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Step-Stress Partially Accelerated Life Tests with Progressive Type-II Censored Sample from Two-Parameter Inverted Exponential Distribution

A. R. Shafay

Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt

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Abstract: In this paper, the topic of step-stress partially accelerated life tests with progressive Type-II censoring scheme is investigated when the lifetimes of test units follow the two-parameter inverted exponential distribution. The maximum likelihood estimation method is used to estimate the unknown parameters and acceleration factor. Under the normal use condition, the maximum likelihood estimators of the reliability and hazard rate functions are also calculated. In addition, the observed Fisher information matrix is produced and employed to determine the approximate confidence intervals of the unknown parameters. Furthermore, for the reliability and hazard rate functions, the delta method is used to construct the approximate confidence interval. To compare and examine the effectiveness of the suggested estimation methods, a Monte Carlo simulation study is conducted with various sample sizes and different censoring schemes. Finally, a numerical example is provided to show how the paper's findings can be applied.

Keywords: Partially accelerated life tests, Progressive Type-II censoring scheme, Two-parameter inverted exponential distribution, Maximum likelihood estimation, Approximate confidence interval

1 Introduction

Nowadays, reliability prediction is commonly employed. Accelerated life testing (ALT) or partially accelerated life testing (PALT) is one of its methods for obtaining rapid failures of any system. To ensure rapid failure and hence minimize the testing duration, stress conditions that are more severe than normal ones are imposed on all or some of the test units. Temperature levels, voltage, pressure, load, and other circumstances might be applied to the test units in order to achieve the accelerated life test. The proposed life-stress model is used to explain the observed results under the design stress. Two methods might be used in such tests to quickly obtain failures. One method is to compress time, in which a device is used more frequently than usual while keeping the loads and stresses constant, and the other method is to increase the device's load capacity. The results of the test in an accelerated environment are then used to predict actual product performance in a normal environment. In ALT, however, the test units are conducted under accelerated settings exclusively, whereas in PALT, the units are tested under both accelerated and normal conditions. For more details see [1]. There are numerous ways to apply the stress. Step-stress and constant-stress are two common methods mentioned by Nelson [2]. For step-stress PALT (SSPALT), the product is operated for a specific amount of time under real-world conditions until it fails. If the product does not fail, it is run under accelerated conditions (stress) until it fails or the observation is censored. In the second method, the product is tested under normal or accelerated conditions. In other words, the product is subjected to a constant level of stress until the test is completed. Several authors have discussed the PALT using the step-stress model, including [3,4,5,6,7,8,9,10].
Due to cost and time constraints, life testing experiments are frequently stopped before all units on test have failed. Failure times are only known for a portion of the sample in such cases, and the data is referred to as censored data. The most commonly used censoring schemes are Type-I (time) and Type-II (failure). The life testing experiment will be terminated using the Type-II censoring scheme when the r^{th} failure is observed. Progressive Type-II censoring is a generalization of Type-II censoring in which n units are placed on the life testing experiment, and only m failures are

* Corresponding author e-mail: a_shafay2013@yahoo.com

observed. When the first failure occurs, R_1 of the remaining units is chosen at random and removed. When the second failure occurs, R_2 of the remaining units are chosen at random and removed. When the m^{th} failure is observed, the experiment will be terminated, and all remaining $R_m = n - R_1 - R_2 - \ldots - R_{m-1} - m$ surviving units will be removed. A progressively Type-II censored sample is denoted by $X_{1:m:n} < X_{2:m:n} < \ldots < X_{m:m:n}$. Readers might refer to [11,12,13] for comprehensive surveys of the literature on progressive censorship.

Because of its simplicity and mathematical viability, the exponential distribution is the most widely employed lifetime model in reliability theory. If X has an exponential distribution, $Y = 1/X$ will have an inverted exponential distribution (IED). Lin et al. [14] used complete samples to calculate the maximum likelihood estimators (MLEs), uniformly minimum variance unbiased estimators and confidence intervals for the unknown parameter and the reliability function for the IED. On the basis of the maintenance data set, they also compared the IED with the inverted Gaussian and log-normal distributions, finding that it gives a better fit than these two distributions.

Abouammoh and Alshingiti [15] added a shape parameter in the IED to generate the generalized two-parameter inverted exponential distributions (GIED). They discussed many distributional properties and reliability characteristics of the GIED. They also calculated the MLEs and least square estimators as well as the confidence intervals of the two parameters. On the basis of a progressively type II censored sample, Krishna and Kumar [16] investigated the GIED's distributional properties and reliability characteristics. They also calculated the MLEs and least square estimators of the two parameters as well as the reliability and failure rate functions. Bakoban [17] investigated the problem of estimating the parameters of the GIED using Type-I censored samples under step-stress PALT. Soliman et al. [18] calculated the MLEs and Bayesain estimators of the GIED parameters, as well as the reliability and hazard rate functions, using an adaptive progressively Type-II censored sample. In this paper, we study the model of step-stress partially accelerated life tests under progressive Type-II censoring scheme when the lifetimes of test units follow the GIED.

