Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/180512

Analysis on α-time scales and its applications to Cauchy-Euler equation

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Received: 12 Jun. 2024, Revised: 2 Jul. 2024, Accepted: 14 Jul. 2024
Published online: 1 Sep. 2024

Abstract: This article is devoted to present the α-power function, calculus on α-time scale, the α-logarithm and their applications on α-difference equations. We introduce the α-power function as an absolutely convergent infinite product. We state that the α-power function verifies the fundamentals of α-time scale and adheres to both the additivity and the power rule for α-derivative. Next, we propose an α-analogue of Cauchy-Euler equation whose coefficient functions are α-polynomials and then construct its solution in terms of α-power function. As illustration, we present examples of the second order α-Cauchy-Euler equation. Consequently, we construct α-analogue of logarithm function which is determined in terms of α-integral. Finally, we propose a second order BVP for α-Cauchy-Euler equation with two point unmixed boundary conditions and compute its solution by the use of Green's function.

Keywords: α-power function; α-logarithm; α-time scale calculus; α-Cauchy-Euler equation; BVP; Green's function

1 Introduction

After the concept of time scales $\mathbb{T}$ was invented [17], not only the studies on the discrete settings

$$h\mathbb{Z} := \{hx : x \in \mathbb{Z}, h > 0\}, \quad \text{or} \quad \mathbb{K}_q := \{q^n : n \in \mathbb{Z}, q \neq 1\} \cup \{0\}, \tag{1}$$

have been unified and extended but also the theories of differential and discrete equations have been gathered under the roof of dynamical equations (see [7,9] and the references therein). Although the notion of time scales has provided significant contributions in pure and applied mathematics, some elementary notions such as polynomials, power functions, Taylor series have not been established in explicit, efficient and applicable forms on a general time scale. In order to overcome these deficiencies, researchers usually prefer to study on particular time scales. For instance, in [12], the general time scale is narrowed to a special time scale, namely (q,h)-*time scale*,

$$\mathbb{T}^{x_0}_{(q,h)} := \{q^n x_0 + h[n]_q :\ n \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\},\ \ h \geq 0,\ q \geq 1,\ q+h > 1,\ x_0 \in \mathbb{R},$$

which covers h- and q-discretizations (1). Here $[n]_q := \frac{q^n-1}{q-1}$. The studies on $\mathbb{T}^{x_0}_{(q,h)}$ attract the attention of many researchers [15,26,27,30,28,20,21] where the delta and nabla (q,h)-analysis have been widely investigated separately.

In [13], motivated by [1], we presented a comprehensive framework which combines and expands the delta and nabla (q,h)-analysis. We introduced *the* α*-time scale*, for $x_0 \in \mathbb{R} \setminus \{\frac{h}{1-q}\}$, $h \in \mathbb{R}^+_0$ and $q \in \mathbb{R}^+ \setminus \{1\}$

$$\mathbb{T}^{x_0}_{\alpha} := \{\alpha^n(x_0) :\ n \in \mathbb{Z}\} \cup \left\{\frac{h}{1-q}\right\}, \tag{2}$$

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where we defined *the α-operator* as

$$\alpha(x) := t\left(\frac{x-h}{q}\right) + (1-t)(qx+h),\ x\in\mathbb{R},\ h\in\mathbb{R}_0^+,\ q\in\mathbb{R}^+,\ t\in[0,1], \tag{3}$$

a convex combination of the backward and forward jump operators of $\mathbb{T}_{(q,h)}^{x_0}$. We emphasize that, the weighted operator α unifies delta and nabla (q,h)-analysis for $t\in\{0,1\}$ and produces their extensions for $t\in(0,1)$. Generated by the operator α, for $k,m\in\mathbb{Z}$, we introduced the $\alpha^{k,m}$-derivative which not only covers various kinds of discrete derivatives such as symmetric (q,h)-/q-/h-derivatives, (p,q)-derivative [14], delta (q,h)-/q-/h-derivatives and nabla (q,h)-/q-/h-derivatives but also provides their extensions. Based on the α-operator, we constructed the α-polynomial which possesses the characteristics of α-time scales. We showed that the α-polynomial recovers discrete polynomials such as delta (q,h)-/q-/h-polynomials, nabla (q,h)-/q-/h-polynomials and produces their extensions.

The concept of power function and its applications on dynamic equations have not been established on a general time scale. Because even the polynomials could have been presented only in implicit and recursive forms by integrals as Δ-polynomials [9,16], ∇-polynomials [2] and diamond-alpha polynomials [25,24], the studies conducted on the exploration of power function on time scales have been notably absent so far. As a result, Cauchy-Euler equation on time scales could have been studied for order two with very specific variable coefficients [8] and its Hyers-Ulam stability was analyzed [3].

The primary motivation of the current work is to address these gaps. For the attainment of this objective, we first introduce the power function on α-time scale. We express the α-power function as an absolutely convergent infinite product. We showed that the α-power function reveals the fundamentals of the α-time scales since it obeys the additivity rule and power rule for α-derivative. Such rules have an additional key role to propose the form of the α-difference equations and to construct their solutions. We plot the graphs of the α-power function with different real powers and for specific q,h,t,k values. This is where the significance of α-operator comes to the forefront. One can observe that the convex combination structure of α-operator provides more balanced approximations to discrete power functions.

The subsequent substantial contribution of the current paper is to present the α-analogue of Cauchy-Euler equation. We first present an α-difference IVP which admits α-power function as a unique solution. We propose an nth order α-Cauchy-Euler equation whose coefficients are α-polynomials. We present the solutions of α-Cauchy-Euler equation in terms of α-power functions unlike the exponential forms studied in the literature [8]. In order to illustrate concrete examples, we focus on second order α-Cauchy-Euler equations whose linearly independent solutions are derived by the reduction of order method. In the first example, we obtain two linearly independent solutions where both are α-power functions with different real powers which can also be constructed from one another by the use of additivity rule. In the second example, similar to the repeated root case, the second linearly independent solution consists of a α-power function and its logarithmic counterpart. Such applications allow us to introduce the α-analogue of logarithm function whose figure is plotted to demonstrate its reductions to (q,h)-/q-/h- and ordinary logarithm functions. After Bohner posed the open problem of defining a "nice" logarithm function [6], numerous articles have been published [18,23,4]. We present a partial answer for this open problem following the second approach given in [6] by introducing the α-logarithm function which is expressed in terms of α-integral presented in the Appendix. We additionally compute the solutions of nonhomogeneous α-Cauchy-Euler equation using the variation of parameters method. Finally, we propose a second order BVP for the α-Cauchy-Euler equation with two point unmixed boundary conditions and derive its solution by the use of Green's function.

2 Preliminaries

In [13], for $k\in\mathbb{Z}$, $n\in\mathbb{N}_0$, $q\in\mathbb{R}_+$ and $t\in[0,1]$, we introduced the α^k-integer as

$$[n]_{\alpha^k} = \begin{cases} \frac{(\frac{t}{q}+(1-t)q)^{nk}-1}{(\frac{t}{q}+(1-t)q)^{k}-1} & \text{if} \quad \frac{t}{q}+(1-t)q\neq 1, \\ n & \text{if} \quad \frac{t}{q}+(1-t)q = 1. \end{cases}$$

Now, we extend it to the real numbers in the following obvious way.

Definition 1. *For $k\in\mathbb{Z}$, $r\in\mathbb{R}$, $q\in\mathbb{R}_+$ and $t\in[0,1]$, the α^k-number is defined by*

$$[r]_{\alpha^k} = \begin{cases} \frac{(\frac{t}{q}+(1-t)q)^{rk}-1}{(\frac{t}{q}+(1-t)q)^{k}-1} & \text{if} \quad \frac{t}{q}+(1-t)q\neq 1, \\ r & \text{if} \quad \frac{t}{q}+(1-t)q = 1. \end{cases}$$

The following properties of α^k-numbers follow from the definition and are very handy in the rest of the manuscript. Since their proofs are almost identical to those in the integer case [13], we state them without proofs.

Proposition 1. (i)$[0]_{\alpha^k}=0$ *and* $[1]_{\alpha^k}=1$.
(ii)*If* $q=1$ *or* $t=\frac{q}{q+1}$, *then* $\frac{t}{q}+(1-t)q=1$ *and hence* $[r]_{\alpha^k}=r$ *for all* $r\in\mathbb{R}$.
(iii)*If* $n\in\mathbb{N}_0$, *then* $[n]_{\alpha^k}=\sum_{j=0}^{n-1}\left(\frac{t}{q}+(1-t)q\right)^{jk}$.
(iv)*The* α^k*-numbers have the following limits*

$$\lim_{r\to\infty}[r]_{\alpha^k}=\begin{cases}\dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)^k} & if\quad 0<\left(\frac{t}{q}+(1-t)q\right)^k<1,\\ \infty & if\quad 1\le\left(\frac{t}{q}+(1-t)q\right)^k,\end{cases}$$

$$\lim_{r\to-\infty}[r]_{\alpha^k}=\begin{cases}-\infty & if\quad 0<\left(\frac{t}{q}+(1-t)q\right)^k\le 1,\\ \dfrac{1}{1-\left(\frac{t}{q}+(1-t)q\right)^k} & if\quad 1<\left(\frac{t}{q}+(1-t)q\right)^k.\end{cases}$$

(v)*For* $r,s\in\mathbb{R}$, *we have* $[s]_{\alpha^k}-[r]_{\alpha^k}=\left(\frac{t}{q}+(1-t)q\right)^{rk}[s-r]_{\alpha^k}$.
(vi)*For* $r,s\in\mathbb{R}$, *we have* $\left(\frac{t}{q}+(1-t)q\right)^{sk}[r]_{\alpha^k}+[s]_{\alpha^k}=[r+s]_{\alpha^k}$.

In [13], we found the explicit form for the integer powers of the α-operator (3).

Proposition 2.*[13] For* $n\in\mathbb{Z}$ *and* $x\in\mathbb{R}$, *the* α*-operator satisfies*

$$\alpha^n(x)=\left(\frac{t}{q}+(1-t)q\right)^n x-h\left(\frac{t}{q}-(1-t)\right)[n]_\alpha. \tag{4}$$

In [13], we introduced the α-time scale as in (2) which unifies delta and nabla (q,h)-analysis for $t\in\{0,1\}$ and allows extensions for $t\in(0,1)$. We extend the convenient form (4) to the real powers of α and present its fundamental properties.

Definition 2.*For* $x\in\mathbb{R}$, $h\ge 0$, $q>0$, $t\in[0,1]$, *and* $r\in\mathbb{R}$, *we introduce the extended* α*-operator, denoted by* α_r, *as follows*

$$\alpha_r(x)=\left(\frac{t}{q}+(1-t)q\right)^r x-h\left(\frac{t}{q}-(1-t)\right)[r]_\alpha.$$

Proposition 3. (i)*If* $r\in\mathbb{Z}$, *then* $\alpha_r(x)=\alpha^r(x)$.
(ii)*For* $r,s\in\mathbb{R}$, *we have* $\alpha_s\circ\alpha_r=\alpha_{s+r}$.
(iii)*For* $r\in\mathbb{R}$, *we have* $\alpha_r(x)=\left(\frac{t}{q}+(1-t)q\right)^r\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}$.
(iv)*For* $r\in\mathbb{R}$, *we have* $\alpha_r(x)=x$ *if and only if* $r=0$ *or* $x=\frac{h}{1-q}$ *or* $\frac{t}{q}+(1-t)q=1$. *The last case occurs when* $q=1$ *or* $t=\frac{q}{q+1}$.
(v)*The extended* α*-operator admits the following limits*

$$\lim_{r\to\infty}\alpha_r(x)=\begin{cases}\infty & if\quad \frac{t}{q}+(1-t)q>1\quad and\quad x>\frac{h}{1-q},\\ -\infty & if\quad \frac{t}{q}+(1-t)q>1\quad and\quad x<\frac{h}{1-q},\\ \frac{h}{1-q} & if\quad \frac{t}{q}+(1-t)q<1,\end{cases}$$

$$\lim_{r\to-\infty}\alpha_r(x)=\begin{cases}\frac{h}{1-q} & if\quad \frac{t}{q}+(1-t)q>1,\\ \infty & if\quad \frac{t}{q}+(1-t)q<1\quad and\quad x>\frac{h}{1-q},\\ -\infty & if\quad \frac{t}{q}+(1-t)q<1\quad and\quad x<\frac{h}{1-q}.\end{cases}$$

