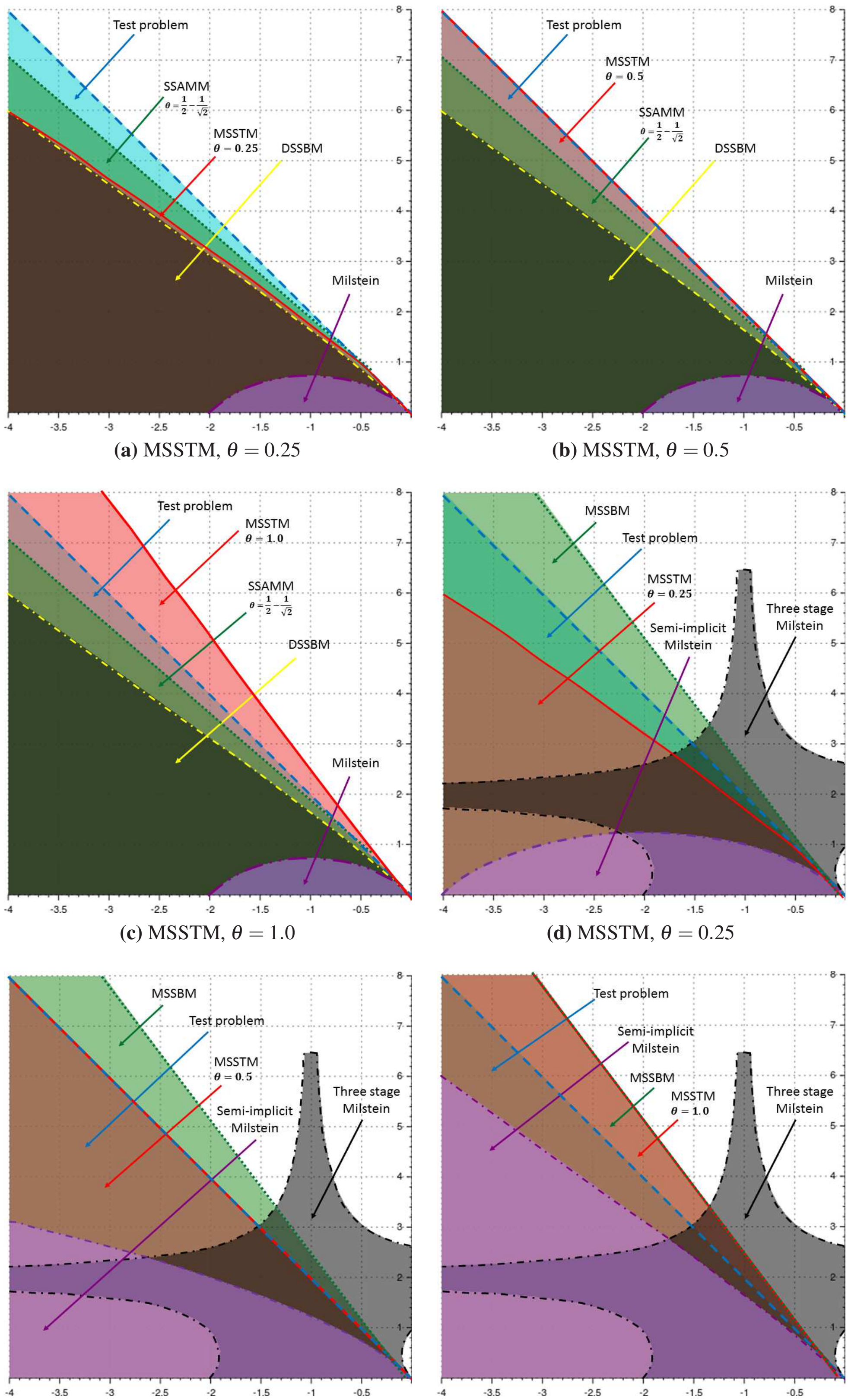
*Remark.*Theorem 7 in Eissa et al., [7], explains that if $\theta \in [0, \frac{1}{2}]$, the DSSTM methods are MS-stable for $h \in (0, h^*(a,b,\theta))$. Moreover, for $\theta > \frac{1}{2}$, if the diffusion term plays the crucial role, the restrict on step-size still holds with some constraints for the parameters. If the drift term plays the crucial role, the methods are MS-stable for all $h > 0$ under some constraints for the parameters a, b and θ. Furthermore, Figure 1 in Eissa et al., [7], shows that the DSSTM methods with $\theta \in [0,1]$ are not A-stable such that the stability regions of the numerical methods can't cover that of the test problem.