The rest of this paper is organized as follows: The model of step-stress partially accelerated life tests under progressive Type-II censoring is provided in Section 2. At the normal use condition, the MLEs of the distribution parameters and acceleration factor, as well as the corresponding reliability and hazard rate functions, are computed in Section 3. In Section 4, approximate confidence intervals for the unknown parameters are constructed using the observed Fisher information matrix. The approximate confidence intervals for the reliability and hazard rate functions are then calculated using the delta method. Finally, some computational results are provided in Section 5 to demonstrate all of the inferential methods discussed in this paper.

2 The model description

The probability density function (PDF) and cumulative distribution function (CDF) of the GIED are given, respectively, by

$$F(x) = 1 - \left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^{\alpha} \tag{1}$$

and

$$f(x) = \frac{\alpha\lambda}{x^2} \exp\left(-\frac{\lambda}{x}\right) \left[1 - \exp\left(-\frac{\lambda}{x}\right)\right]^{\alpha-1}, \quad x > 0, \tag{2}$$

where $\lambda > 0$ is the scale parameter and $\alpha > 0$ is the shape parameter.

The corresponding reliability and hazard rate functions are given, respectively, by

$$R(t) = \left[1 - \exp\left(-\frac{\lambda}{t}\right)\right]^{\alpha}, \quad t > 0, \tag{3}$$

and

$$h(t) = \frac{\alpha\lambda}{t^2} \exp\left(-\frac{\lambda}{t}\right) \left[1 - \exp\left(-\frac{\lambda}{t}\right)\right]^{-1}, \quad t > 0. \tag{4}$$

In step-stress PALT, the normal condition is employed from the start of the test until the time τ arrives. If the component does not fail or is removed from the test before τ, the test is shifted to a higher level of stress and the component is run in an accelerated manner. This switch has an effect that may be calculated by multiplying the component's remaining lifetime by the inverse of the acceleration factor β. This effect may be described as the ratio of the hazard rate in the accelerated condition to the hazard rate in the normal use condition, which reduces the component's lifetime. Furthermore, under the tampered random variable model proposed by DeGroot and Goel [19], the total lifetime of the component in step-stress PALT is given by:

$$Y = \begin{cases} T, & T \le \tau; \\ \tau + \beta^{-1}(T - \tau), & T > \tau, \end{cases} \tag{5}$$

where T is the lifetime of the unit at use condition, τ is the stress change time and $\beta > 1$ is the acceleration factor.

Assume that the lifetime of the test unit follows GIED. Then, the PDF of the total lifetime Y is given by

$$f(y) = \begin{cases} 0, & y < 0; \\ f_1(y), & 0 < y < \tau; \\ f_2(y), & y > \tau, \end{cases} \tag{6}$$

where

$$f_1(y) = \left(\frac{\alpha\lambda}{y^2}\right) \exp\left(-\frac{\lambda}{y}\right) \left[1 - \exp\left(-\frac{\lambda}{y}\right)\right]^{\alpha-1} \tag{7}$$

and

$$f_2(y) = \left(\frac{\alpha\lambda\beta}{(\tau+\beta(y-\tau))^2}\right) \exp\left(\frac{-\lambda}{\tau+\beta(y-\tau)}\right) \left[1 - \exp\left(\frac{-\lambda}{\tau+\beta(y-\tau)}\right)\right]^{\alpha-1}. \tag{8}$$

3 The maximum likelihood estimation

Assume that n identical and independent units are tested under the normal use condition. If n_1 indicates the number of units that fail up to the time τ under the normal use condition, and $m - n_1$ represents the number of units that fail after τ under the accelerated condition. The total lifetimes using the progressive Type-II censoring with censored scheme $R = (R_1, R_2, ..., R_m)$, are then

$$Y^R_{1:m:n} < ... < Y^R_{n_1:m:n} < \tau < Y^R_{n_1+1:m:n} < ... < Y^R_{m:m:n}$$

Let $y_i = y^R_{i:m:n}$, $i = 1, 2, ..., m$, be the observed values of the total lifetimes. Then, the joint density function of $Y^R_{1:m:n}, ..., Y^R_{n_1:m:n}, Y^R_{n_1+1:m:n}, ..., Y^R_{m:m:n}$ is given by

$$f_{1,2,\cdots,m:m:n}(y_1, y_2, \cdots, y_m) = C\left\{\prod_{i=1}^{n_1} f_1(y_i)[1-F_1(y_i)]^{R_i}\right\} \times \left\{\prod_{i=n_1+1}^{m} f_2(y_i)[1-F_2(y_i)]^{R_i}\right\}, \tag{9}$$

where $C = n(n-1-R_1)(n-2-R_1-R_2)...(n-m+1-\sum_{i=1}^{m-1} R_i)$.