(vi)*For $r,s\in\mathbb{R}$, we have*

$$\alpha_s(x)-\alpha_r(x)=\left(\frac{t}{q}+(1-t)q\right)^r[s-r]_\alpha(\alpha(x)-x)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^r[s-r]_\alpha\left(\frac{t}{q}+(1-t)q-1\right)\left(x-\frac{h}{1-q}\right).$$

Proof. (i)Follows from (4).
(ii)By Definition 2 and Proposition 1/(vi) with $k=1$, we have

$$(\alpha_s\circ\alpha_r)(x)=\alpha_s(\alpha_r(x))=\left(\frac{t}{q}+(1-t)q\right)^s\alpha_r(x)-h\left(\frac{t}{q}-(1-t)\right)[s]_\alpha$$
$$=\left(\frac{t}{q}+(1-t)q\right)^s\left(\left(\frac{t}{q}+(1-t)q\right)^r x-h\left(\frac{t}{q}-(1-t)\right)[r]_\alpha\right)-h\left(\frac{t}{q}-(1-t)\right)[s]_\alpha$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{s+r}x-h\left(\frac{t}{q}-(1-t)\right)\left(\left(\frac{t}{q}+(1-t)q\right)^s[r]_\alpha+[s]_\alpha\right)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^{s+r}x-h\left(\frac{t}{q}-(1-t)\right)[s+r]_\alpha=\alpha_{s+r}(x).$$

(iii)By Definition 2, we have

$$\alpha_r(x)=\left(\frac{t}{q}+(1-t)q\right)^r x-h\left(\frac{t}{q}-(1-t)\right)[r]_\alpha=\left(\frac{t}{q}+(1-t)q\right)^r\left(x-\frac{h\left(\frac{t}{q}-(1-t)\right)}{\frac{t}{q}+(1-t)q-1}\right)+\frac{h\left(\frac{t}{q}-(1-t)\right)}{\frac{t}{q}+(1-t)q-1}.$$

Note that $(1-q)\left(\frac{t}{q}-(1-t)\right)=\frac{t}{q}+(1-t)q-1,$ hence we conclude with

$$\alpha_r(x)=\left(\frac{t}{q}+(1-t)q\right)^r\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}.$$

(iv)By (iii), we get

$$\alpha_r(x)-x=\left(\left(\frac{t}{q}+(1-t)q\right)^r-1\right)\left(x-\frac{h}{1-q}\right), \tag{5}$$

which implies that $\alpha_r(x)=x$ if and only if $r=0$ or $x=\frac{h}{1-q}$ or $\frac{t}{q}+(1-t)q=1$.
(v)Obvious from (iii).
(vi)By (iii) and (5) with $r=1$, we derive

$$\alpha_s(x)-\alpha_r(x)=\left(\left(\frac{t}{q}+(1-t)q\right)^s\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}\right)-\left(\left(\frac{t}{q}+(1-t)q\right)^r\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}\right)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^r\left(\left(\frac{t}{q}+(1-t)q\right)^{s-r}-1\right)\left(x-\frac{h}{1-q}\right)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^r[s-r]_\alpha\left(\frac{t}{q}+(1-t)q-1\right)\left(x-\frac{h}{1-q}\right)$$
$$=\left(\frac{t}{q}+(1-t)q\right)^r[s-r]_\alpha(\alpha(x)-x).$$

By Proposition 3/(i), from now on we prefer to use the notation α^r instead of α_r for any $r\in\mathbb{R}$. Now, we recall α^k-factorial [13] and extending α^k-permutation coefficient [13] to real numbers which will be necessary in the remaining of the text.

Definition 3. *For $j \in \mathbb{N}_0$, the α^k-factorial is introduced as*

$$[j]_{\alpha^k}! := [j]_{\alpha^k}[j-1]_{\alpha^k}[j-2]_{\alpha^k}\cdots[2]_{\alpha^k}[1]_{\alpha^k}$$

with convention $[0]_{\alpha^k}! = 1$. For $n \in \mathbb{N}$, α^k-permutation coefficient is defined by

$$P_{\alpha^k}[r,j] := \begin{cases} [r]_{\alpha^k}[r-1]_{\alpha^k}\cdots[r-j+1]_{\alpha^k} & \text{if} \quad j \in \mathbb{N}, \\ 1 & \text{if} \quad j = 0. \end{cases}$$

The notions for $k = 1$ in Definition 3, are called as *α-factorial and α-permutation*, respectively. Note that by Proposition (3)/(iii), $x > \frac{h}{1-q}$ if and only if $\alpha_r(x) > \frac{h}{1-q}$. In the rest of the article, any function under consideration is real valued and either defined for $x \in [\frac{h}{1-q}, \infty)$ or for $x \in (-\infty, \frac{h}{1-q}]$ depending on whether the seed element $x_0 > \frac{h}{1-q}$ or $x_0 < \frac{h}{1-q}$.

Definition 4. *[13] For any $k \in \mathbb{Z}$, we define the α^k-derivative of f by*

$$D_{\alpha^k} f(x) := \begin{cases} \dfrac{f(\alpha^k(x)) - f(x)}{\alpha^k(x) - x} & \text{if} \quad x \neq \frac{h}{1-q}, \\ \lim\limits_{\substack{s \to \frac{h}{1-q} \\ s \in \mathbb{T}_\alpha^{x_0}}} \dfrac{f(s) - f\left(\frac{h}{1-q}\right)}{s - \frac{h}{1-q}} = f'\left(\dfrac{h}{1-q}\right) & \text{if} \quad x = \frac{h}{1-q}, \end{cases} \tag{6}$$

if the limit exists.

Note that, the α^k-derivative (6) is a comprehensive derivative. For $t \in \{0,1\}$, it leads to the (q,h)-derivative generator which provides delta/nabla (q,h)-derivatives [26,30], q-derivative generator which covers delta/nabla q-derivatives [19] and h-derivative generator which produces delta/nabla h-derivatives [7]. Moreover, for $t \in (0,1)$, α^k-derivative provides extensions for the (q,h)-derivative generator, q-derivative generator and h-derivative generator. For details see [13].

Definition 5. *For any $r \in \mathbb{R}$, we define the shift operator on f, denoted by $S_{\alpha^r}(f)$, as*

$$(S_{\alpha^r} f)(x) := f(\alpha^r(x)). \tag{7}$$

The following simple form of the chain rule will be repeatedly used especially in applications.

Proposition 4. *For any $r \in \mathbb{R}$, $k \in \mathbb{Z}$, we have*

$$D_{\alpha^k}(S_{\alpha^{kr}} f(x)) = \left(\frac{t}{q} + (1-t)q\right)^{kr} (D_{\alpha^k} f)(S_{\alpha^{kr}}(x)).$$

Proof. Using Definition 4, Definition 5 and Proposition 3/(vi), we have

$$\begin{aligned} D_{\alpha^k}(S_{\alpha^{kr}} f(x)) &= \frac{f(\alpha^{k(r+1)}(x)) - f(\alpha^{kr}(x))}{\alpha^k(x) - x} = \frac{f(\alpha^{k(r+1)}(x)) - f(\alpha^{kr}(x))}{\alpha^{k(r+1)}(x) - \alpha^{kr}(x)} \frac{\alpha^{k(r+1)}(x) - \alpha^{kr}(x)}{\alpha^k(x) - x} \\ &= \left(\frac{t}{q} + (1-t)q\right)^{kr} \frac{[k(r+1) - kr]_\alpha (\alpha(x) - x)}{[k]_\alpha (\alpha(x) - x)} (D_{\alpha^k} f)(\alpha^{kr}(x)) \\ &= \left(\frac{t}{q} + (1-t)q\right)^{kr} (D_{\alpha^k} f)(\alpha^{kr}(x)) = \left(\frac{t}{q} + (1-t)q\right)^{kr} (D_{\alpha^k} f)(S_{\alpha^{kr}}(x)). \end{aligned}$$

We finish this section, by rewriting the product rule [13] for the α^k-derivative in terms of shift operator (7).

Proposition 5. *The product rule for the α^k-derivative is given by*

$$D_{\alpha^k}(f(x)g(x)) = S_{\alpha^k} f(x) D_{\alpha^k} g(x) + g(x) D_{\alpha^k} f(x) = S_{\alpha^k} g(x) D_{\alpha^k} f(x) + f(x) D_{\alpha^k} g(x).$$

3 α-Power function

This section is devoted to introduce the power function and its properties. Our objective is to formulate an explicit α-power function such that it is compatible with an additivity rule and power rule for α^k-derivative given in Definition 4 and it is consistent with the ordinary power function. To achieve this, the convex combination structure has to function on the level of forward and backward jump operators rather than on the level of delta and nabla (q,h)-derivatives. This approach yields an explicit, efficient and applicable α-power function which exhibits a structure that is in harmony with $\mathbb{T}_\alpha^{x_0}$. The α-power function recovers nabla (q,h)-power function [15] and delta (q,h)-polynomials [26,27], nabla (q,h)-polynomials [30] for nonnegative powers.

Definition 6. *If $x \geq \frac{h}{1-q}$, $r \in \mathbb{R}$, $k \in \mathbb{Z}$, $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ and $a \geq -\frac{h}{1-q}$, the α^k-power function is defined by*

$$(a+x)^r_{\alpha^k} := \begin{cases} \left(a+\frac{h}{1-q}\right)^r \prod\limits_{j=0}^{\infty} \frac{a+\alpha^{kj}(x)}{a+\alpha^{k(r+j)}(x)} & \text{if} \quad a > -\frac{h}{1-q}, \\ \left(\frac{t}{q}+(1-t)q\right)^{\frac{kr(r-1)}{2}} \left(-\frac{h}{1-q}+x\right)^r & \text{if} \quad a = -\frac{h}{1-q}. \end{cases}$$

If $x \leq \frac{h}{1-q}$, $r \in \mathbb{R}$, $k \in \mathbb{Z}$, $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ and $a \leq -\frac{h}{1-q}$, the α^k-power function is defined by

$$(a+x)^r_{\alpha^k} := \begin{cases} \left|a+\frac{h}{1-q}\right|^r e^{ir\pi} \prod\limits_{j=0}^{\infty} \frac{a+\alpha^{kj}(x)}{a+\alpha^{k(r+j)}(x)} & \text{if} \quad a < -\frac{h}{1-q}, \\ \left(\frac{t}{q}+(1-t)q\right)^{\frac{kr(r-1)}{2}} \left|-\frac{h}{1-q}+x\right|^r e^{ir\pi} & \text{if} \quad a = -\frac{h}{1-q}. \end{cases}$$

Proposition 6. *The infinite products in Definition 6 converge and the α^k-power function is well defined.*

Proof. Using Proposition 3/(iii), for any $s \in \mathbb{R}$, we obtain

$$a+\alpha^s(x) = \left(a+\frac{h}{1-q}\right) + \left(\frac{t}{q}+(1-t)q\right)^s \left(x-\frac{h}{1-q}\right).$$

As we showed in [13, Proposition 5], if $x \neq \frac{h}{1-q}$, then $x > \frac{h}{1-q}$ if and only if $x_0 > \frac{h}{1-q}$. Therefore, for all $s \in \mathbb{R}$, $a+\alpha^s(x) > 0$ if $x_0 > \frac{h}{1-q}$ and $a+\alpha^s(x) < 0$ if $x_0 < \frac{h}{1-q}$ and hence the terms of the infinite products are positive real numbers. Now, to prove the convergence of the infinite product, it is enough to show that the series

$$\sum_{j=0}^{\infty} \left(\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(r+j)}(x)} - 1\right) = \sum_{j=0}^{\infty} \frac{\alpha^{kj}(x)-\alpha^{k(r+j)}(x)}{a+\alpha^{k(r+j)}(x)} \tag{8}$$

converges absolutely, see [22, p. 54]. Using Proposition 3/(vi), we have

$$L := \lim_{j\to\infty} \left| \frac{\alpha^{k(j+1)}(x)-\alpha^{k(r+j+1)}(x)}{a+\alpha^{k(r+j+1)}(x)} \, \frac{a+\alpha^{k(r+j)}(x)}{\alpha^{kj}(x)-\alpha^{k(r+j)}(x)} \right| = \lim_{j\to\infty} \left(\frac{t}{q}+(1-t)q\right)^k \left| \frac{a+\alpha^{k(r+j)}(x)}{a+\alpha^{k(r+j+1)}(x)} \right|.$$

The condition $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ implies that $k > 0$ when $\frac{t}{q}+(1-t)q < 1$ and $k < 0$ when $\frac{t}{q}+(1-t)q > 1$, then it follows from Proposition 3/(v) that

$$\lim_{j\to\infty} \alpha^{k(r+j)}(x) = \lim_{j\to\infty} \alpha^{k(r+j+1)}(x) = \frac{h}{1-q},$$

and hence

$$\lim_{j\to\infty} \left| \frac{a+\alpha^{k(r+j)}(x)}{a+\alpha^{k(r+j+1)}(x)} \right| = 1,$$

which implies that

$$L = \lim_{j\to\infty} \left(\frac{t}{q}+(1-t)q\right)^k < 1$$

and the series (8) converges absolutely by the ratio test.

