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On Fractional Differential Equation with Complex Order

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Abstract: In the present article, we investigate a fractional boundary value problem (FBVP) of complex order $\theta = m + i\alpha$, where $1 < m \leq 2$ and $\alpha \in \mathbb{R}^+$ is studied. By applying two fixed point theorems Banach and Schauder, we achieved some new existence and uniqueness conclusions of complex solutions. We present an example to express our results.

Keywords: fixed-point theorem, complex order, existence.

1 Introduction

Study of fractional differential equations has been considerably progressed in recent years that implies importance and place of the fractional calculus in the sciences and engineering. On the other side, according to extensive applications of fractional calculus in natural phenomena like chemical physics, electrical networks, viscoelasticity, porous media, electrical networks, it got many scholar's attention, see articles like [1,2,3,4].

In the past few years, solvability of BVPs for nonlinear fractional differential equations were studied that in these types of problems usually existence and multiplicity of solutions is discussed with fixed point theorems, see [7,8,9,11]. Also the existence and uniqueness of positive solutions of FBVPs by applying some fixed point theorems on cone were acquired, as [5,6,7,8,10].

Bai and Lu [5] considered the following BVP of nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,u(t)) = 0,\ 0 \leq t \leq 1; \\ u(0) = u(1) = 0, \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order $\alpha \in (1,2]$ and f is a continuous function. By applying fixed point theorems on cone existence and multiplicity of positive solutions to the problem achieved.

Agarwal and his co-authors [6] studied existence of positive solutions for the singular fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t,u(t),D^{\mu}u(t)) = 0,\ 0 \leq t \leq 1; \\ u(0) = u(1) = 0, \end{cases}$$

in which $1 < \alpha < 2, 0 \leq \mu \leq \alpha - 1$ and the positive function f satisfied the Caratheodory conditions on $[0,1] \times [0,\infty] \times \mathbb{R}$ and f has singularity at $x = 0$.

In all above mentioned papers, order of differentiation was real and in the knowledge of authors there isn't any problem containing fractional differential operator of complex order.

The novelty of the present work is to consider the fractional BVP for differential equation of complex-order, namely:

$$D^{\theta}_{0+}q(\tau) = h(\tau,q(\tau)), \tau \in [0,1], \theta = m + i\alpha, \tag{1}$$

subject to boundary conditions

$$q(0) = q(1) = 0, \tag{2}$$

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in which, $1 < m \le 2, \alpha \in \mathbb{R}^+$ and D_{0+}^{θ} is the Riemann-Liouville fractional derivative of order $\theta \in \mathbb{C}$ and $h \in C([0,1] \times [0,\infty),[0,\infty))$. Our aim is to prove the existence of complex solution for the differential equation of complex order which has not been studied as mentioned above.

The remainder article is divided as follows: in section 2, we give some preliminaries of fractional derivatives and integrals. Then the integral equation pertaining to the problem (1)-(2) and the corresponding Green's function obtained. By assuming conditions and using fixed point theorems existence and uniqueness of complex solutions will be obtained in the last section. Furthermore an example is given.

2 Preliminaries and Notations

This section contains definitions and lemmas of fractional calculus that is needed to prove our results. The presentation here can be found in [1,2,5].

Definition 2.1. ([5]) The Riemann-Liouville fractional integral of order $\mu \in \mathbb{C}$, $(\Re(\mu) > 0)$ of a function $h : (0,\infty) \longrightarrow \mathbb{R}$ is

$$I_{0^+}^{\mu} h(\rho) = \frac{1}{\Gamma(\mu)} \int_0^{\rho} (\rho - s)^{\mu-1} h(s) ds.$$

Definition 2.2. ([5]) The Riemann-Liouville fractional derivative of order $\mu \in \mathbb{C}$, $(\Re(\mu) > 0)$ of a function $h : (0,\infty) \longrightarrow \mathbb{R}$ has the form

$$D_{0+}^{\mu} h(\rho) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{d\rho^n} \int_0^{\rho} \frac{h(s)}{(\rho - s)^{\mu-n+1}} ds,$$

where $n = [\Re(\mu)] + 1$.