Upon using (1) and (2) in (9), the likelihood function of α, λ and β is given by

$$L(\alpha,\lambda,\beta) = C\alpha^m\lambda^m\beta^{m-n_1}\left\{\prod_{i=1}^{n_1}\frac{1}{y_i^2}\exp\left(-\frac{\lambda}{y_i}\right)\left[1-\exp\left(-\frac{\lambda}{y_i}\right)\right]^{(R_i+1)\alpha-1}\right\}$$
$$\times\left\{\prod_{i=n_1+1}^{m}\frac{1}{(\tau+\beta(y_i-\tau))^2}\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left[1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right]^{(R_i+1)\alpha-1}\right\}. \tag{10}$$

The natural logarithm of the likelihood function may then be calculated as

$$\ln L = \ln C + m\ln\alpha + m\ln\lambda + (m-n_1)\ln\beta - 2\sum_{i=1}^{n_1}\ln y_i - \lambda\sum_{i=1}^{n_1}\frac{1}{y_i}$$
$$+\sum_{i=1}^{n_1}((R_i+1)\alpha-1)\ln\left(1-\exp\left(-\frac{\lambda}{y_i}\right)\right) - 2\sum_{i=n_1+1}^{m}\ln(\tau+\beta(y_i-\tau))$$
$$-\lambda\sum_{i=n_1+1}^{m}\frac{1}{\tau+\beta(y_i-\tau)} + \sum_{i=n_1+1}^{m}((R_i+1)\alpha-1)\ln\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right). \tag{11}$$

By differentiating (11) with respect to α, λ and β and equating to zero, the likelihood equations are obtained as

$$\frac{\partial\ln L}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{n_1}(R_i+1)\ln\left(1-\exp\left(-\frac{\lambda}{y_i}\right)\right) + \sum_{i=n_1+1}^{m}(R_i+1)\ln\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right) = 0, \tag{12}$$

$$\frac{\partial\ln L}{\partial\lambda} = \frac{m}{\lambda} - \sum_{i=1}^{n_1}\frac{1}{y_i} - \sum_{i=n_1+1}^{m}\frac{1}{\tau+\beta(y_i-\tau)} + \sum_{i=1}^{n_1}\frac{(R_i+1)\alpha-1}{y_i}\exp\left(-\frac{\lambda}{y_i}\right)\left(1-\exp\left(-\frac{\lambda}{y_i}\right)\right)^{-1}$$
$$+\sum_{i=n_1+1}^{m}\frac{(R_i+1)\alpha-1}{\tau+\beta(y_i-\tau)}\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1} = 0, \tag{13}$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{m-n_1}{\beta} - 2\sum_{i=n_1+1}^{m} \frac{y_i-\tau}{\tau+\beta(y_i-\tau)} + \lambda \sum_{i=n_1+1}^{m} \frac{y_i-\tau}{(\tau+\beta(y_i-\tau))^2}$$
$$-\lambda \sum_{i=n_1+1}^{m} \frac{((R_i+1)\alpha-1)(y_i-\tau)}{(\tau+\beta(y_i-\tau))^2} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1} = 0. \tag{14}$$

From (12), the ML estimator $\hat{\alpha}$ of the unknown parameter α can be obtained as

$$\hat{\alpha} = -\frac{m}{\sum_{i=1}^{n_1}(R_i+1)\ln\left(1-\exp\left(-\frac{\hat{\lambda}}{y_i}\right)\right) + \sum_{i=n_1+1}^{m}(R_i+1)\ln\left(1-\exp\left(-\frac{\hat{\lambda}}{\tau+\hat{\beta}(y_i-\tau)}\right)\right)}, \tag{15}$$

where $\hat{\lambda}$ and $\hat{\beta}$ are the ML estimators of the unknown parameters λ and β, respectively, which can be calculated by substituting the value of $\hat{\alpha}$ in the two equations (13) and (14) and then solving the two nonlinear equations using an iterative numerical method such as Newton-Raphson.

Under the normal use condition, the ML estimators of the corresponding reliability function $\hat{R}(t)$ and hazard rate function $\hat{h}(t)$ may be computed using the invariance property of the ML estimator as follows:

$$\hat{R}(t) = \left[1-\exp\left(-\frac{\hat{\lambda}}{t}\right)\right]^{\hat{\alpha}} \tag{16}$$

and

$$\hat{h}(t) = \frac{\hat{\alpha}\hat{\lambda}}{t^2} \exp\left(-\frac{\hat{\lambda}}{t}\right)\left[1-\exp\left(-\frac{\hat{\lambda}}{t}\right)\right]^{-1}. \tag{17}$$

4 The asymptotic confidence intervals

The elements of the inverse of the following Fisher information matrix provide the asymptotic variances and covariances of the ML estimates of the parameters (α, λ, β)

$$I(\hat{\alpha},\hat{\lambda},\hat{\beta}) = \begin{bmatrix} E\left[-\partial^2 \ln L/\partial\alpha^2\right] & E\left[-\partial^2 \ln L/\partial\alpha\partial\lambda\right] & E\left[-\partial^2 \ln L/\partial\alpha\partial\beta\right] \\ E\left[-\partial^2 \ln L/\partial\lambda\partial\alpha\right] & E\left[-\partial^2 \ln L/\partial\lambda^2\right] & E\left[-\partial^2 \ln L/\partial\lambda\partial\beta\right] \\ E\left[-\partial^2 \ln L/\partial\beta\partial\alpha\right] & E\left[-\partial^2 \ln L/\partial\beta\partial\lambda\right] & E\left[-\partial^2 \ln L/\partial\beta^2\right] \end{bmatrix} \tag{18}$$