*Remark.*Note that when $\left(\frac{t}{q}+(1-t)q\right)^k>1$, it follows from Proposition 3/(iii) that

$$\lim_{j\to\infty}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(r+j)}(x)}=\lim_{j\to\infty}\frac{a+\left(\frac{t}{q}+(1-t)q\right)^{kj}(x-\frac{h}{1-q})+\frac{h}{1-q}}{a+\left(\frac{t}{q}+(1-t)q\right)^{k(r+j)}(x-\frac{h}{1-q})+\frac{h}{1-q}}=\frac{1}{\left(\frac{t}{q}+(1-t)q\right)^{kr}}\neq 1$$

unless $r=0$ or $x=\frac{h}{1-q}$, and hence the infinite product $\prod_{j=0}^{\infty}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(r+j)}(x)}$ diverges.

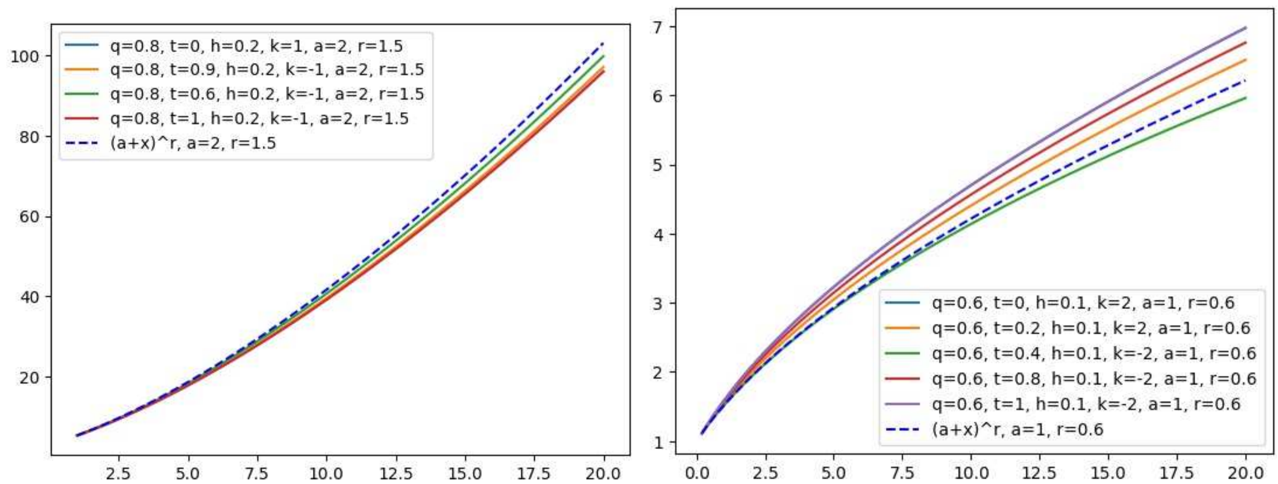

Fig. 1: Graph of α-power function for several parameters

To gain a deeper understanding of the α-power function, we plot Figure 1. It is evident that, the green graphs in both sides provide the closest approximation to the ordinary power function. It is noteworthy that in the left graph, the blue and red graphs (when $t=0$ and $t=1$) coincide, while in the right graph, the blue and purple graphs (when $t=0$ and $t=1$) coincide. Moreover, the graphs (For Python codes see [31]) in Figure 1, can be considered as approximations to discrete power functions.

Theorem 1.*The following additivity rule holds for the α^k-power function*

$$(a+x)^{s+r}_{\alpha^k}=(a+x)^s_{\alpha^k}(a+\alpha^{ks}(x))^r_{\alpha^k},\quad s,r\in\mathbb{R},\quad k\in\mathbb{Z}. \tag{9}$$

In particular, the shifted form of the additivity rule (9) is given by

$$(a+\alpha^{jk}(x))^{s+r}_{\alpha^k}=(a+\alpha^{jk}(x))^s_{\alpha^k}(a+\alpha^{(s+j)k}(x))^r_{\alpha^k},\quad s,r,j\in\mathbb{R},\quad k\in\mathbb{Z}. \tag{10}$$

*Proof.*We prove the theorem for $x_0>\frac{h}{1-q}$, the proof of the case $x_0<\frac{h}{1-q}$ is similar. We first consider the case when $a>-\frac{h}{1-q}$

$$\begin{aligned}(a+x)^{s+r}_{\alpha^k}&=\left(a+\frac{h}{1-q}\right)^{s+r}\prod_{j=0}^{\infty}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(s+r+j)}(x)}=\left(a+\frac{h}{1-q}\right)^{s}\left(a+\frac{h}{1-q}\right)^{r}\lim_{n\to\infty}\prod_{j=0}^{n}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(s+r+j)}(x)}\\&=\left(a+\frac{h}{1-q}\right)^{s}\lim_{n\to\infty}\prod_{j=0}^{n}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(s+j)}(x)}\cdot\left(a+\frac{h}{1-q}\right)^{r}\lim_{n\to\infty}\prod_{j=0}^{n}\frac{a+\alpha^{k(s+j)}(x)}{a+\alpha^{k(s+r+j)}(x)}\\&=(a+x)^s_{\alpha^k}\left(a+\frac{h}{1-q}\right)^{r}\lim_{n\to\infty}\prod_{j=0}^{n}\frac{a+\alpha^{kj}(\alpha^{ks}(x))}{a+\alpha^{k(r+j)}(\alpha^{ks}(x))}=(a+x)^s_{\alpha^k}(a+\alpha^{ks}(x))^r_{\alpha^k}.\end{aligned}$$

Using Definition 6 for $a=-\frac{h}{1-q}$ and Proposition 3/(iii), we derive

$$\begin{aligned}\left(-\frac{h}{1-q}+x\right)_{\alpha^k}^{s+r} &= \left(\frac{t}{q}+(1-t)q\right)^{\frac{k(s+r)(s+r-1)}{2}}\left(-\frac{h}{1-q}+x\right)^{s+r}\\ &= \left(\frac{t}{q}+(1-t)q\right)^{\frac{ks(s-1)}{2}}\left(-\frac{h}{1-q}+x\right)^{s}\left(\frac{t}{q}+(1-t)q\right)^{\frac{kr(r-1)}{2}}\left(\frac{t}{q}+(1-t)q\right)^{ksr}\left(-\frac{h}{1-q}+x\right)^{r}\\ &= \left(\frac{t}{q}+(1-t)q\right)^{\frac{ks(s-1)}{2}}\left(-\frac{h}{1-q}+x\right)^{s}\left(\frac{t}{q}+(1-t)q\right)^{\frac{kr(r-1)}{2}}\left(-\frac{h}{1-q}+\alpha^{ks}(x)\right)^{r}\\ &= \left(-\frac{h}{1-q}+x\right)_{\alpha^k}^{s}\left(-\frac{h}{1-q}+\alpha^{ks}(x)\right)_{\alpha^k}^{r}.\end{aligned}$$

The identity (10) follows from the additivity rule (9).

Proposition 7. *If n is a nonnegative integer, then*

$$(a+x)_{\alpha^k}^{n}=\prod_{j=0}^{n-1}\left(a+\alpha^{kj}(x)\right). \tag{11}$$

Proof. We present the proof when $x_0>\frac{h}{1-q}$, the proof of the case $x_0<\frac{h}{1-q}$ is similar. If $n=0$, the statement is obvious. If n is a positive integer, we have

$$\begin{aligned}(a+x)_{\alpha^k}^{n} &= \left(a+\frac{h}{1-q}\right)^{n}\lim_{N\to\infty}\prod_{j=0}^{N}\frac{a+\alpha^{kj}(x)}{a+\alpha^{k(n+j)}(x)} = \left(a+\frac{h}{1-q}\right)^{n}\lim_{N\to\infty}\frac{\prod_{j=0}^{n-1}\left(a+\alpha^{kj}(x)\right)\prod_{j=n}^{N}\left(a+\alpha^{kj}(x)\right)}{\prod_{j=0}^{N-n}\left(a+\alpha^{k(n+j)}(x)\right)\prod_{j=N-n+1}^{N}\left(a+\alpha^{k(n+j)}(x)\right)}\\ &= \left(a+\frac{h}{1-q}\right)^{n}\prod_{j=0}^{n-1}\left(a+\alpha^{kj}(x)\right)\prod_{j=1}^{n}\lim_{N\to\infty}\left(a+\alpha^{k(N+j)}(x)\right)^{-1}.\end{aligned}$$

If $1\le j\le n$, then by Proposition 3/(v), we have $\lim_{N\to\infty}\alpha^{k(N+j)}(x)=\frac{h}{1-q}$ and the result follows as

$$(a+x)_{\alpha^k}^{n}=\left(a+\frac{h}{1-q}\right)^{n}\prod_{j=0}^{n-1}\left(a+\alpha^{kj}(x)\right)\prod_{j=1}^{n}\left(a+\frac{h}{1-q}\right)^{-1}=\prod_{j=0}^{n-1}\left(a+\alpha^{kj}(x)\right).$$

If $a=-\frac{h}{1-q}$, the statement is straightforward by Proposition 3/(iii).

Theorem 2. *If $n\in\mathbb{N}_0$, then*

$$D_{\alpha^k}^{n}(a+x)_{\alpha^k}^{r}=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kn(n-1)}{2}}P_{\alpha^k}[r,n]\left(a+\alpha^{kn}(x)\right)_{\alpha^k}^{r-n}.$$

Proof. The assertion obviously holds when $n=0$. Now suppose it holds for n and prove it for $n+1$. Note that

$$D_{\alpha^k}^{n+1}(a+x)_{\alpha^k}^{r}=D_{\alpha^k}(D_{\alpha^k}^{n}(a+x)_{\alpha^k}^{r})=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kn(n-1)}{2}}P_{\alpha^k}[r,n]D_{\alpha^k}\left(a+\alpha^{kn}(x)\right)_{\alpha^k}^{r-n}.$$

Using Theorem 1, we have

$$\begin{aligned}(a+\alpha^{k(n+1)}(x))_{\alpha^k}^{r-n} &= (a+\alpha^{k(n+1)}(x))_{\alpha^k}^{r-(n+1)}(a+\alpha^{kr}(x)),\\ (a+\alpha^{kn}(x))_{\alpha^k}^{r-n} &= (a+\alpha^{kn}(x))(a+\alpha^{k(n+1)}(x))_{\alpha^k}^{r-(n+1)}\end{aligned}$$

which implies that

$$D_{\alpha^k}\left(a+\alpha^{kn}(x)\right)^{r-n}_{\alpha^k}=\frac{\left(a+\alpha^{k(n+1)}(x)\right)^{r-n}_{\alpha^k}-\left(a+\alpha^{kn}(x)\right)^{r-n}_{\alpha^k}}{\alpha^k(x)-x}=\frac{\left(a+\alpha^{k(n+1)}(x)\right)^{r-(n+1)}_{\alpha^k}\left(\alpha^{kr}(x)-\alpha^{kn}(x)\right)}{\alpha^k(x)-x}.$$

It follows from Proposition 3/(vi) that

$$\frac{\alpha^{kr}(x)-\alpha^{kn}(x)}{\alpha^k(x)-x}=\left(\frac{t}{q}+(1-t)q\right)^{kn}[r-n]_{\alpha^k}.$$

Noting $P_{\alpha^k}[r,n+1]=[r-n]_{\alpha^k}P_{\alpha^k}[r,n]$, the result follows. As in [15], the derivative at the accumulation point $\frac{h}{1-q}$ can be derived by applying l'Hôpital's rule and the logarithmic differentiation.