*Remark.*To improve the stability properties of DSSTM methods for SDEs (3), we constructed the MSSTM methods. Theorem 3 shows that, for any $\theta \in [0, \frac{1}{2})$, if the drift term plays the main rule, then the MSSTM methods are MS-stable for $h \in (0, \tilde{h}(a,b,\theta))$. If the diffusion term plays the crucial role, then the MSSTM methods are MS-stable for all $h > 0$ with some restrict on the parameters. Furthermore, the MSSTM methods are GMS-stable with $\theta \in [\frac{1}{2}, 1]$ for all $h > 0$ with no restrict on the parameters. Figure 1 shows that stability regions of the MSSTM methods with $\theta \in [\frac{1}{2}, 1]$ covered that of the test problem (i.e. the numerical methods are A-stable.).

Table 1: MS-stability conditions of various methods

Numerical methods	***MS-stability condition***
Milstein [21]	$x^2 + 2x + +y + \frac{1}{2}y^2 < 0$
Semi-implicit Milstein [12]	$(1-2\theta)x^2 + 2x + y + \frac{1}{2}y^2 < 0$
DSSBM [34]	$(1 + y + \frac{1}{2}y^2)/(1-x)^2 < 1$
MSSBM [34]	$(1 + 2y + \frac{3}{4}y^2)/(1 - x + \frac{1}{2}x^2)^2 < 1$
SSAMM [32]	$(1 + y + \frac{1}{2}y^2)(1 + 2\theta x)^2/(1 - (\frac{1}{2} - \theta)x)^4 < 1$
Three stage Milstein [35]	$(1+x)^2(1 - \frac{1}{2}y)^2(1 + 2y + \frac{3}{4}y^2) < 1$

Figure 2 shows the MS-stability regions of MSSTM methods (54) compared with that of test problem (53), and the existing numerical methods which are presented in Table 1. We examine the MS-stability regions of the MSSTM methods with $\theta = 0.25, 0.5, 1.0$, respectively. Figure 2 when $\theta = 1.0$, the MS-stability regions of the MSSTM methods and that of MSSBM methods, identical to some extent (see Figure 2f). With $\theta = 0.5$, the MS-stability regions of the MSSTM methods match exactly the test problem stability region. Finally, we can conclude that the MSSTM methods are more flexible than others and the MS-stability regions of MSSTM methods are better than that of all others in Table 1.



(a) MSSTM, $\theta = 0.25$ **(b)** MSSTM, $\theta = 0.5$

(c) MSSTM, $\theta = 1.0$ **(d)** MSSTM, $\theta = 0.25$

(e) MSSTM, $\theta = 0.5$ **(f)** MSSTM, $\theta = 1.0$

Fig. 2: Real MS-stability regions

5 Numerical results

We consider several illustrative numerical examples for showing the strong convergence order and MS-stability of MSSTM methods for SDEs. The mean-square errors at time T versus the step-size h are analyzed in a log-log diagram. The mean-square errors of the numerical approximations are defined by [3], as follows:

$$\varepsilon = \frac{1}{N}\sum_{i=1}^{N} |y_n^{(i)} - Z^{(i)}(T)|^2, \qquad (55)$$

where $y_n^{(i)}$ is a numerical approximation to $Z^{(i)}(T)$ and $Z^{(i)}(T)$ is the value of the exact solution of SDEs at time T. The superscript i means the i^{th} samples path, $i = 1,2,\ldots,N$.