Unfortunately, deriving these expectations is quite difficult, thus the asymptotic distribution of $(\hat{\alpha},\hat{\lambda},\hat{\beta})$, under some regularity criteria, is the normal distribution with mean (α,λ,β) and variance-covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})$

$$\left(\hat{\alpha},\hat{\lambda},\hat{\beta}\right) \sim N\left((\alpha,\lambda,\beta), I_0^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})\right).$$

The approximate asymptotic variance-covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})$ for the ML estimates of the parameters (α,λ,β) can be obtained as the inverse of the observed Fisher information matrix

$$I_0^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta}) = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial\alpha^2} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\lambda} & -\frac{\partial^2 \ln L}{\partial\alpha\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\lambda\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\lambda^2} & -\frac{\partial^2 \ln L}{\partial\lambda\partial\beta} \\ -\frac{\partial^2 \ln L}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} & -\frac{\partial^2 \ln L}{\partial\beta^2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\lambda},\hat{\beta})} = \begin{bmatrix} Var(\widehat{\alpha}) & Cov(\widehat{\alpha},\widehat{\lambda}) & Cov(\widehat{\alpha},\widehat{\beta}) \\ Cov(\widehat{\lambda},\widehat{\alpha}) & Var(\widehat{\lambda}) & Cov(\widehat{\lambda},\widehat{\beta}) \\ Cov(\widehat{\beta},\widehat{\alpha}) & Cov(\widehat{\beta},\widehat{\lambda}) & Var(\widehat{\beta}) \end{bmatrix} \tag{19}$$

where

$$\frac{\partial^2 \ln L}{\partial\alpha^2} = -\frac{m}{\alpha^2}, \tag{20}$$

$$\frac{\partial^2 \ln L}{\partial\lambda^2} = -\frac{m}{\lambda^2} - \sum_{i=1}^{n_1} \frac{(R_i+1)\alpha-1}{y_i^2} \exp\left(-\frac{\lambda}{y_i}\right)\left(1-\exp\left(-\frac{\lambda}{y_i}\right)\right)^{-2}$$
$$-\sum_{i=n_1+1}^{m} \frac{(R_i+1)\alpha-1}{(\tau+\beta(y_i-\tau))^2} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-2}, \tag{21}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{m-n_1}{\beta^2} + 2\sum_{i=n_1+1}^{m} \frac{(y_i-\tau)^2}{\tau+\beta(y_i-\tau)} - 2\lambda \sum_{i=n_1+1}^{m} \frac{(y_i-\tau)^2}{(\tau+\beta(y_i-\tau))^3}$$
$$+2\lambda \sum_{i=n_1+1}^{m} \frac{((R_i+1)\alpha-1)(y_i-\tau)^2}{(\tau+\beta(y_i-\tau))^3} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1}$$
$$-\lambda^2 \sum_{i=n_1+1}^{m} \frac{((R_i+1)\alpha-1)(y_i-\tau)^2}{(\tau+\beta(y_i-\tau))^4} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-2}, \tag{22}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n_1} \frac{R_i+1}{y_i} \exp\left(-\frac{\lambda}{y_i}\right)\left(1-\exp\left(-\frac{\lambda}{y_i}\right)\right)^{-1}$$
$$+\sum_{i=n_1+1}^{m} \frac{R_i+1}{\tau+\beta(y_i-\tau)} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1}, \tag{23}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = -\lambda \sum_{i=n_1+1}^{m} \frac{(R_i+1)(y_i-\tau)}{(\tau+\beta(y_i-\tau))^2} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1}, \tag{24}$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = \sum_{i=n_1+1}^{m} \frac{y_i-\tau}{(\tau+\beta(y_i-\tau))^2}$$
$$-\sum_{i=n_1+1}^{m} \frac{((R_i+1)\alpha-1)(y_i-\tau)}{(\tau+\beta(y_i-\tau))^2} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-1}$$
$$+\lambda \sum_{i=n_1+1}^{m} \frac{((R_i+1)\alpha-1)(y_i-\tau)}{(\tau+\beta(y_i-\tau))^3} \exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\left(1-\exp\left(-\frac{\lambda}{\tau+\beta(y_i-\tau)}\right)\right)^{-2}. \tag{25}$$

Thus, the $100(1-\gamma)\%$ approximate confidence intervals for α, λ and β are , respectively, given by

$$\widehat{\alpha} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\widehat{\alpha})}, \qquad \widehat{\lambda} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\widehat{\lambda})} \qquad \text{and} \qquad \widehat{\beta} \pm z_{\frac{\gamma}{2}} \sqrt{Var(\widehat{\beta})} \tag{26}$$

where $Var(\widehat{\alpha})$, $Var(\widehat{\lambda})$ and $Var(\widehat{\beta})$ are the first, second and third elements on the main diagonal of the approximate asymptotic variance-covariance matrix $I_0^{-1}(\hat{\alpha},\hat{\lambda},\hat{\beta})$ and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