4 Applications

Cauchy-Euler equation models various real life dynamical problems and it has numerous applications in physics and applied mathematics such as time-harmonic vibrations of a thin elastic rod, in engineering such as wave mechanics and computer algorithm analysis and in finance such as Black-Scholes PDE's [5].

We aim to propose nth order α-Cauchy-Euler equation whose coefficients are α^k-polynomials (11) and whose solutions are expressed as α^k-power functions as in Definition 6. For this purpose, we start this section by presenting the following nth order IVP.

Theorem 3. *The power function $(a+x)^r_{\alpha^k}$ is the unique solution of the IVP*

$$(a+x)^n_{\alpha^k}D^n_{\alpha^k}y(x)=\gamma y(x), \tag{12}$$

$$D^j_{\alpha^k}y\left(\frac{h}{1-q}\right)=\delta_j,\qquad j=0,1,\ldots,n-1, \tag{13}$$

if γ,δ_j respectively satisfy

$$\gamma=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kn(n-1)}{2}}P_{\alpha^k}[r,n] \tag{14}$$

and

$$\delta_j=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kj(j-1)}{2}}P_{\alpha^k}[r,j]\left(a+\frac{h}{1-q}\right)^{r-j}. \tag{15}$$

Proof. Consider $y=(a+x)^r_{\alpha^k}$. We use Theorem 2 to compute $D^n_{\alpha^k}y$ and obtain

$$\begin{aligned}(a+x)^n_{\alpha^k}D^n_{\alpha^k}y-\gamma y&=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kn(n-1)}{2}}P_{\alpha^k}[r,n](a+x)^n_{\alpha^k}(a+\alpha^{kn}(x))^{r-n}_{\alpha^k}-\gamma(a+x)^r_{\alpha^k}\\&=\left(\left(\frac{t}{q}+(1-t)q\right)^{\frac{kn(n-1)}{2}}P_{\alpha^k}[r,n]-\gamma\right)(a+x)^r_{\alpha^k}=0,\end{aligned}$$

where we utilized Theorem 1. Hence the power function $y=(a+x)^r_{\alpha^k}$ satisfies the α-difference equation (12) provided that r satisfies the equation (14). Now we impose the initial conditions (13). By Proposition 3/(iv), it is clear that for $r\in\mathbb{R}$, $\alpha^r(\frac{h}{1-q})=\frac{h}{1-q}$ and by Definition 6 we have

$$y\left(\frac{h}{1-q}\right)=\left(a+\frac{h}{1-q}\right)^r_{\alpha^k}=\left(a+\frac{h}{1-q}\right)^r,$$

which implies that the first initial condition is satisfied for $j=0$ if $\delta_0=\left(a+\frac{h}{1-q}\right)^r$. Inductively, we obtain

$$D^j_{\alpha^k}y\left(\frac{h}{1-q}\right)=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kj(j-1)}{2}}P_{\alpha^k}[r,j]\left(a+\alpha^{kj}\left(\frac{h}{1-q}\right)\right)^{r-j}_{\alpha^k}=\left(\frac{t}{q}+(1-t)q\right)^{\frac{kj(j-1)}{2}}P_{\alpha^k}[r,j]\left(a+\frac{h}{1-q}\right)^{r-j}.$$

Therefore, if (15) hold for $j=0,1,\ldots,n-1$, the initial conditions (13) are satisfied. Hence, by [7, Theorem 3.36], $y=(a+x)^r_{\alpha^k}$ is the unique solution of the IVP (12)-(13).

4.1 α-Cauchy-Euler equation

In order not to repeat the similar calculations, in this section we restrict x to the case $x > \frac{h}{1-q}$. Motivated by the IVP (12)-(13), we propose the following nth order α-difference equation

$$L_n(y) := c_n(a+x)^n_{\alpha^k} D^n_{\alpha^k} y + c_{n-1}(a+x)^{n-1}_{\alpha^k} D^{n-1}_{\alpha^k} y + \ldots + c_1(a+x)^1_{\alpha^k} D^1_{\alpha^k} y + c_0 y = f(x), \tag{16}$$

where $c_j \in \mathbb{R}$ for $j = 0,1,2,\ldots,n$, with the leading coefficient $c_n \neq 0$ and the coefficients $(a+x)^j_{\alpha^k}$ are determined in terms of the polynomials (11) for $j = 0,1,2,\ldots,n$.

As $(a,q,h) \to (0,1,0)$, the equation (16) reduces to the ordinary Cauchy-Euler equation

$$c_n x^n \frac{d^n y}{dx^n} + c_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + c_1 x \frac{dy}{dx} + c_0 y = f(x).$$

Therefore, the equation (16) can be regarded as an α-analogue of Cauchy-Euler equation. The rest of its reductions will be examined in Remark 4.1.

Proposition 8. *If r is a real root of the following characteristic equation*

$$\sum_{j=0}^{n} c_j \left(\frac{t}{q} + (1-t)q\right)^{\frac{kj(j-1)}{2}} P_{\alpha^k}[r,j] = 0, \tag{17}$$

then the power function $y_1 = (a+x)^r_{\alpha^k}$ is a solution of $L_n(y) = 0$.

Proof. We use Theorem 2 to compute $D^j_{\alpha^k}(a+x)^r_{\alpha^k}$, $j = 1,2,\ldots,n$.

$$L_n(a+x)^r_{\alpha^k} = \sum_{j=0}^{n} c_j \left(\frac{t}{q} + (1-t)q\right)^{\frac{kj(j-1)}{2}} P_{\alpha^k}[r,j](a+x)^j_{\alpha^k}(a+\alpha^{kj}(x))^{r-j}_{\alpha^k}.$$

Then by Theorem 1, if (17) holds, we end up with

$$L_n(a+x)^r_{\alpha^k} = (a+x)^r_{\alpha^k} \sum_{j=0}^{n} c_j \left(\frac{t}{q} + (1-t)q\right)^{\frac{kj(j-1)}{2}} P_{\alpha^k}[r,j] = 0.$$

In order to analyze linearly independent solutions of α-difference equations, we introduce the α-analogue of Wronskian in terms of the shift operator (7).

Definition 7. *Let $f_1, f_2, \ldots, f_n$ be any real valued functions defined on $\mathbb{T}^{x_0}_{\alpha}$. We define the Wronskian of the functions $f_1, f_2, \ldots, f_n$, as follows*

$$W_{\alpha^k}(f_1,f_2,\ldots,f_n) := \begin{vmatrix} S_{\alpha^k}(f_1) & S_{\alpha^k}(f_2) & \cdots & S_{\alpha^k}(f_n) \\ S_{\alpha^k}(D_{\alpha^k} f_1) & S_{\alpha^k}(D_{\alpha^k} f_2) & \cdots & S_{\alpha^k}(D_{\alpha^k} f_n) \\ \vdots & \vdots & \vdots & \vdots \\ S_{\alpha^k}(D^{n-1}_{\alpha^k} f_1) & S_{\alpha^k}(D^{n-1}_{\alpha^k} f_2) & \cdots & S_{\alpha^k}(D^{n-1}_{\alpha^k} f_n) \end{vmatrix}.$$

Moreover, the functions $f_1, f_2, \ldots f_n$ as linearly independent if and only if $W_{\alpha^k}(f_1,f_2,\ldots,f_n)(x) \neq 0$ for all $x \in \mathbb{T}^{x_0}_{\alpha}$.

In order to present concrete examples for α-Cauchy-Euler equation, we focus on $n = 2$. By Proposition 8, if r is a root of

$$c_0 + c_1[r]_{\alpha^k} + c_2 \left(\frac{t}{q} + (1-t)q\right)^k [r]_{\alpha^k}[r-1]_{\alpha^k} = 0, \tag{18}$$

then $y_1 = (a+x)^r_{\alpha^k}$ is a solution of

$$L_2(y) = c_2(a+x)^2_{\alpha^k} D^2_{\alpha^k} y + c_1(a+x)^1_{\alpha^k} D^1_{\alpha^k} y + c_0 y = 0. \tag{19}$$

Theorem 4.*Let r be a root of (18). Then $y_1 = (a+x)^r_{\alpha^k}$ and $y_2 = v(x)(a+x)^r_{\alpha^k}$ are linearly independent solutions of (19), if the non-constant function $v(x)$ solves the α-difference equation*

$$c_2(a+x)^{r+2}_{\alpha^k} D^2_{\alpha^k} v(x) + \left(c_1 + c_2[r]_{\alpha^k}\left(1+\left(\frac{t}{q}+(1-t)q\right)^k\right)\right)\cdot(a+x)^{r+1}_{\alpha^k} D_{\alpha^k} v(x) = 0 \tag{20}$$

and $W_{\alpha^k}(y_1,y_2)(x) \neq 0$.

Note that, setting $\bar{v}(x) := D_{\alpha^k} v(x)$, the equation (20) turns out to be a first order α-difference equation.

*Proof.*By Proposition 8, $y_1 = (a+x)^r_{\alpha^k}$ is a solution of (19). In order to derive a second linearly independent solution, we use the method of reduction of order. Let $y_2 = v(x)(a+x)^r_{\alpha^k}$ be the second solution of (19). Using Proposition 5 and Proposition 4, we calculate $D^j_{\alpha^k}(y_2)$, $j = 1,2$ and using Theorem 1, the equation (19) turns out to be

$$v(x)(a+x)^r_{\alpha^k}\sum_{j=0}^{2} c_j\left(\frac{t}{q}+(1-t)q\right)^{\frac{kj(j-1)}{2}} P_{\alpha^k}[r,j] + c_2(a+x)^{r+2}_{\alpha^k} D^2_{\alpha^k} v(x)$$
$$+\left(c_1 + c_2[r]_{\alpha^k}\left(1+\left(\frac{t}{q}+(1-t)q\right)^k\right)\right)(a+x)^{r+1}_{\alpha^k} D_{\alpha^k} v(x) = 0,$$

whose first term vanishes since r is a root of the characteristic equation (18). Therefore, we conclude that the function $v(x)$ has to solve the equation (20). Moreover, $y_1 = (a+x)^r_{\alpha^k}$ and $y_2 = v(x)(a+x)^r_{\alpha^k}$ are linearly independent solutions of (19). Indeed, using Definition 7 and Proposition 5, we compute the Wronskian as

$$W_{\alpha^k}(y_1,y_2)(x) = \begin{vmatrix} S_{\alpha^k}(y_1) & S_{\alpha^k}(y_2) \\ S_{\alpha^k}(D_{\alpha^k} y_1) & S_{\alpha^k}(D_{\alpha^k} y_2) \end{vmatrix} = (a+\alpha^k(x))^r_{\alpha^k}(a+\alpha^{2k}(x))^r_{\alpha^k} S_{\alpha^k}(D_{\alpha^k} v(x)) \neq 0,$$

provided that $S_{\alpha^k}(D_{\alpha^k} v(x)) \neq 0$.