Definition 2.3. ([2]) The Stirling asymptotic formula of the Gamma function for $z \in \mathbb{C}$ is following

$$\Gamma(z) = (2\pi)^{1/2} z^{z-1/2} e^{-z} \left[1 + O\left(\frac{1}{z}\right)\right] \qquad (|arg(z)| < \pi; |z| \to \infty), \tag{3}$$

and its result for $|\Gamma(a+ib)|$, $(a,b \in \mathbb{R})$ is

$$|\Gamma(a+ib)| = (2\pi)^{1/2} |b|^{a-1/2} e^{-a-\pi|b|/2} \left[1 + O\left(\frac{1}{b}\right)\right] \qquad (b \to \infty). \tag{4}$$

Lemma 2.1. ([5]) Let $q \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\mu \in \mathbb{C}$, $(\Re(\mu) > 0)$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$I_{0^+}^{\mu} D_{0^+}^{\mu} q(\tau) = q(\tau) + \sum_{i=1}^{n} c_i \tau^{\mu-i},$$

for some $c_i \in \mathbb{R}$, $i = 1,...,n$, where $n = [\Re(\mu)] + 1$.

Proof. From [5] we can conclude for $\mu \in \mathbb{C}$, $(\Re(\mu) > 0)$ that is true.

Lemma 2.2. Let $y(s) \in C[0,1]$. Then, the following FBVP

$$\begin{cases} D_{0+}^{\theta} q(\tau) = y(\tau), \ \tau \in [0,1]; \\ q(0) = q(1) = 0, \ \theta = m + i\alpha. \end{cases}$$

for $1 < m \le 2$, $\alpha \in \mathbb{R}^+$, is equivalent to

$$q(\tau) = \int_0^1 G(\tau,s) y(s) ds,$$

where

$$G(\tau,s) = \frac{1}{\Gamma(\theta)} \begin{cases} (\tau - s)^{\theta-1} - \tau^{\theta-1}(1-s)^{\theta-1}, & 0 \le s \le \tau, \\ -\tau^{\theta-1}(1-s)^{\theta-1}, & \tau \le s \le 1. \end{cases}$$

Proof. Lemma 2.1. leads to

$$q(\tau) = c_1 \tau^{\theta-1} + c_2 \tau^{\theta-2} + I_{0+}^{\theta} y(\tau) = c_1 \tau^{\theta-1} + c_2 \tau^{\theta-2} + \frac{1}{\Gamma(\theta)} \int_0^{\tau} (\tau - s)^{\theta-1} y(s) ds, \tag{5}$$

for some $c_1, c_2 \in \mathbb{R}$. By using condition $q(0) = 0$ suppose $c_2 = 0$. Also, the boundary condition $q(1) = 0$ gives

$$c_1 = -\frac{1}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} y(s) ds.$$

Substituting c_1 into (5) yield

$$q(\tau) = \frac{-\tau^{\theta-1}}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} y(s) ds + \frac{1}{\Gamma(\theta)} \int_0^\tau (\tau-s)^{\theta-1} y(s) ds.$$

We obtain the Green's function

$$\begin{aligned} q(\tau) &= -\frac{1}{\Gamma(\theta)} \left(\int_0^\tau + \int_\tau^1 \right) \tau^{\theta-1} (1-s)^{\theta-1} y(s) ds + \frac{1}{\Gamma(\theta)} \int_0^\tau (\tau-s)^{\theta-1} y(s) ds \\ &= \frac{1}{\Gamma(\theta)} \int_0^\tau [(\tau-s)^{\theta-1} - \tau^{\theta-1}(1-s)^{\theta-1}] y(s) ds - \frac{1}{\Gamma(\theta)} \int_\tau^1 \tau^{\theta-1}(1-s)^{\theta-1} y(s) ds \\ &= \int_0^1 G(\tau,s) y(s) ds. \end{aligned}$$

3 Existence Results

Let $B = C[0,1]$ be a Banach space of continuous functions endowed with the norm $\|q\| = \max_{0\le\tau\le1} |q(\tau)|$.
Define the operator $\mathscr{A} : B \to B$ as follows:

$$(\mathscr{A}q)(\tau) = \frac{1}{\Gamma(\theta)} \int_0^\tau (\tau-s)^{\theta-1} h(s,q(s)) ds - \frac{\tau^{\theta-1}}{\Gamma(\theta)} \int_0^1 (1-s)^{\theta-1} h(s,q(s)) ds.$$

According to Lemma 2.2, the fixed points of the operator $\mathscr{A}$ are the same solutions of the BVP (1)-(2). To obtain necessary results, we assume:

$(\boldsymbol{H}_1)$ $h \in C([0,1] \times [0,\infty), [0,\infty))$.

$(\boldsymbol{H}_2)$ $\forall \tau \in [0,1]$, $q, \acute{q} \in \mathbb{R}$, there is a constant $K > 0$ so that

$$|h(\tau,q) - h(\tau,\acute{q})| \le K|q - \acute{q}|.$$

$(\boldsymbol{H}_3)$ $\gamma_1 = \frac{2K}{m|\Gamma(\theta)|} < 1$.