4.1 The asymptotic approach using the delta method

In this subsection, at the normal use condition, the approximate confidence intervals for $R(t)$ and $h(t)$ are generated using the delta method proposed by Greene [20]. Let

$$V_1 = \begin{bmatrix} \frac{\partial R(t)}{\partial \alpha} & \frac{\partial R(t)}{\partial \lambda} & \frac{\partial R(t)}{\partial \beta} \end{bmatrix} \text{ and } V_2 = \begin{bmatrix} \frac{\partial h(t)}{\partial \alpha} & \frac{\partial h(t)}{\partial \lambda} & \frac{\partial h(t)}{\partial \beta} \end{bmatrix},$$

where

$$\frac{\partial R(t)}{\partial \alpha} = \left[1-\exp\left(\frac{-\lambda}{t}\right)\right]^{\alpha} \ln\left[1-\exp\left(\frac{-\lambda}{t}\right)\right],$$

$$\frac{\partial R(t)}{\partial \lambda} = \frac{\alpha}{x} \exp\left(\frac{-\lambda}{x}\right)\left[1-\exp\left(\frac{-\lambda}{x}\right)\right]^{\alpha-1},$$

$$\frac{\partial R(t)}{\partial \beta} = 0,$$

$$\frac{\partial h(t)}{\partial \alpha} = \frac{\lambda}{t^2}\exp\left(-\frac{\lambda}{t}\right)\left[1-\exp\left(-\frac{\lambda}{t}\right)\right]^{-1},$$

$$\frac{\partial h(t)}{\partial \lambda} = \frac{\alpha}{t^2}\exp\left(-\frac{\lambda}{t}\right)\left[1-\exp\left(-\frac{\lambda}{t}\right)\right]^{-1} - \frac{\alpha\lambda}{t^3}\exp\left(-\frac{\lambda}{t}\right)\left[1-\exp\left(-\frac{\lambda}{t}\right)\right]^{-2},$$

$$\frac{\partial h(t)}{\partial \beta} = 0.$$

Then, the approximate estimates of $Var(\hat{R}(t))$ and $Var(\hat{h}(t))$ are given, respectively, by

$$Var(\hat{R}(t)) \simeq [V_1^T I_0^{-1} V_1]_{(\hat{\alpha},\hat{\lambda})} \quad \text{and} \quad Var(\hat{h}(t)) \simeq [V_2^T I_0^{-1} V_2]_{(\hat{\alpha},\hat{\lambda})}$$

where V_i^T is the transpose of the vector V_i, $i = 1,2$. These results yield the approximate confidence intervals for $R(t)$ and $h(t)$ as

$$\left(\hat{R}(t) \pm z_{\frac{\gamma}{2}}\sqrt{Var\ (\hat{R}(t))}\right) \quad \text{and} \quad \left(\hat{h}(t) \pm z_{\frac{\gamma}{2}}\sqrt{Var(\hat{h}(t))}\right). \tag{27}$$

5 Computational Results

Some computational results are reported in this section. A Monte Carlo simulation study is carried out to evaluate the performance of the inferential procedures presented in the paper. Finally, a numerical example is provided to demonstrate all of the inferential results.

5.1 Monte Carlo simulation study

In this section, Monte Carlo simulation is used to illustrate the theoretical results discussed in the previous sections. In this simulation study, we used different choices of the sample size n, effective sample size m, and censoring scheme R as shown in Table 1.

Table 1: The different censoring schemes with different choices of n, m and R.

Censoring Scheme	n	m	R
CS1	40	10	$(4,2,4,3,2,4,2,4,3,2)$
CS2	40	15	$(1,2,1,3,2,1,2,1,3,2,1,2,1,1,2)$
CS3	40	20	$(1,2,1,0,2,0,0,0,0,2,1,2,1,0,2,1,2,1,0,2)$
CS4	50	10	$(4,3,4,3,5,4,5,4,3,5)$
CS5	50	15	$(4,2,4,3,2,1,2,1,3,2,1,2,3,3,2)$
CS6	50	20	$(4,2,0,3,2,0,2,0,3,2,0,2,0,3,2,0,0,0,3,2)$
CS7	60	20	$(4,2,4,3,2,4,2,0,3,2,0,2,0,3,2,0,2,0,3,2)$
CS8	60	30	$(1,2,1,0,0,1,0,1,0,2,1,0,1,0,2,1,2,1,0,2,1,2,1,0,2,1,2,1,0,2)$

By using the algorithm described in [21], we generate 1000 progressively Type-II censored samples from the generalized inverted exponential distribution, with parameters $\alpha = 0.4$ and $\lambda = 2$, under the step-stress partially accelerated life test model with acceleration factor $\beta = 2$ and stress change time $\tau = 0.7$ or 1. The following algorithm is used to generate a progressive Type-II censored sample from the generalised inverted exponential distribution under the step-stress partially accelerated life test model