*Example 1.*Let $c_0 = 1$, $c_1 = -1$ and $c_2 = 1+\left(\frac{t}{q}+(1-t)q\right)^{k/2}$. Then, one can observe that $r_1 = 1/2$ is a root of (18), i.e., $y_1 = (a+x)^{1/2}_{\alpha^k}$ is a solution of

$$\left(1+\left(\frac{t}{q}+(1-t)q\right)^{k/2}\right)(a+x)^2_{\alpha^k} D^2_{\alpha^k} y - (a+x)^1_{\alpha^k} D^1_{\alpha^k} y + y = 0. \tag{21}$$

For the second solution, we use reduction of order method, i.e., we let $y_2 = v(x)(a+x)^{1/2}_{\alpha^k}$ and we use Theorem 4. Here the equation (20) reduces to

$$\left(1+\left(\frac{t}{q}+(1-t)q\right)^{k/2}\right)(a+x)^{5/2}_{\alpha^k} D^2_{\alpha^k} v(x) + \left(\frac{t}{q}+(1-t)q\right)^k (a+x)^{3/2}_{\alpha^k} D_{\alpha^k} v(x) = 0. \tag{22}$$

One can observe that (22) admits the following solution

$$v(x) = (a+\alpha^{k/2}(x))^{1/2}_{\alpha^k},$$

by the use of Proposition 4. By Theorem 1, we conclude that

$$y_2 = v(x)(a+x)^{1/2}_{\alpha^k} = (a+\alpha^{k/2}(x))^{1/2}_{\alpha^k}(a+x)^{1/2}_{\alpha^k} = (a+x)^1_{\alpha^k},$$

is the second solution of (21). One can check also that $r_2 = 1$ is another root of (18). Moreover, $y_1 = (a+x)^1_{\alpha^k}$ and $y_2 = (a+x)^{1/2}_{\alpha^k}$ are linearly independent by Theorem 4. Alternatively observing that 1 is a root of (18), one can use a similar procedure with $\bar{y}_2 = w(x)(a+x)^1_{\alpha^k}$ and conclude that $1/2$ is another root of (18) from $\bar{y}_2 = (a+\alpha^k(x))^{-1/2}_{\alpha^k}(a+x)^1_{\alpha^k} = (a+x)^{1/2}_{\alpha^k}$. Similar to this example, once we have a solution of $L_n(y) = 0$, it is possible to derive other linearly independent solutions by the reduction of order method.

Note that, if $y_1 = (a+x)^r_{\alpha^k}$ and $y_2 = v(x)(a+x)^r_{\alpha^k}$ are linearly independent solutions of (18), but y_2 fails to be another power function with a power different than r, this situation can be regarded as repeated root case for which $v(x)$ has a logarithmic counterpart. We discuss such a case and introduce α-analogue of logarithm function in the forthcoming example.

*Example 2.*Let $c_0 = 1$, $c_1 = -1$ and $c_2 = 1$. Then, one can observe that $r = 1$ is a root of (18), i.e., $y_1 = (a+x)^1_{\alpha^k} = a+x$ is a solution of

$$(a+x)^2_{\alpha^k} D^2_{\alpha^k} y - (a+x)^1_{\alpha^k} D^1_{\alpha^k} y + y = 0. \tag{23}$$

By Theorem 4, if $v(x)$ satisfies (20), then $y_2 = v(x)(a+x)$ is another solution of (23). Here (20) reduces to

$$(a+x)^3_{\alpha^k} D^2_{\alpha^k} v(x) + \left(\frac{t}{q} + (1-t)q\right)^k (a+x)^2_{\alpha^k} D_{\alpha^k} v(x) = 0,$$

which can be rewritten by Theorem 1 and Proposition 5

$$(a+x)^2_{\alpha^k} \left((a+\alpha^{2k}(x)) D^2_{\alpha^k} v(x) + \left(\frac{t}{q} + (1-t)q\right)^k D_{\alpha^k} v(x) \right) = (a+x)^2_{\alpha^k} D_{\alpha^k} \left((a+\alpha^k(x)) D_{\alpha^k} v(x) \right) = 0.$$

Therefore, the function $v(x)$ needs to satisfy

$$D_{\alpha^k} v(x) = \frac{c}{a+\alpha^k(x)}, \quad c \in \mathbb{R}. \tag{24}$$

Hence, $y_2 = v(x)(a+x)$ is the second solution if (24) holds. Moreover $y_1 = a+x$ and $y_2 = v(x)(a+x)$ are linearly independent since

$$W_{\alpha^k}(y_1, y_2)(x) = \begin{vmatrix} a+\alpha^k(x) & v(\alpha^k(x))(a+\alpha^k(x)) \\ 1 & S_{\alpha^k}\left((a+\alpha^k(x)) D_{\alpha^k} v(x) + v(x)\right) \end{vmatrix} = c(a+\alpha^k(x)) \neq 0,$$

where we used Proposition 5 and (24).

Inspired by (24) and Example 4, we introduce the α-analogue of logarithm function.

Definition 8.*The α^k-logarithm function is defined by*

$$\mathrm{Log}_{\alpha^k}(a+x) := \begin{cases} \dfrac{[k]_\alpha \left(\frac{t}{q} + (1-t)q - 1\right)}{k \ln\left(\frac{t}{q} + (1-t)q\right)} \ln\left| x - \dfrac{h}{1-q} \right| & \text{if} \quad a = -\frac{h}{1-q}, \\ \displaystyle\int_{[\frac{h}{1-q}, x]} \frac{d_{\alpha^k}\tau}{a+\tau} & \text{if} \quad a \neq -\frac{h}{1-q}, \end{cases} \tag{25}$$

where α^k-integral (see Definition 10) and its properties are presented in Appendix.

Remark. (i)If $a = -\frac{h}{1-q}$, then using l'Hôpital's rule, we have the following limit

$$\lim_{(q,h)\to(1,0)} \mathrm{Log}_{\alpha^k}\left(-\frac{h}{1-q} + x\right) = \ln|x|.$$

(ii)If $h = 0$, $t = 0$, $k = 1$, (25) reduces to q-logarithm

$$\mathrm{Log}_{\alpha^k}(x) = \frac{(q-1)}{\ln q} \ln|x|,$$

which coincides with the logarithm function presented in [23].

(iii)If $a \neq -\frac{h}{1-q}$, then $\frac{1}{a+x}$ is bounded near $\frac{h}{1-q}$ and it follows from Definition 10 that α-logarithm function is a well-defined function with $\mathrm{Log}_{\alpha^k}(a+\frac{h}{1-q})=0$ and can be presented as the following series

$$\mathrm{Log}_{\alpha^k}(a+x)=\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\cdot\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj}\frac{1}{a+\alpha^{kj}(x)},$$

which is absolutely convergent for $\left(\frac{t}{q}+(1-t)q\right)^k<1$. Furthermore, Theorem 10 implies that $D_{\alpha^k}\mathrm{Log}_{\alpha^k}(a+x)=\frac{1}{a+x}$. To visualize the α-logarithm function, we refer to Figure 2. One can observe that, the green graphs (for $t=0.5$ on the left graph and for $t=0.4$ on the right graph) best approximate the ordinary logarithm function. Note that, on the left graph the blue and red graphs (when $t=0$ and $t=1$) coincide while on the right graph, blue and purple graphs (when $t=0$ and $t=1$) coincide. Furthermore, the left and the right graphs can be considered as approximations to h- and q-logarithm functions, respectively. For Python codes see [32].

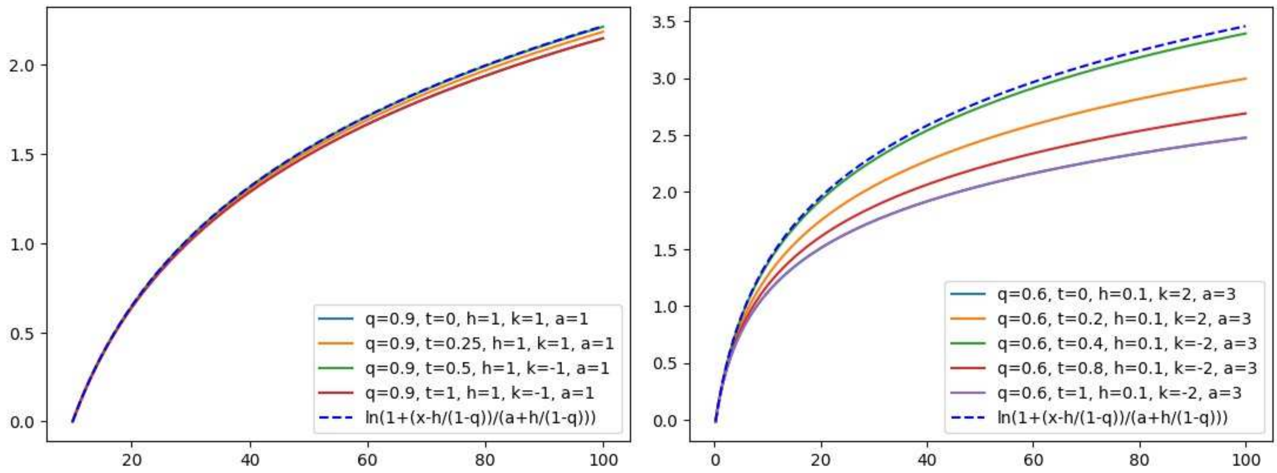

Fig. 2: Graph of α-logarithm function for several parameters

(iv)The equation (23) can be understood as a repeated root example whose linearly independent solutions are

$$y_1=(a+x)^1_{\alpha^k},\qquad y_2=\mathrm{Log}_{\alpha^k}(a+\alpha^k(x))(a+x)^1_{\alpha^k},$$

since by Proposition (4) $D_{\alpha^k}\mathrm{Log}_{\alpha^k}(a+\alpha^k(x))=\frac{(\frac{t}{q}+(1-t)q)^k}{a+\alpha^k(x)}$ which is consistent with (25).

Theorem 5.*Let $y_j,\ j=1,2,\ldots,n$, be linearly independent solutions of the homogeneous equation $L_n(y)=0$. Then the particular solution y_p of the nonhomogeneous equation (16) is presented as*

$$y_p(x)=\sum_{j=1}^{n}y_j\int\frac{\det(A_j)d_{\alpha^k}x}{W_{\alpha^k}(y_1,y_2,\ldots,y_n)(x)}. \tag{26}$$

Here A_j is the matrix which is determined by interchanging the jth column of the matrix

$$A=\begin{bmatrix} S_{\alpha^k}(y_1) & S_{\alpha^k}(y_2) & \cdots & S_{\alpha^k}(y_n)\\ S_{\alpha^k}(D_{\alpha^k}y_1) & S_{\alpha^k}(D_{\alpha^k}y_2) & \cdots & S_{\alpha^k}(D_{\alpha^k}y_n)\\ \vdots & \vdots & \vdots & \vdots\\ S_{\alpha^k}(D^{n-1}_{\alpha^k}y_1) & S_{\alpha^k}(D^{n-1}_{\alpha^k}y_2) & \cdots & S_{\alpha^k}(D^{n-1}_{\alpha^k}y_n)\end{bmatrix}$$

with the vector $\left(0,0,\ldots,\frac{f(x)}{c_n(a+x)^n_{\alpha^k}}\right)^T$.

*Proof.*For the proof, we use the method of variation of parameters. Assume that

$$y_p(x) = \sum_{j=1}^{n} u_j(x) y_j(x),$$

is a solution of (16), where u_j's are the functions to be determined for $j = 1,2,\ldots,n$. Proposition 5 implies that

$$D_{\alpha^k}(y_p) = \sum_{j=1}^{n} u_j D_{\alpha^k}(y_j) + S_{\alpha^k}(y_j) D_{\alpha^k}(u_j).$$

We assume that the functions u_j, $j = 1,2,\ldots,n$ satisfy the equation

$$\sum_{j=1}^{n} S_{\alpha^k}(y_j) D_{\alpha^k}(u_j) = 0. \tag{27}$$

Under the assumption (27),

$$D_{\alpha^k}(y_p) = \sum_{j=1}^{n} u_j D_{\alpha^k}(y_j). \tag{28}$$

Inductively, for $l = 0,1,\ldots,n-2$ we assume

$$\sum_{j=1}^{n} S_{\alpha^k}(D^l_{\alpha^k} y_j) D_{\alpha^k}(u_j) = 0, \tag{29}$$

which implies

$$D^{l+1}_{\alpha^k}(y_p) = \sum_{j=1}^{n} \left(u_j D^{l+1}_{\alpha^k}(y_j) + S_{\alpha^k}(D^l_{\alpha^k} y_j) D_{\alpha^k}(u_j) \right) = \sum_{j=1}^{n} u_j D^{l+1}_{\alpha^k}(y_j). \tag{30}$$

Accordingly, $D^n_{\alpha^k}(y_p)$ is derived as

$$D^n_{\alpha^k}(y_p) = \sum_{j=1}^{n} \left(u_j D^n_{\alpha^k}(y_j) + S_{\alpha^k}(D^{n-1}_{\alpha^k} y_j) D_{\alpha^k}(u_j) \right). \tag{31}$$