Theorem 3.1. Under assumptions ($\mathbf{H}_1$-$\mathbf{H}_3$), the FBVP (1)-(2) has a unique solution.

Proof. Since h and $G(\tau,s)$ are continuous, then $\mathscr{A}$ is continuous. Regarding the condition **($\mathbf{H}_1$)**, put $M = \max_{\tau\in[0,1]} |h(\tau,0)|$. Define a ball $D_r = \{q \in B : \|q\| \le r\}$, where $\frac{\gamma_2}{1-\gamma_1} \le r$ and $\gamma_2 = \frac{2M}{m|\Gamma(\theta)|}$. First, we show $\mathscr{A}D_r \subset D_r$, for $q \in D_r$

$$\begin{aligned} |\mathscr{A}q(\tau)| &\le \frac{|\tau^{\theta-1}|}{|\Gamma(\theta)|} \int_0^1 |(1-s)^{\theta-1}||h(s,q(s))| ds + \frac{1}{|\Gamma(\theta)|} \int_0^\tau |(\tau-s)^{\theta-1}||h(s,q(s))| ds \\ &\le \frac{1}{|\Gamma(\theta)|} \int_0^1 |(1-s)^{\theta-1}|(|h(s,q(s)) - h(s,0)| + |h(s,0)|) ds \\ &\quad + \frac{1}{|\Gamma(\theta)|} \int_0^\tau |(\tau-s)^{\theta-1}|(|h(s,q(s)) - h(s,0)| + |h(s,0)|) ds \\ &\le (K|q(s)| + M) \left\{ \frac{1}{|\Gamma(\theta)|} \int_0^1 |(1-s)^{\theta-1}| ds + \frac{1}{|\Gamma(\theta)|} \int_0^\tau |(\tau-s)^{\theta-1}| ds \right\} \\ &\le \frac{(K\|q\| + M)}{|\Gamma(\theta)|} \left\{ \int_0^1 (1-s)^{m-1} ds + \int_0^\tau (\tau-s)^{m-1} ds \right\} \\ &\le \frac{Kr + M}{|\Gamma(\theta)|} \left\{ \frac{1}{m} + \frac{\tau^m}{m} \right\} \\ &\le \frac{2(Kr + M)}{m|\Gamma(\theta)|} = \gamma_1 r + \gamma_2 \le r. \end{aligned}$$

Now, for $q,\acute{q} \in B$, $\tau \in [0,1]$, we obtain

$$\begin{aligned}
|\mathscr{A}q(\tau)-\mathscr{A}\acute{q}(\tau)| &\leq \frac{|\tau^{\theta-1}|}{|\Gamma(\theta)|}\int_0^1 |(1-s)^{\theta-1}||h(s,q(s))-h(s,\acute{q}(s))|ds \\
&\quad +\frac{1}{|\Gamma(\theta)|}\int_0^\tau |(\tau-s)^{\theta-1}||h(s,q(s))-h(s,\acute{q}(s))|ds \\
&\leq \frac{K\|q-\acute{q}\|}{|\Gamma(\theta)|}\left\{\int_0^1 |(1-s)^{\theta-1}|ds+\int_0^\tau |(\tau-s)^{\theta-1}|ds\right\} \\
&= \frac{K\|q-\acute{q}\|}{|\Gamma(\theta)|}\left\{\int_0^1 (1-s)^{m-1}ds+\int_0^\tau (\tau-s)^{m-1}ds\right\} \\
&\leq \frac{2K}{m|\Gamma(\theta)|}\|q-\acute{q}\| \\
&= \gamma_1\|q-\acute{q}\|,
\end{aligned}$$

where $\gamma_1 < 1$. Therefore, $\mathscr{A}$ is a contraction. By the contraction mapping principle, we conclude that FBVP (1)-(2) has a unique solution.

Theorem 3.2. Under assumptions ($\mathbf{H}_1$) and ($\mathbf{H}_3$) the FBVP (1)-(2) has a solution on [0,1].

Proof. Let us consider a convex, bounded and closed subset $D_r = \{q \in B : \|q\| \leq r\}$, of the Banach space B. We showed that $\mathscr{A}$ maps D_r into D_r.