1. Specify the values of n, m, $R_i, i = 1,2,...,m$, τ and the values of the parameters α, λ and β.
2. Compute the actual values of the reliability and hazard rate functions with $t = 1$.
3. Generate progressive Type-II censored sample from the generalized inverted exponential distribution by setting $Y = -\frac{\lambda}{\ln\left(1-(1-u)^{\frac{1}{\alpha}}\right)}$ if $Y \le \tau$, and $Y = \frac{-\lambda/\ln\left(1-(1-u)^{\frac{1}{\alpha}}\right)-\tau}{\beta} + \tau$ if $Y > \tau$, where U represents a uniform (0,1) random variable.

4.Use the progressive Type-II censored sample obtained in Step 3 to compute the ML estimates of the parameters α, λ and β and then construct the corresponding approximate confidence intervals with confidence level $1-\gamma = 0.95$.
5.Use the values of $\hat{\alpha}$ and $\hat{\lambda}$ obtained in Step 3 to compute the ML estimates of the reliability and hazard rate functions with $t = 1$ and then construct the corresponding approximate confidence intervals with confidence level $1-\gamma = 0.95$.
6.Repeat Steps 3-5 $N = 1000$ times.
7.Compute the expected value (EV) and mean square error (MSE) of $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\beta}$, $\hat{R}(t=1)$ and $\hat{h}(t=1)$.
8.Compute the expected values of the lower bound, upper bound and the width of the corresponding approximate confidence intervals.

Tables 2-6 present the EV and MSE of $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\beta}$, $\hat{R}(t=1)$ and $\hat{h}(t=1)$, respectively, and the lower bound and upper bound and the width of the corresponding approximate confidence interval based on different choices of n, m, R and τ.

Table 2: The EV and MSE of $\hat{\alpha}$, and the lower bound and upper bound and the width of the corresponding approximate confidence interval.

			$\tau = 0.7$					$\tau = 1$				
n	m	Scheme	EV	MSE	lower	upper	width	EV	MSE	lower	upper	width
40	10	CS1	0.7963	0.4500	-0.7831	2.3756	3.1587	0.7559	0.3735	-0.5302	2.0420	2.5722
40	15	CS2	0.6212	0.1291	-0.0471	1.2896	1.3367	0.5822	0.1400	0.0417	1.1226	1.0809
40	20	CS3	0.5500	0.0776	0.1146	0.9855	0.8709	0.5491	0.0726	0.1766	0.9215	0.7449
50	10	CS4	0.7486	0.3483	0.8896	2.3867	3.2763	0.9443	0.6647	-0.9164	2.8050	3.7214
50	15	CS5	0.5573	0.0986	-0.0894	1.2040	1.2935	0.6262	0.1860	-0.0252	1.2777	1.3029
50	20	CS6	0.5479	0.0779	0.1127	0.9830	0.8703	0.5708	0.0942	0.1800	0.9615	0.7815
60	20	CS7	0.5565	0.0818	0.1064	1.0065	0.9001	0.5315	0.0762	0.1953	0.8677	0.6723
60	30	CS8	0.4824	0.0310	0.2200	0.7449	0.5249	0.5096	0.0387	0.2889	0.7303	0.4415

Table 3: The EV and MSE of $\hat{\lambda}$, and the lower bound and upper bound and the width of the corresponding approximate confidence interval.

			$\tau = 0.7$					$\tau = 1$				
n	m	Scheme	EV	MSE	lower	upper	width	EV	MSE	lower	upper	width
40	10	CS1	2.2381	0.2314	0.8226	3.6536	2.8309	2.8977	1.1421	1.2347	4.5607	3.3260
40	15	CS2	2.1780	0.1351	0.9950	3.3610	2.3660	2.8520	0.9630	1.5232	4.1808	2.6576
40	20	CS3	2.1384	0.1218	0.9789	3.2980	2.3191	2.8213	0.8454	1.5860	4.0566	2.4706
50	10	CS4	2.3214	0.2803	0.8631	3.7797	2.9165	3.3069	2.0864	1.5046	5.1091	3.6045
50	15	CS5	2.2078	0.1575	1.0415	3.3741	2.3326	2.9945	1.2594	1.6905	4.2985	2.6080
50	20	CS6	2.2227	0.1417	1.1252	3.3203	2.1952	3.0010	1.1837	1.8096	4.1924	2.3828
60	20	CS7	2.3362	0.2157	1.2830	3.3895	2.1065	3.0368	1.2509	1.9690	4.1046	2.1356
60	30	CS8	2.2966	0.1694	1.2907	3.3025	2.0118	3.1262	1.4055	2.1306	4.1219	2.1356

From the results shown in Tables 2-6, it is observed that:

1.For fixed τ, the MSEs of the ML estimates and the widths of the corresponding approximate confidence intervals decrease in all cases as sample size n increases. As a result, the estimators that depend on large samples perform better than those that depend on small samples.
2.For fixed n and τ, in most cases, the MSEs of the ML estimates and the widths of the corresponding approximate confidence intervals decrease as the effective sample size m increases. As a result, the estimators that depend on large effective samples perform better than those that depend on small samples.
3.For fixed n and m, in most cases, the small value of stress change time τ gives a better result in the sense of having smaller MSE.
4.The approximate confidence intervals always include the ML estimate of the unknown parameter in all cases.