Finally, plugging the related derivatives (28), (30), (31) of y_p in (16), we end up with

$$L_n(y_p) = \sum_{j=1}^{n} u_j L_n(y_j) + c_n (a+x)^n_{\alpha^k} \sum_{j=1}^{n} S_{\alpha^k}(D^{n-1}_{\alpha^k} y_j) D_{\alpha^k}(u_j) = f(x)$$

from which we derive

$$\sum_{j=1}^{n} S_{\alpha^k}(D^{n-1}_{\alpha^k} y_j) D_{\alpha^k}(u_j) = \frac{f(x)}{c_n (a+x)^n_{\alpha^k}} \tag{32}$$

since $L_n(y_j) = 0$ for $j = 1,2,\ldots,n$. Taking equations (27), (29) and (32) into account, we conclude with

$$\begin{bmatrix} S_{\alpha^k}(y_1) & S_{\alpha^k}(y_2) & \cdots & S_{\alpha^k}(y_n) \\ S_{\alpha^k}(D_{\alpha^k} y_1) & S_{\alpha^k}(D_{\alpha^k} y_2) & \cdots & S_{\alpha^k}(D_{\alpha^k} y_n) \\ \vdots & \vdots & \vdots & \vdots \\ S_{\alpha^k}(D^{n-1}_{\alpha^k} y_1) & S_{\alpha^k}(D^{n-1}_{\alpha^k} y_2) & \cdots & S_{\alpha^k}(D^{n-1}_{\alpha^k} y_n) \end{bmatrix} \begin{bmatrix} D_{\alpha^k}(u_1) \\ D_{\alpha^k}(u_2) \\ \vdots \\ D_{\alpha^k}(u_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{f(x)}{c_n (a+x)^n_{\alpha^k}} \end{bmatrix}.$$

By Cramer's Rule, each unknown $D_{\alpha^k}(u_j)$ can be determined as follows

$$D_{\alpha^k}(u_j(x)) = \frac{\det(A_j)}{W_{\alpha^k}(y_1, y_2, \ldots, y_n)(x)},$$

since $\det A = W_{\alpha^k}(y_1, y_2, \ldots, y_n)(x) \neq 0$. To sum up, the particular solution of (16) is computed as (26).

*Example 3.*Consider the nonhomogenous problem

$$L_2(y) := \frac{1}{[2]_{\alpha^k}}(a+x)^2_{\alpha^k} D^2_{\alpha^k} y - (a+x)^1_{\alpha^k} D^1_{\alpha^k} y + y = (a+x)^s_{\alpha^k}, \qquad s \in \mathbb{R}. \tag{33}$$

One can show that $y_1(x) = (a+x)^1_{\alpha^k}$ and $y_2(x) = (a+x)^2_{\alpha^k}$ are solutions of $L_2(y) = 0$ and they are linearly independent since

$$W_{\alpha^k}(y_1, y_2)(x) = \left(\frac{t}{q} + (1-t)q\right)^k (a + \alpha^k(x))^2_{\alpha^k} \neq 0.$$

Assuming $y_p(x) = u_1(x)(a+x)^1_{\alpha^k} + u_2(x)(a+x)^2_{\alpha^k}$ and using Theorem 5, we obtain

$$D_{\alpha^k} u_1 = -\frac{[2]_{\alpha^k}(a+\alpha^{2k}(x))^{s-2}_{\alpha^k}}{\left(\frac{t}{q} + (1-t)q\right)^k} \text{ and } D_{\alpha^k} u_2 = \frac{[2]_{\alpha^k}(a+\alpha^{3k}(x))^{s-3}_{\alpha^k}}{\left(\frac{t}{q} + (1-t)q\right)^k}$$

from which we acquire

$$u_1 = -\frac{[2]_{\alpha^k}(a+\alpha^{k}(x))^{s-1}_{\alpha^k}}{\left(\frac{t}{q} + (1-t)q\right)^{2k}[s-1]_{\alpha^k}} \text{ and } u_2 = \frac{[2]_{\alpha^k}(a+\alpha^{2k}(x))^{s-2}_{\alpha^k}}{\left(\frac{t}{q} + (1-t)q\right)^{3k}[s-2]_{\alpha^k}}.$$

Then by the use of Theorem 1, the particular solution of (33) is derived as

$$y_p(x) = [2]_{\alpha^k}(a+x)^s_{\alpha^k}\left(\frac{1}{\left(\frac{t}{q} + (1-t)q\right)^{3k}[s-2]_{\alpha^k}} - \frac{1}{\left(\frac{t}{q} + (1-t)q\right)^{2k}[s-1]_{\alpha^k}}\right).$$

*Remark.*The α-Cauchy-Euler equation (16) is a generic equation which unifies well-known discrete Cauchy-Euler equations and their generators for $t \in \{0,1\}$ and it allows their extensions for $t \in (0,1)$.

(i)If $t = 0$, the coefficients (11) and the α^k-derivative (6) turn out to be

$$(a+x)^n_{(q,h,k)} = \prod_{j=0}^{n-1}(a + q^{kj}x + h[kj]_q), \qquad D_{\alpha^k} f(x) = D_{(q,h,k)} f(x)$$

respectively. These reductions demonstrate that the α-Cauchy-Euler equation (16) reduces to

$$\sum_{j=0}^{n} c_j (a+x)^j_{(q,h,k)} D^j_{(q,h,k)} y = f(x)$$

which can be regarded as the *generator* of delta-(q,h)-Cauchy-Euler equation since for $k = 1$, it recovers the delta-(q,h)-Cauchy-Euler equation $\sum_{j=0}^{n} c_j (a+x)^j_{(q,h)} D^j_{(q,h)} y = f(x)$.

(ii)If $t = 1$, the coefficients (11) and the α^k-derivative (6) result in respectively

$$(a+x)^n_{\widetilde{(q,h,k)}} = \prod_{j=0}^{n-1}\left(a + \frac{x - h[jk]_q}{q^{jk}}\right), \qquad D_{\alpha^k} f(x) = \widetilde{D}_{(q,h,k)} f(x),$$

providing the *generator* of nabla-(q,h)-Cauchy-Euler equation

$$\sum_{j=0}^{n} c_j (a+x)^j_{\widetilde{(q,h,k)}} \widetilde{D}^j_{(q,h,k)} y = f(x)$$

since for $k = 1$ it produces the nabla-(q,h)-Cauchy-Euler equation $\sum_{j=0}^{n} c_j (a+x)^j_{\widetilde{(q,h)}} \widetilde{D}_{(q,h)} y = f(x)$.

(iii)If $h=0$, then $\alpha^k(x)=\left(\frac{t}{q}+(1-t)q\right)^k x$, which leads the q-polynomial and q-derivative generators:

$$(a+x)^n_{q,t,k}:=\prod_{j=0}^{n-1}\left(a+\left(\frac{t}{q}+(1-t)q\right)^{kj}x\right),\qquad D_{q,t,k}f(x):=\frac{f\left(\left(\frac{t}{q}+(1-t)q\right)^k x\right)-f(x)}{\left(\frac{t}{q}+(1-t)q\right)^k x-x}.$$

Hence (16) evolves into

$$\sum_{j=0}^{n} c_j(a+x)^j_{q,k,t}D^j_{q,k,t}y=f(x). \tag{34}$$

(a)Additionally, if $t=0$, then $\alpha^k(x)=q^k x$, $(a+x)^n_{\alpha^k}=(a+x)^n_{q,k}$, $D_{\alpha^k}f(x)=D_{q,k}f(x)$. In this case (34) takes the form $\sum_{j=0}^{n} c_j(a+x)^j_{q,k}D^j_{q,k}y=f(x)$ which can be considered as the generator of delta q-Cauchy-Euler equation. The case $k=1$, $a=0$ $n=2$ and $c_2=q$ leads to delta q-Cauchy-Euler equation whose second order form is studied in [10]

$$x\sigma(x)D^2_q y+c_1xD_q y+c_0y=0.$$

(b)On the other hand, if $t=1$, then $\alpha^k(x)=q^{-k}x$, $(a+x)^n_{\alpha^k}=(a+x)^n_{\widetilde{q,k}}$ and $D_{\alpha^k}f(x)=\widetilde{D}_{q,k}f(x)$. Here, the equation (34) yields to the generator of nabla q-Cauchy-Euler equation $\sum_{j=0}^{n} c_j(a+x)^j_{\widetilde{q,k}}\widetilde{D}^j_{q,k}y=f(x)$ which provides nabla q-Cauchy-Euler equation for $k=1$.

(iv)If $q=1$, then $\alpha^k(x)=x-(2t-1)kh$, and the h-polynomial and h-derivative generators follow:

$$(a+x)^n_{\alpha^k}=(a+x)^n_{h,k,t}\quad\text{and}\quad D_{\alpha^k}f(x)=D_{h,k,t}f(x).$$

These reductions result in the equation (16) to be in the form

$$\sum_{j=0}^{n} c_j(a+x)^j_{h,k,t}D^j_{h,k,t}y=f(x). \tag{35}$$

If $t=0$, then (35) reduces to the generator of delta h-Cauchy-Euler equation $\sum_{j=0}^{n} c_j(a+x)^j_{h,k}D^j_{h,k}y=f(x)$ which produces delta h-Cauchy-Euler equation for $k=1$. On the other hand, if $t=1$, then the equation (35) conforms into $\sum_{j=0}^{n} c_j(a+x)^j_{\widetilde{h,k}}\widetilde{D}^j_{h,k}y=f(x)$ which may be interpreted as the generator of nabla h-Cauchy-Euler equation, since for $k=1$, it leads to nabla h-Cauchy-Euler equation. The differential difference analogue of Cauchy-Euler equation is studied in [11].

(v)If $t\in(0,1)$, new extension for discrete derivatives and polynomials are accomplished. For instance, if $|s|<1$ and $t=\frac{q^2-q^{s+1}}{q^2-1}$, then $t\in(0,1)$ for which (4) and (6) imply novel extensions respectively

$$\alpha^k(x)=q^{sk}x+h[s]_q[k]_\alpha,\qquad D_{\alpha^k}f(x)=\frac{f(q^{sk}x+h[s]_q[k]_\alpha)-f(x)}{(q^{sk}-1)x+h[s]_q[k]_\alpha}, \tag{36}$$

for $k\in\mathbb{Z}$, $\quad |s|<1$. The α-Cauchy-Euler equation (16) equipped with (36) produces new extensions.

4.2 BVP with two point unmixed boundary conditions

We propose the following BVP for the second order α-Cauchy-Euler equation with two point unmixed boundary conditions

$$L_2(y):=c_2(a+x)^2_{\alpha^k}D^2_{\alpha^k}y+c_1(a+x)^1_{\alpha^k}D_{\alpha^k}y+c_0y=f(x), \tag{37}$$

$$B_1y(\gamma):=\beta_{11}y(\gamma)+\beta_{12}D_{\alpha^k}y(\gamma)=0, \tag{38}$$

$$B_2y(\delta):=\beta_{21}y(\delta)+\beta_{22}D_{\alpha^k}y(\delta)=0, \tag{39}$$

where $\gamma,\delta\in\mathbb{T}_{\alpha}^{x_0}$, the constants $\beta_{ij}\in\mathbb{R}$ satisfy the conditions $\beta_{11}^2+\beta_{12}^2>0$ and $\beta_{21}^2+\beta_{22}^2>0$. Using Theorem 5 and the forthcoming lemma, we present the solution of BVP (37)-(39) by means of Green's functions.

Lemma 1. *Let*

$$G(x,s):=\begin{cases}\dfrac{y_1(x)S_{\alpha^k}y_2(s)}{c_2W_{\alpha^k}(y_1,y_2)(s)(a+s)_{\alpha^k}^2} & \text{if } x\in(\alpha^{2k}(\gamma),s),\\[2ex] \dfrac{y_2(x)S_{\alpha^k}y_1(s)}{c_2W_{\alpha^k}(y_1,y_2)(s)(a+s)_{\alpha^k}^2} & \text{if } \quad x\in(x,\delta].\end{cases} \tag{40}$$

Then $G(x,s)$ solves the BVP (37)-(39) *for $f(x)=0$.*

Proof. Since y_1 and y_2 form the fundamental set of solutions, then $L_2(y_j)=0$, for $j=1,2$. It is straightforward to conclude that $L_2(G(x,s))=0$. Moreover, boundary conditions (38) and (39) allow us to have $B_1y_1(\gamma)=0$ and $B_2y_2(\delta)=0$, respectively. Hence

$$B_1G(\gamma,s)=B_2G(\delta,s)=0. \tag{41}$$

The function $G(x,s)$ can be referred as the *α-analogue of Green's function* [29].