5.1 One-dimensional SDEs

Example 1: A scalar linear SDEs

We apply the MSSTM methods to the scalar linear SDEs (3)

$$dZ(t) = aZ(t)dt + bZ(t)dW(t), \qquad t \in [0,T],$$
$$Z(0) = 1,$$

whose exact solution is

$$Z(t) = Z(0)exp\left(\left(a - \frac{1}{2}b^2\right)t + bW(t)\right).$$

We use the parameters $a = -\frac{1}{2}$ and $b = \frac{1}{2}$ as in [3, 32] and demonstrate the strong convergence rate of the MSSTM methods at the terminal time $T = 1$. We compute 4000 different discretized Brownian paths over $[0,1]$ with step size $dt = 2^{-9}$. For each path, the MSSTM methods are applied with five different step sizes: $\Delta t = 2^{p-1}dt, 1 \le p \le 5$. Table 2 compares the mean of absolute errors over the sample paths for SST methods [3], SSAMM methods [32], and DSSTM methods [7, 38] with the proposed MSSTM methods (Note that, the value of parameter θ for SST, SSAMM has been chosen to give the best absolute error of that methods according to [3, 32]). We find that the proposed MSSTM methods are more efficient than SST, SSAMM, DSSTM methods. Figure 3 shows the results of the mean of absolute errors using a log log plot.

Example 2: A scalar nonlinear SDEs

We consider the nonlinear SDEs

$$dZ(t) = -(\alpha + \beta^2 Z)(1 - Z^2)dt + \beta(1 - Z^2)dW(t), \quad (56)$$

where $t \in [0,T]$, $Z(0) = Z_0$ with α and β are real constants. The exact solution is given by [18]

$$Z(t) = \frac{(1+Z_0)exp(-2\alpha t + 2\beta W(t)) + Z_0 - 1}{(1+Z_0)exp(-2\alpha t + 2\beta W(t)) - Z_0 + 1}.$$

Table 2: Mean of absolute errors for (3) with $a = -\frac{1}{2}$ and $b = \frac{1}{2}$

Step size	*SST* $\theta = 0.1$	*SSAMM* $\theta = -\frac{1}{2} - \frac{1}{\sqrt{2}}$	*DSSTM* $\theta = 0.5$	*MSSTM* $\theta = 0.5$
2^{-5}	4.27E-04	7.12E-04	1.09E-05	7.14E-06
2^{-6}	2.32E-04	3.80E-04	2.68E-06	1.76E-06
2^{-7}	1.20E-04	1.90E-04	7.31E-07	4.85E-07
2^{-8}	5.63E-05	9.52E-05	1.83E-07	1.22E-07
2^{-9}	2.64E-05	4.80E-05	4.53E-08	3.01E-08

Table 3: Errors for (56) with $\alpha = 1$ and $\beta = 0.5$

Step size	*Milstein*	*SSAMM* $\theta = -\frac{1}{2} - \frac{1}{\sqrt{2}}$	*DSSTM* $\theta = 0.5$	*MSSTM* $\theta = 0.5$
2^{-5}	1.38E-03	3.64E-04	5.99E-07	5.81E-07
2^{-6}	6.76E-04	6.18E-05	1.46E-07	1.42E-07
2^{-7}	3.34E-04	4.40E-06	3.62E-08	3.54E-08
2^{-8}	1.66E-04	7.20E-06	9.01E-09	8.82E-09
2^{-9}	8.28E-05	5.40E-06	2.24E-09	2.20E-09

Errors of Milstein [18], SSAMM [32], and DSSTM [7, 38] methods with our MSSTM methods are displayed in Table 3 at the terminal time $T = 1$. The numerical results provide a comparison of these methods for fixed parameter $\alpha = 1$ and $\beta = 0.5$. We compute 4000 different discretized Brownian paths over $[0,1]$ with step size $dt = 2^{-9}$. For each path, the methods are applied with five different step size: $\Delta t = 2^{p-1}dt, 1 \le p \le 5$. For both values of β, our methods returns the most accurate solution for these step-sizes. Figure 4 shows the results of the mean of absolute errors using a log log plot.

Remark. Though the functions f and g in the test problem (56) are not satisfy the Assumption 2 (Lipschitz condition

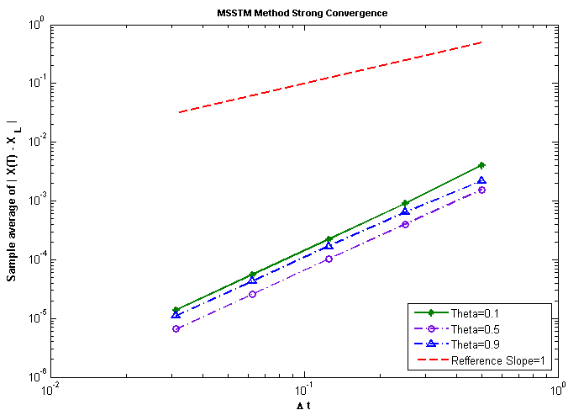


Fig. 3: Strong convergence of MSSTM methods with $a = -0.5$ and $b = 0.5$

Table 4: Error and convergence rates for (57) with $u = 5$, $v = 1$.