Table 4: The EV and MSE of $\hat{\beta}$, and the lower bound and upper bound and the width of the corresponding approximate confidence interval.

			$\tau = 0.7$					$\tau = 1$				
n	m	Scheme	EV	MSE	lower	upper	width	EV	MSE	lower	upper	width
40	10	CS1	1.4177	0.7854	-0.8918	3.7273	4.6191	1.8138	0.8995	-0.7984	4.4261	5.2245
40	15	CS2	1.4343	0.6700	-0.4362	3.3047	3.7408	2.3809	2.7804	-0.3165	5.0783	5.3948
40	20	CS3	1.6847	0.6578	-0.2986	3.6680	3.9666	2.0255	0.8522	0.0406	4.0045	3.9639
50	10	CS4	1.5937	0.6878	-0.9956	4.1831	5.1787	1.7285	0.7220	-0.6703	4.1273	4.7976
50	15	CS5	1.7749	0.5550	-0.5387	4.0885	4.6272	2.2106	1.3602	-0.2105	4.6316	4.8421
50	20	CS6	1.7178	0.5678	-0.2090	3.6446	3.8537	2.0286	0.6469	0.1502	3.9071	3.7569
60	20	CS7	1.7786	0.4398	-0.1109	3.6682	3.7791	2.4231	1.3258	0.3860	4.4602	4.0741
60	30	CS8	1.9462	0.4265	0.2023	3.6900	3.4877	2.2566	0.6455	0.6986	3.8145	3.1159

Table 5: The EV and MSE of $\hat{R}(t=1)$, and the lower bound and upper bound and the width of the corresponding approximate confidence interval.

			$\tau = 0.7$					$\tau = 1$				
n	m	Scheme	EV	MSE	lower	upper	width	EV	MSE	lower	upper	width
40	10	CS1	0.9251	0.0276	0.8231	1.0270	0.2038	0.9633	0.0220	0.9196	1.0070	0.0874
40	15	CS2	0.9319	0.0202	0.8494	1.0143	0.1649	0.9684	0.0263	0.9309	1.0059	0.0750
40	20	CS3	0.9359	0.0179	0.8592	1.0126	0.1535	0.9676	0.0258	0.9304	1.0049	0.0745
50	10	CS4	0.9348	0.0186	0.8474	1.0223	0.1750	0.9699	0.0275	0.9370	1.0029	0.0659
50	15	CS5	0.9409	0.0150	0.8699	1.0119	0.1420	0.9712	0.0288	0.9393	1.0031	0.0638
50	20	CS6	0.9415	0.0145	0.8752	1.0077	0.1325	0.9723	0.0298	0.9418	1.0028	0.0610
60	20	CS7	0.9472	0.0137	0.8898	1.0046	0.1148	0.9754	0.0325	0.9511	0.9997	0.0486
60	30	CS8	0.9507	0.0142	0.8994	1.0020	0.1026	0.9775	0.0345	0.9559	0.9991	0.0432

Table 6: The EV and MSE of $\hat{h}(t=1)$, and the lower bound and upper bound and the width of the corresponding 95% approximate confidence interval.

			$\tau = 0.7$					$\tau = 1$				
n	m	Scheme	EV	MSE	lower	upper	width	EV	MSE	lower	upper	width
40	10	CS1	0.1882	0.0959	-0.0797	0.4561	0.5358	0.1104	0.0346	-0.0084	0.2291	0.2375
40	15	CS2	0.1637	0.0609	0.0065	0.3338	0.3404	0.0930	0.0402	0.0052	0.1807	0.1755
40	20	CS3	0.1503	0.0484	0.0093	0.2914	0.2821	0.0939	0.0385	0.0109	0.1770	0.1661
50	10	CS4	0.1677	0.0734	-0.0809	0.4164	0.4972	0.1017	0.0355	-0.0062	0.2095	0.2157
50	15	CS5	0.1430	0.0464	-0.0087	0.2947	0.3034	0.0886	0.0437	0.0076	0.1697	0.1622
50	20	CS6	0.1418	0.0434	0.0136	0.2700	0.2565	0.0849	0.0453	0.0119	0.1580	0.1461
60	20	CS7	0.1330	0.0373	0.0147	0.2513	0.2366	0.0766	0.0520	0.0175	0.1357	0.1182
60	30	CS8	0.1214	0.0290	0.0265	0.2163	0.1898	0.0715	0.0560	0.0192	0.1238	0.1047

5.2 Numerical example

To illustrate the inferential procedures discussed in this paper, we chose $n = 40$, $m = 10$ and $R = (4,2,4,3,2,4,2,4,3,2)$ and generated the following progressively Type-II censored sample from the generalized inverted exponential distribution, with parameters $\alpha = 0.4$ and $\lambda = 2$, under the step-stress partially accelerated life test model with acceleration factor $\beta = 2$ and stress change time $\tau = 0.7$.