Now, let $M = \max_{\tau\in[0,1],q\in D_r}|h(\tau,q(\tau))|+1$, for $q \in D_r$, we give

$$\begin{aligned}
|\mathscr{A}q(\tau)| &\leq \frac{|\tau^{\theta-1}|}{|\Gamma(\theta)|}\int_0^1 |(1-s)^{\theta-1}||h(s,q(s))|ds+\frac{1}{|\Gamma(\theta)|}\int_0^\tau |(\tau-s)^{\theta-1}||h(s,q(s))|ds \\
&\leq \frac{M}{|\Gamma(\theta)|}\left\{\int_0^1 (1-s)^{m-1}ds+\int_0^\tau (\tau-s)^{m-1}ds\right\} \\
&\leq \frac{2M}{m|\Gamma(\theta)|}.
\end{aligned}$$

Thus, $\mathscr{A}$ is uniformly bounded on D_r. Next, we show that $\mathscr{A}(D_r)$ is equicontinuous. For $q \in D_r$ and $\tau_1,\tau_2 \in [0,1]$ such that $\tau_1 < \tau_2$ we get,

$$\begin{aligned}
|\mathscr{A}q(\tau_2)-\mathscr{A}q(\tau_1)| &\leq \left|\frac{\tau_2^{\theta-1}-\tau_1^{\theta-1}}{\Gamma(\theta)}\int_0^1 (1-s)^{\theta-1}h(s,q(s))ds\right| \\
&\quad +\left|\frac{1}{\Gamma(\theta)}\int_0^{\tau_2}(\tau_2-s)^{\theta-1}h(s,q(s))ds-\frac{1}{\Gamma(\theta)}\int_0^{\tau_1}(\tau_1-s)^{\theta-1}h(s,q(s))ds\right| \\
&\leq \left|\frac{\tau_2^{\theta-1}-\tau_1^{\theta-1}}{\Gamma(\theta)}\right|\left|\int_0^1 (1-s)^{\theta-1}h(s,q(s))ds\right| \\
&\quad +\left|\frac{1}{\Gamma(\theta)}\int_0^{\tau_1}[(\tau_2-s)^{\theta-1}-(\tau_1-s)^{\theta-1}]h(s,q(s))ds\right. \\
&\qquad \left.+\frac{1}{\Gamma(\theta)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\theta-1}h(s,q(s))ds\right| \\
&\leq \frac{M|\tau_2^{\theta-1}-\tau_1^{\theta-1}|}{|\Gamma(\theta)|}\left|\int_0^1 (1-s)^{\theta-1}ds\right| \\
&\quad +\frac{M}{|\Gamma(\theta)|}\left|\int_0^{\tau_1}[(\tau_2-s)^{\theta-1}-(\tau_1-s)^{\theta-1}]ds+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\theta-1}ds\right| \\
&\leq \frac{M}{|\Gamma(\theta+1)|}|\tau_2^{\theta-1}-\tau_1^{\theta-1}|+\frac{2M}{|\Gamma(\theta+1)|}|(\tau_2-\tau_1)^{\theta}|+\frac{M}{|\Gamma(\theta+1)|}|\tau_2^{\theta}-\tau_1^{\theta}|.
\end{aligned}$$

It is easy to see that functions τ^θ and $\tau^{\theta-1}$ are uniformly continuous on [0,1]. Then, $\mathscr{A}(D_r)$ is equicontinuous. By the Arzela-Ascoli theorem $\overline{\mathscr{A}(D_r)}$ is compact and so $\mathscr{A} : D_r \to D_r$, is completely continuous. The Schauder fixed point theorem now causes that the BVP (1)-(2) has a solution.

4 Illustrative Example

Example 4.1 Let us consider fractional BVP of complex order:

$$\begin{cases} D_{0+}^{\frac{3}{2}+i}q(\tau) = \frac{\tan^{-1}q}{\tau+2},\ \tau\in[0,1]; \\ q(0)=q(1)=0. \end{cases} \tag{6}$$

where, $m=\frac{3}{2}$, $\alpha=1$ and

$$h(\tau,q) = \frac{\tan^{-1}q}{\tau+2},\ \ (\tau,q)\in[0,1]\times[0,\infty).$$

It is clear that h is a continuous function. Now, for $(\tau,q),(\tau,\acute{q})\in[0,1]\times[0,\infty)$ we get

$$\begin{aligned} |h(\tau,q)-h(\tau,\acute{q})| &= \frac{1}{\tau+2}|\tan^{-1}q-\tan^{-1}\acute{q}| \\ &\le \frac{1}{2}|\tan^{-1}q-\tan^{-1}\acute{q}| \\ &\le \frac{1}{2}|q-\acute{q}|. \end{aligned}$$

Thus, $K=\frac{1}{2}$, and in view of relation (4), since $|\Gamma(\frac{3}{2}+i)|>1$, we have

$$\gamma_1 = \frac{2K}{m|\Gamma(\theta)|} = \frac{1}{\frac{3}{2}|\Gamma(\frac{3}{2}+i)|} = \frac{2}{3|\Gamma(\frac{3}{2}+i)|} < 1.$$

By applying Theorem 3.1, we deduced that BVP (6) has a unique solution.

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