Theorem 6. *Let*

$$\phi(x):=\int_{[\gamma,\delta]}G(x,s)f(s)d_{\alpha^k}s, \tag{42}$$

where G is defined by (40). *Then $\phi(x)$ solves the BVP* (37)-(39).

Proof. We define

$$K_j(s):=\frac{S_{\alpha^k}y_j(s)f(s)}{c_2W_{\alpha^k}(y_1,y_2)(s)(a+s)_{\alpha^k}^2},\qquad j=1,2.$$

Then (42) can be written as

$$\phi(x)=\int_{[\gamma,\delta]}G(x,s)f(s)d_{\alpha^k}s=\int_{[\gamma,x]}y_2(x)K_1(s)d_{\alpha^k}s+\int_{(x,\delta]}y_1(x)K_2(s)d_{\alpha^k}s. \tag{43}$$

Using Proposition 9, we derive

$$\begin{aligned}D_{\alpha^k}\phi(x)&=D_{\alpha^k}\left\{\int_{[\gamma,x]}y_2(x)K_1(s)d_{\alpha^k}s\right\}-D_{\alpha^k}\left\{\int_{[\delta,x)}y_1(x)K_2(s)d_{\alpha^k}s\right\}\\&=S_{\alpha^k}y_2(x)K_1(x)+\int_{[\gamma,x]}D_{\alpha^k}y_2(x)K_1(s)d_{\alpha^k}s-S_{\alpha^k}y_1(x)K_2(x)-\int_{[\delta,x)}D_{\alpha^k}y_1(x)K_2(s)d_{\alpha^k}s\\&=\int_{[\gamma,x]}D_{\alpha^k}y_2(x)K_1(s)d_{\alpha^k}s+\int_{(x,\delta]}D_{\alpha^k}y_1(x)K_2(s)d_{\alpha^k}s.\end{aligned} \tag{44}$$

Similarly, we also achieve

$$D_{\alpha^k}^2\phi(x)=\int_{[\gamma,x]}D_{\alpha^k}^2y_2(x)K_1(s)d_{\alpha^k}s+S_{\alpha^k}(D_{\alpha^k}y_2(x))K_1(x)+\int_{(x,\delta]}D_{\alpha^k}^2y_1(x)K_2(s)d_{\alpha^k}s-S_{\alpha^k}(D_{\alpha^k}y_1(x))K_2(x). \tag{45}$$

Using Definition 7, we obtain

$$S_{\alpha^k}(D_{\alpha^k}y_2(x))K_1(x)-S_{\alpha^k}(D_{\alpha^k}y_1(x))K_2(x)=\frac{f(x)}{c_2(a+s)_{\alpha^k}^2},$$

from which the equation (45) becomes

$$D_{\alpha^k}^2\phi(x)=\int_{[\gamma,x]}D_{\alpha^k}^2y_2(x)K_1(s)d_{\alpha^k}s+\int_{(x,\delta]}D_{\alpha^k}^2y_1(x)K_2(s)d_{\alpha^k}s+\frac{f(x)}{c_2(a+s)_{\alpha^k}^2}. \tag{46}$$

Substituting (43), (44) and (46) in (37), we end up with $L_2\phi=f$. We emphasize that the boundary conditions $B_1\phi(\gamma)=B_2\phi(\delta)=0$ hold by (41). Hence, $\phi(x)$ defined by (42), solves the BVP (37)-(39).

5 Appendix: α-Integration

This section is dedicated to α^k-integration and its analysis. Let f be any real valued function with $D_{\alpha^k}F(x) = f(x)$ for all $x \neq \frac{h}{1-q}$, it follows from the definition of the α^k-derivative and the shift operator (7) that

$$(1 - S_{\alpha^k})F(x) = (x - \alpha^k(x))f(x)$$

and hence by Proposition 3/(iii), we get

$$F(x) = \frac{1}{1-S_{\alpha^k}}((x-\alpha^k(x))f(x)) = \sum_{j=0}^{\infty} S_{\alpha^k}^j((x-\alpha^k(x))f(x)) = \sum_{j=0}^{\infty} S_{\alpha^k}^j\left(\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)f(x)\right)$$
$$= \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(x)). \tag{47}$$

To be more precise, F can be written more explicitly as,

$$F(x) = \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right) \times \sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f\left(\left(\frac{t}{q}+(1-t)q\right)^{kj}\left(x-\frac{h}{1-q}\right)+\frac{h}{1-q}\right)$$

Now we show the convergence of the series (47) under certain conditions.

Theorem 7. *Suppose that $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ and $\left|\left(x-\frac{h}{1-q}\right)^r f(x)\right| < M$ on $0 < \left|x-\frac{h}{1-q}\right| < a$ for some $0 \leq r < 1$, $M > 0$ and $a > 0$. Then the following properties hold.*

(i) *For any $b > a$, the series (47) is uniformly absolutely convergent on $\left|x-\frac{h}{1-q}\right| < b$. Hence it is absolutely convergent for all x.*

(ii) *The function F given by (47) is continuous at the accumulation point $\frac{h}{1-q}$ with $F\left(\frac{h}{1-q}\right) = 0$.*

(iii) *The function F is an α^k-antiderivative of f for $x \neq \frac{h}{1-q}$. If $\lim_{x\to\frac{h}{1-q}} f(x) = L \in \mathbb{R}$, then F is also differentiable at $\frac{h}{1-q}$ with $(D_{\alpha^k}F)\left(\frac{h}{1-q}\right) = L$. In particular, if f is continuous at $\frac{h}{1-q}$, then $(D_{\alpha^k}F)\left(\frac{h}{1-q}\right) = f\left(\frac{h}{1-q}\right)$.*

Proof. (i) Let $b > a$ be arbitrary and $|x-\frac{h}{1-q}| < b$. Set

$$c_j(x) := \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(x)).$$

Our aim is to show that $|c_j(x)|$ is dominated by a convergent numerical series. Note that

$$\left|\alpha^{kj}(x)-\frac{h}{1-q}\right| = \left(\frac{t}{q}+(1-t)q\right)^{kj}\left|x-\frac{h}{1-q}\right| < \left(\frac{t}{q}+(1-t)q\right)^{kj} b < b \quad \text{for all} \quad j \in \mathbb{N}_0,$$

and therefore either we have $|\alpha^{kj}(x)-\frac{h}{1-q}| < a$ or $a \leq |\alpha^{kj}(x)-\frac{h}{1-q}| < b$. Since $\left(\frac{t}{q}+(1-t)q\right)^{kj} \to 0$ as $j\to\infty$, the latter case occurs only for finitely many j's. More precisely, if we choose m so that $\left(\frac{t}{q}+(1-t)q\right)^{kj} < \frac{a}{b}$ for $j \geq m$, then $|\alpha^{kj}(x)-\frac{h}{1-q}| < a$ for all $j \geq m$ and so $a \leq |\alpha^{kj}(x)-\frac{h}{1-q}| < b$ holds for at most m times. When $|\alpha^{kj}(x)-\frac{h}{1-q}| < a$, the condition implies

$$\left|\left(\alpha^{kj}(x)-\frac{h}{1-q}\right)^r f(\alpha^{kj}(x))\right| = \left(\frac{t}{q}+(1-t)q\right)^{kjr}\left|x-\frac{h}{1-q}\right|^r |f(\alpha^{kj}(x))| < M, \tag{48}$$

and so

$$|c_j(x)| < \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(\frac{t}{q}+(1-t)q\right)^{(1-r)kj} b^{1-r}M.$$

When $a \le |\alpha^{kj}(x)-\frac{h}{1-q}| < b$,

$$|c_j(x)| < \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right) bC,$$

where $C=\max\{|f(\alpha^{kj}(x))| : \alpha^{kj}(x)\in(\frac{h}{1-q}-b,\frac{h}{1-q}+b)\setminus(\frac{h}{1-q}-a,\frac{h}{1-q}+a)\}$. Note that such a maximum exists because for only finitely many j's, $\alpha^{kj}(x)$ lie in $(\frac{h}{1-q}-b,\frac{h}{1-q}+b)\setminus(\frac{h}{1-q}-a,\frac{h}{1-q}+a)$. If we set

$$c_j := \begin{cases} \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(\frac{t}{q}+(1-t)q\right)^{(1-r)kj} b^{1-r}M & \text{if} \quad |\alpha^{kj}(x)-\frac{h}{1-q}|<a, \\ \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right) bC & \text{if} \quad a\le|\alpha^{kj}(x)-\frac{h}{1-q}|<b, \end{cases}$$

we obtain $|c_j(x)| < c_j$ and

$$\sum_{j=0}^{\infty} c_j \le \left(\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right) b^{1-r}M\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{(1-r)kj}\right)+m\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)bC<\infty.$$

Therefore the series converges uniformly by the Weierstrass M-test.

(ii)It follows from (47) that

$$F\left(\frac{h}{1-q}\right)=\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(\frac{h}{1-q}-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f\left(\frac{h}{1-q}\right)=0.$$

When $|x-\frac{h}{1-q}|<a$, we have $|\alpha^{kj}(x)-\frac{h}{1-q}|<a$ for all j and hence as in (48), we get

$$|F(x)| \le \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left|x-\frac{h}{1-q}\right|^{1-r} M\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{k(1-r)j},$$

and this shows F tends to 0 as x tends to $\frac{h}{1-q}$ and hence F is continuous at $\frac{h}{1-q}$.

(iii)First, let us show that $D_{\alpha^k}F(x)=f(x)$ for $x\neq\frac{h}{1-q}$. Using Proposition 3/(iii),(vi), we obtain $\alpha^k(x)-\frac{h}{1-q}=\left(\frac{t}{q}+(1-t)q\right)^k(x-\frac{h}{1-q})$ and $\alpha^k(x)-x=[k]_\alpha(\frac{t}{q}+(1-t)q-1)(x-\frac{h}{1-q})$, and so

$$\begin{aligned} D_{\alpha^k}F(x) &= \frac{F(\alpha^k(x))-F(x)}{\alpha^k(x)-x} \\ &= -\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{k(j+1)} f(\alpha^{k(j+1)}(x))+\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(x))=f(x). \end{aligned}$$

Now, suppose $\lim_{x\to\frac{h}{1-q}} f(x)=L\in\mathbb{R}$. By definition of the derivative at the limit point and using uniform convergence proved in part (i), we obtain

$$\begin{aligned}(D_{\alpha^k}F)\left(\frac{h}{1-q}\right) &= \lim_{\substack{x\to\frac{h}{1-q}\\ x\in\mathbb{T}_\alpha^{x_0}}} \frac{F(x)-F(\frac{h}{1-q})}{x-\frac{h}{1-q}} \\ &= \lim_{\substack{x\to\frac{h}{1-q}\\ x\in\mathbb{T}_\alpha^{x_0}}} \frac{\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(x)).}{x-\frac{h}{1-q}} \\ &= \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj}\lim_{\substack{x\to\frac{h}{1-q}\\ x\in\mathbb{T}_\alpha^{x_0}}} f(\alpha^{kj}(x)) \\ &= \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj} L = L.\end{aligned}$$

The following example shows that the condition $\left|\left(x-\frac{h}{1-q}\right)^r f(x)\right|<M$ in Theorem 7 cannot be weakened. To be more precise, the sufficiency condition $r<1$ cannot be replaced with $r\le 1$.