Step size	*SST* $\theta = 0.1$	*SSAMM* $\theta = -\frac{1}{2} - \frac{1}{\sqrt{2}}$	*DSSTM* $\theta = 0.5$	*MSSTM* $\theta = 0.5$
2^{-5}	$2.21E-1$	$2.97E-2$	$2.06E-2$	$1.98E-2$
2^{-6}	$1.52E-1$	$1.71E-2$	$1.07E-2$	$1.04E-2$
2^{-7}	$1.10E-1$	$7.31E-3$	$5.26E-3$	$4.75E-3$
2^{-8}	$7.64E-2$	$3.98E-3$	$2.80E-3$	$2.42E-3$
2^{-9}	$5.36E-2$	$2.13E-3$	$1.37E-3$	$1.22E-3$

and linear growth condition), the approximate solution of the MSSTM methods converge to the exact solution. In the future we will discuss the strong convergence order under weaker conditions of the methods.

5.2 Two-dimensional SDEs

Example 3: A two-dimensional linear SDEs

We consider the two-dimensional SDEs [32]

$$\begin{aligned} dZ(t) &= UZ(t)dt + VZ(t)dW(t), \qquad t \in [0,1], \\ Z(0) &= Z_0, \end{aligned} \tag{57}$$

where $U = \begin{pmatrix} -u & u \\ u & -u \end{pmatrix}$ and $V = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$. The exact solution of test equation (57) is

$$Z(t) = P \begin{pmatrix} \exp(\rho^{+}(t)) & 0 \\ 0 & \exp(\rho^{-}(t)) \end{pmatrix} P^{-1} Z_0, \tag{58}$$

where $P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\rho^{\pm}(t) = (-u - \frac{1}{2}v^2 \pm u)t + vW(t)$, and $p^{-1} = p$. With initial $Z_0 = [1,2]^T$ and parameters $u = 5, v = 1$, Table 4 shows that the proposed methods MSSTM return a more accurate solution than that of the SST methods [3], SSAMM methods [32], DSSTM methods [7,38].

Example 4: A two-dimensional nonlinear SDEs

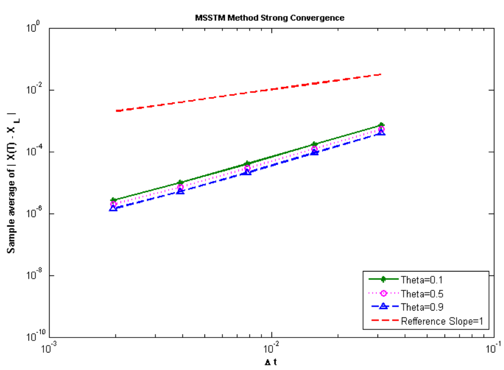


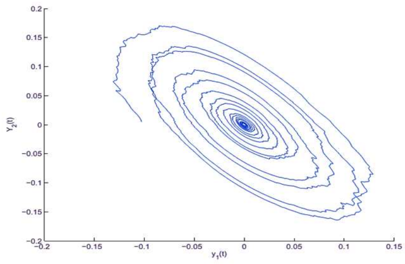
Fig. 4: Strong convergence rate of MSSTM methods with $\alpha = 1$ and $\beta = 0.5$

The next problem is the Brusselator system of SDEs [33], which unforced periodic oscillations in certain chemical reaction.