*Example 4.*Let $F(x)=A\ln\left|x-\frac{h}{1-q}\right|$ with $A=\frac{[k]_\alpha\left(\frac{t}{q}+(1-t)q-1\right)}{k\ln\left(\frac{t}{q}+(1-t)q\right)}$. Then, we have

$$D_{\alpha^k}F(x)=\frac{1}{x-\frac{h}{1-q}}.$$

This shows that F is an α^k-antiderivative of $f(x)=\frac{1}{x-\frac{h}{1-q}}$ for $x\ne\frac{h}{1-q}$. It follows from Proposition 3/(iii) that

$$f(\alpha^{kj}(x))=\frac{1}{\left(\frac{t}{q}+(1-t)q\right)^{kj}\left(x-\frac{h}{1-q}\right)},$$

and hence the series

$$\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj} f\left(\alpha^{kj}(x)\right)=\sum_{j=0}^\infty\frac{1}{x-\frac{h}{1-q}}$$

diverges. Clearly, for any $a>0$, f does not satisfy the condition $\left|\left(x-\frac{h}{1-q}\right)^r f(x)\right|<M$ on the interval $0<\left|x-\frac{h}{1-q}\right|<a$ for some $M>0$ and $0<r<1$, but it satisfies for $r=1$. Note also that, F is not continuous at $\frac{h}{1-q}$.

Definition 9.*For any function f satisfying the condition of Theorem 7, the indefinite α^k-integral of f is defined by*

$$\int f(x)d_{\alpha^k}x:=\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\times\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj} f\left(\alpha^{kj}(x)\right)+C$$

where C is a constant.

*Example 5.*Let $f(x)=\left(x-\frac{h}{1-q}\right)^r$, $r\in\mathbb{R}^+$. Then, we have

$$\begin{aligned}\int f(x)d_{\alpha^k}x &= \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)\times\sum_{j=0}^\infty\left(\frac{t}{q}+(1-t)q\right)^{kj}\left(\alpha^{kj}(x)-\frac{h}{1-q}\right)^r+C \\ &= \frac{\left(x-\frac{h}{1-q}\right)^{r+1}}{[r+1]_{\alpha^k}}+C.\end{aligned}$$

Before we introduce the definite α^k-integral, we want to make a convention on interval notation. Since we deal with functions either defined on $(-\infty, \frac{h}{1-q}]$ or $[\frac{h}{1-q}, \infty)$, for a,b with $|a-\frac{h}{1-q}| \le |b-\frac{h}{1-q}|$, we define open, closed, and half-open intervals as follows

$$\begin{aligned}
[a,b] &:= \left\{x : \left|a-\frac{h}{1-q}\right| \le \left|x-\frac{h}{1-q}\right| \le \left|b-\frac{h}{1-q}\right|\right\},\\
(a,b] &:= \left\{x : \left|a-\frac{h}{1-q}\right| < \left|x-\frac{h}{1-q}\right| \le \left|b-\frac{h}{1-q}\right|\right\},\\
[a,b) &:= \left\{x : \left|a-\frac{h}{1-q}\right| \le \left|x-\frac{h}{1-q}\right| < \left|b-\frac{h}{1-q}\right|\right\},\\
(a,b) &:= \left\{x : \left|a-\frac{h}{1-q}\right| < \left|x-\frac{h}{1-q}\right| < \left|b-\frac{h}{1-q}\right|\right\}.
\end{aligned}$$

Note that when $a,b \ne \frac{h}{1-q}$, if $\left(\frac{t}{q}+(1-t)q\right) < 1$, then $(a,b] = [\alpha^{-1}(a), b]$, $[a,b] = (\alpha(a), b]$, etc. and if $\left(\frac{t}{q}+(1-t)q\right) > 1$, then $(a,b] = [\alpha(a), b]$, $[a,b] = (\alpha^{-1}(a), b]$, etc.

Definition 10. *For f satisfying the condition of Theorem 7 and $a,b,c \ne \frac{h}{1-q}$, the definite α^k-integral of f on $(\frac{h}{1-q}, c]$ or $[\frac{h}{1-q}, c]$ is defined by*

$$\int_{(\frac{h}{1-q},c]} f(x) d_{\alpha^k} x := \int_{[\frac{h}{1-q},c]} f(x) d_{\alpha^k} x = \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(c-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(c)),$$

and the definite α^k-integral of f on $(a,b]$ is defined by

$$\int_{(a,b]} f(x) d_{\alpha^k} x := \int_{[\frac{h}{1-q},b]} f(x) d_{\alpha^k} x - \int_{[\frac{h}{1-q},a]} f(x) d_{\alpha^k} x.$$

Theorem 8. *If F is an α^k-antiderivative of f that is continuous at $\frac{h}{1-q}$, $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ and a,b,c are as in Definition 10, then*

$$\int_{(\frac{h}{1-q},c]} f(x) d_{\alpha^k} x = F(c) - F\left(\frac{h}{1-q}\right) \quad \text{and} \quad \int_{(a,b]} f(x) d_{\alpha^k} x = F(b) - F(a).$$

Proof. For any $c \ne \frac{h}{1-q}$, it follows from Proposition 3/(v),(vi) and the continuity of F at $\frac{h}{1-q}$ that

$$\begin{aligned}
\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(c)) &= \lim_{N\to\infty}\sum_{j=0}^{N}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(c))\\
&= \lim_{N\to\infty}\sum_{j=0}^{N}\left(\frac{t}{q}+(1-t)q\right)^{kj} \frac{F(\alpha^k(\alpha^{kj}(c))) - F(\alpha^{kj}(c))}{\alpha^k(\alpha^{kj}(c)) - \alpha^{kj}(c)}\\
&= \lim_{N\to\infty}\sum_{j=0}^{N}\left(\frac{t}{q}+(1-t)q\right)^{kj} \frac{F(\alpha^{k(j+1)}(c))) - F(\alpha^{kj}(c))}{\left(\frac{t}{q}+(1-t)q\right)^{kj}[k]_\alpha(\frac{t}{q}+(1-t)q-1)(c-\frac{h}{1-q})}\\
&= \frac{1}{[k]_\alpha(\frac{t}{q}+(1-t)q-1)(c-\frac{h}{1-q})}\lim_{N\to\infty}\sum_{j=0}^{N}\left(F(\alpha^{k(j+1)}(c))) - F(\alpha^{kj}(c))\right)\\
&= \frac{1}{[k]_\alpha(\frac{t}{q}+(1-t)q-1)(c-\frac{h}{1-q})}\lim_{N\to\infty}\left(F(\alpha^{k(N+1)}(c)) - F(c)\right)\\
&= \frac{1}{[k]_\alpha(\frac{t}{q}+(1-t)q-1)(c-\frac{h}{1-q})}\left(F\left(\frac{h}{1-q}\right) - F(c)\right),
\end{aligned}$$

and hence

$$\int_{(\frac{h}{1-q},c]} f(x)d_{\alpha^k}x = \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(c-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} f(\alpha^{kj}(c)) = F(c)-F\left(\frac{h}{1-q}\right).$$

Similary, $\int_{(\frac{h}{1-q},b]} f(x)d_{\alpha^k}x = F(b)-F\left(\frac{h}{1-q}\right)$ and $\int_{(\frac{h}{1-q},a]} f(x)d_{\alpha^k}x = F(a)-F\left(\frac{h}{1-q}\right)$ and hence

$$\int_{(a,b]} f(x)d_{\alpha^k}x = \left(F(b)-F\left(\frac{h}{1-q}\right)\right)-\left(F(a)-F\left(\frac{h}{1-q}\right)\right) = F(b)-F(a).$$

Theorem 9(Fundamental theorem of α^k-integral I). *If f is continuous at $\frac{h}{1-q}$, $\left(\frac{t}{q}+(1-t)q\right)^k < 1$ and a,b,c are as in Definition 10, then*

$$\int_{(\frac{h}{1-q},c]} D_{\alpha^k}f(x)d_{\alpha^k}x = f(c)-f\left(\frac{h}{1-q}\right) \quad and \quad \int_{(a,b]} D_{\alpha^k}f(x)d_{\alpha^k}x = f(b)-f(a).$$

*Proof.*We apply Theorem 8 fo $D_{\alpha^k}f$.

Theorem 10(Fundamental theorem of α^k-integral II). *Let f be continuous at $\frac{h}{1-q}$, then*

$$D_{\alpha^k}\left(\int_{[\frac{h}{1-q},x]} f(s)d_{\alpha^k}s\right) = f(x).$$

*Proof.*By Theorem 47, f has an antiderivative F that is also continuous at $\frac{h}{1-q}$ and then by Theorem 8,

$$\int_{[\frac{h}{1-q},x]} f(x)d_{\alpha^k}x = F(x)-F\left(\frac{h}{1-q}\right).$$

The result follows by taking the α^k-derivative of each side.

We need the following form of Leibniz's rule of differentiation under the integral sign. Below we use the notation $\partial^x_{\alpha^k}$ to denote the partial α^k-derivative with respect to the variable x.

Proposition 9.*If g and $\partial^x_{\alpha^k}g$ are continuous with respect to s at $\frac{h}{1-q}$, for $x \neq \frac{h}{1-q}$,*

$$D_{\alpha^k}\left(\int_{[\frac{h}{1-q},x]} g(x,s)d_{\alpha^k}s\right) = g\left(\alpha^k(x),x\right) + \int_{[\frac{h}{1-q},x]} \partial^x_{\alpha^k}g(x,s)d_{\alpha^k}s.$$

*Proof.*By definition of the α^k derivative, we have

$$D_{\alpha^k}\left(\int_{[\frac{h}{1-q},x]} g(x,s)d_{\alpha^k}s\right) = \frac{\int_{[\frac{h}{1-q},\alpha^k(x)]} g(\alpha^k(x),s)d_{\alpha^k}s - \int_{[\frac{h}{1-q},x]} g(x,s)d_{\alpha^k}s}{\alpha^k(x)-x}.$$

If $\left(\frac{t}{q}+(1-t)q\right)^k < 1$, then $|\alpha^k(x)-\frac{h}{1-q}| < |x-\frac{h}{1-q}|$, and so

$$\int_{[\frac{h}{1-q},\alpha^k(x)]} g(\alpha^k(x),s)d_{\alpha^k}s = \int_{[\frac{h}{1-q},x]} g(\alpha^k(x),s)d_{\alpha^k}s - \int_{(\alpha^k(x),x]} g(\alpha^k(x),s)d_{\alpha^k}s.$$

Noting

$$\int_{[\frac{h}{1-q},x]} \frac{g(\alpha^k(x),s)-g(x,s)}{\alpha^k(x)-x} d_{\alpha^k}s = \int_{[\frac{h}{1-q},x]} \partial^x_{\alpha^k} g(x,s) d_{\alpha^k}s,$$

we're done if we show that

$$\frac{\int_{(\alpha^k(x),x]} g(\alpha^k(x),s)d_{\alpha^k}s}{\alpha^k(x)-x} = -g\left(\alpha^k(x),x\right).$$

By Definition 10,

$$\int_{(\alpha^k(x),x]} g(\alpha^k(x),s)d_{\alpha^k}s = \left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)(I_1-I_2)$$

where

$$I_1 := \left(x-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} g(\alpha^k(x),\alpha^{kj}(x)),$$

$$I_2 := \left(\alpha^k(x)-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{kj} g(\alpha^k(x),\alpha^{kj}(\alpha^k(x))).$$

Using Proposition 3/(iii), we get

$$I_2 = \left(x-\frac{h}{1-q}\right)\sum_{j=0}^{\infty}\left(\frac{t}{q}+(1-t)q\right)^{k(j+1)} g(\alpha^k(x),\alpha^{k(j+1)}(x)),$$

and so $I_1-I_2 = \left(x-\frac{h}{1-q}\right)g(\alpha^k(x),x)$. Finally, applying Proposition 3/(vi), we obtain

$$\frac{\int_{(\alpha^k(x),x]} g(\alpha^k(x),s)d_{\alpha^k}s}{\alpha^k(x)-x} = \frac{\left(1-\left(\frac{t}{q}+(1-t)q\right)^k\right)\left(x-\frac{h}{1-q}\right)g(\alpha^k(x),x)}{[k]_\alpha(\frac{t}{q}+(1-t)q-1)\left(x-\frac{h}{1-q}\right)} = -g\left(\alpha^k(x),x\right).$$

6 Conclusion

In this article, our primary objectives are to accomplish α^k-power function, to propose and solve α-Cauchy-Euler equation. We discovered α-logarithm function which arises in the solution of a second order α-Cauchy-Euler equation. We will address the α-analogue of exponential function, the relation between α-analogues of logarithm and exponential functions, further algebraic properties of α-logarithm and some special functions (gamma, beta, hypergeometric functions) as a separate paper.

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