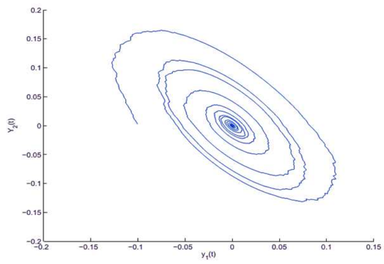
$$\begin{aligned} dZ_1(t) &= ((\alpha - 1)Z_1(t) + \alpha Z_1^2(t) + (Z_1(t) + 1)^2 Z_2(t))dt \\ &\quad + \gamma Z_1(t)(1 + Z_1(t))dW(t), \end{aligned} \tag{59}$$

$$\begin{aligned} dZ_2(t) &= (-\alpha Z_1(t) - \alpha Z_1^2(t) - (Z_1(t) + 1)^2 Z_2(t))dt \\ &\quad - \gamma Z_1(t)(1 + Z_1(t))dW(t). \end{aligned} \tag{60}$$

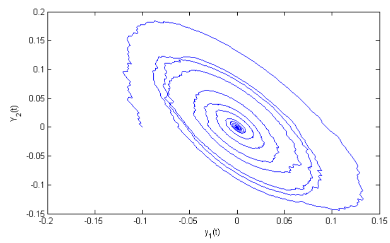
Figure 5, the numerical simulation of (59-60) with $h = 0.025$, $\alpha = 1.9$, and $\gamma = 0.1$ for $0 \le t \le 125$ starting at $(Z_1(0), Z_2(0)) =$



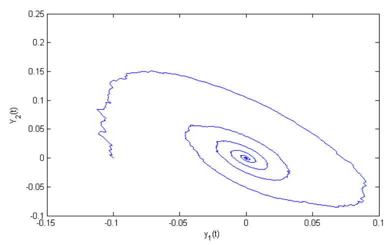
(a) Milstein method



(b) SSAMM methods



(c) DSSTM methods with $\theta = 0.5$



(d) MSSTM methods with $\theta = 0.5$

Fig. 5: Numerical simulation of (59-59) with $h = 0.025, \alpha = 1.9$, and $\gamma = 0.1$.

$(-1,0)$ using Milstein [18], SSAMM methods [32], and DSSTM [7, 38] with proposed method MSSTM.

Similar to the semi-implicit balanced Milstein (SIBM) method in [33], we see that the MSSTM methods stay close to the origin replicating the behavior of the true solution. Furthermore, the proposed method a better approximation than other methods.

5.3 Stability experiment results

In the following, we illustrate stability properties of MSSTM methods by simulating SDEs (3). The set of coefficients satisfy the condition (8). The trivial solution of the test equation is MS-stable. The data used in the following figures is obtained by the mean-square of data from 4000 trajectories

$$\frac{1}{4000}\sum_{i=1}^{4000} E|y_n(w_i)|^2.$$

We test the MS-stability of MSSTM methods when $a=-15$, $b=1$ and the initial value $x_0=0.5$. For $\theta=0.1$ and 0.3, we obtain $\tilde{h}(-15,1,0.1)=0.1752$ and $\tilde{h}(-15,1,0.3)=0.3841$. We first fix the parameter $\theta=0.1$ and 0.3 and change the step size h (see Figure 6 and 7 respectively). It show that the MSSTM methods are MS-stable for any $\theta\in[0,\frac{1}{2})$ if $h\in(0,\tilde{h}(a,b,\theta))$. Figure 8 explains that for any $\theta\in[\frac{1}{2},1]$, the MSSTM methods are MS-stable for all $h>0$.

6 Conclusion

We are interested in the Itô stochastic differential equations (SDEs). With respect to using split-step techniques, split-step Milstein type methods can be classified in two families: the split-step Milstein methods and modified split-step Milstein methods to approximate solutions of SDEs. Based on Split-Step Theta (SST) methods which have advantages in flexibility and stability, the split-step theta Milstein type methods have been provided to improve stability properties and stability regions of Milstein type methods. However, we can see that the existing numerical methods based on Milstein scheme still have restrictions on the parameters and step-size to be A-stable. In this paper, we constructed the new general methods; the modified spit-step theta Milstein (MSSTM) methods with strong convergence order 1.0 for solving m-dimensional SDEs in order to remove the existing restrictions on the parameters and step-size to share the mean-square stability. The proposed methods are derived based on the split-step technique by combining the SST method with Milstein scheme. Then, the MS-stability is investigated for linear SDEs. We proved that the proposed methods MSSTM can share the MS-stability of the exact solution for all step-size with no restrictions on parameters. Furthermore, We extended the MSSTM methods for SDEs with Poisson jump (SDEwPJ) which has been called the compensated modified spit-step theta Milstein (CMSSTM) methods. The MS-stability was investigated for linear SDEwPJ, as follows: For $\theta\in[\frac{1}{2},1]$, the CMSSTM methods are general mean-square (GMS) stable and If $\theta\in[0,\frac{1}{2})$, the CMSSTM methods are MS-stable for all step-size with restrictions on parameters. Finally, the stability regions were discussed to explain that the proposed methods were A-stable. In addition, several examples were given to show the effectiveness of the proposed method in approximation of one and two dimensional SDEs compared to some existing methods.

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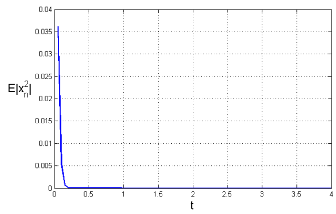
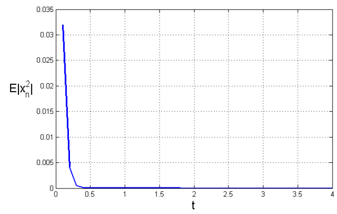
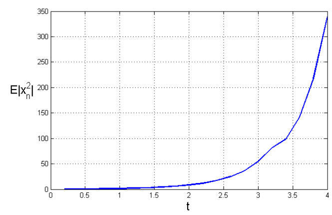


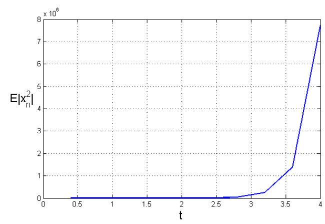
(a) $h=0.05<h^*(-15,1,0.1)=0.1620$.



(b) $h=0.1<h^*(-15,1,0.1)=0.1620$.

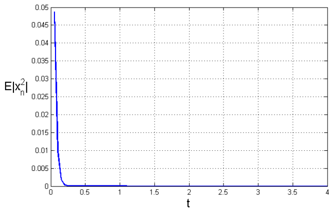


(c) $h=0.2>h^*(-15,1,0.1)=0.1620$.

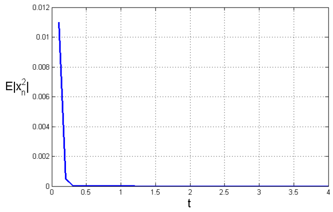


(d) $h=0.4>h^*(-15,1,0.1)=0.1620$.

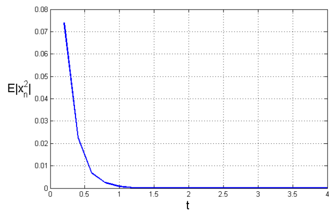
Fig. 6: Simulating of MSSTM with fixed parameter $\theta=0.1$, $a=-15$ and $b=1$ for (3)



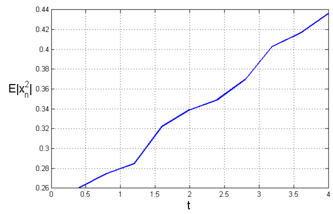
(a) $h = 0.05 < h^*(-15,1,0.1) = 0.3372$.



(b) $h = 0.1 < h^*(-15,1,0.1) = 0.3372$.



(c) $h = 0.2 < h^*(-15,1,0.1) = 0.3372$.



(d) $h = 0.4 > h^*(-15,1,0.1) = 0.3372$.

Fig. 7: Simulating of MSSTM with fixed parameter $\theta = 0.3$, $a = -15$ and $b = 1$ for (3)

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

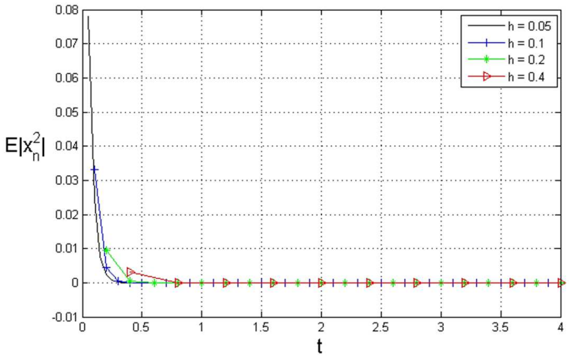

Fig. 8: Simulating MSSTM with $\theta = 0.5$, $a = -15$ and $b = 1$ for (3)

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