0.3980 0.5722 0.7582 0.8235 0.8763 1.7621 1.9303 2.4606 4.6408 8.9036

Based on this generated progressively Type-II censored sample, we calculated the ML estimates $\hat{\alpha}$, $\hat{\lambda}$, $\hat{\beta}$, $\hat{R}(1)$ and $\hat{h}(1)$. Also, we obtained the 95% approximate confidence intervals of α, λ, β, $R(1)$ and $h(1)$. The obtained results are presented in Table 7.

Table 7: The ML estimates of α, λ, β, $R(1)$ and $h(1)$, and the lower bounds, upper bounds and widths of the corresponding 95% approximate confidence intervals.

Parameter	*ML Estimate*	*Lower Bound*	*Upper Bound*	*Width*
α	1.5634	0.6116	2.5151	1.9035
λ	0.3180	-0.1067	0.7428	0.8495
β	1.7828	-1.7126	5.2781	6.9907
$R(1)$	0.9280	0.8527	1.0033	0.1506
$h(1)$	0.1317	-0.0104	0.2738	0.2842

6 Conclusions

In this paper, when the lifetimes of test units follow the generalized inverted exponential distribution, the problem of step-stress partially accelerated life tests with progressive Type-II censoring is discussed. The ML estimators of the unknown parameters α, λ, and the acceleration factor β, as well as the ML estimators of the reliability and hazard rate functions, have been developed. In addition, we obtained the observed Fisher information matrix and used it to construct the approximate confidence intervals of the unknown parameters. In addition, for the reliability and hazard rate functions, we employed the delta method to generate an approximate confidence interval. We use a simulation study with varied sample sizes and different censoring schemes to compare and examine the effectiveness of the suggested estimate methods.

From the obtained numerical results, we note that the estimators that depend on large samples perform better than those that depend on small samples. Also, the estimators that depend on large effective samples perform better than those that depend on small samples. Moreover, the small value of stress change time τ gives a better result in the sense of having smaller MSE.

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Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] H.F. Martz, Reliability Theory, *Encyclopedia of Physical Science and Technology*, **14**, 457-474 (1992).
[2] W. Nelson, *Accelerated Life Testing: Statistical Models, Test Plans and Data Analysis*. Wiley, New York, (1990).
[3] A.A. Ismail, Inference in the generalized exponential distribution under partially accelerated tests with progressive Type-II censoring, *Theoretical and Applied Fracture Mechanics*, **59**, 49-56 (2012).
[4] A.A. Ismail, Likelihood inference for a step-stress partially accelerated life test model with Type-I progressively hybrid censored data from Weibull distribution, *Journal of Statistical Computation and Simulation*, **84**, 2486-2494 (2014).
[5] D. Han and D. Kundu, Inference for a Step-Stress Model With Competing Risks for Failure From the Generalized Exponential Distribution Under Type-I Censoring, *IEEE Transactions on Reliability*, **64**, 31-43 (2014).
[6] A. Ganguly, D. Kundu and S. Mitra, Bayesian Analysis of a Simple Step-Stress Model Under Weibull Lifetimes, *IEEE Transactions on Reliability*, **64**, 473-485 (2015).
[7] M.M. Mohie El-Din, M.M Amein, H.E. El-Attar and E.H. Hafez, Estimation in Step-Stress Accelerated Life Testing for Lindely Distribution with Progressive First-Failure Censoring, *Journal of Statistics Applications & Probability*, **5**, 393-398 (2016).
[8] A.R. Shafay, Exact Inference for a Simple Step-stress Model with Generalized Type-I Hybrid Censored Data from the Exponential Distribution, *Communications in Statistics - Simulation and Computation*, **45**, 181-206 (2016).
[9] C. Zhang and Y. Shi, Statistical prediction of failure times under generalized progressive hybrid censoring in a simple step-stress accelerated competing risks model, *Journal of Systems Engineering and Electronics*, **28**, 282-291 (2017).
[10] A.A. Soliman, E.A. Ahmed, N.A. Abou-Elheggag and S.M. Ahmed, Step-Stress Partially Accelerated Life Tests Model in Estimation of Inverse Weibull Parameters under Progressive Type-II Censoring, *Applied Mathematics & Information Sciences*, **11**, 1369-1381 (2017).

[11] N. Balakrishnan and R. Aggarwala, *Progressive Censoring: Theory, Methods and Applications*, Birkhäuser, Boston, (2000).
[12] N. Balakrishnan, Progressive censoring methodology: An appraisal (with discussions), *Test*, **16**, 211-296 (2007).
[13] M.M. Mohie El-Din and A.R. Shafay, One- and Two-Sample Bayesian Prediction Intervals Based on Progressively Type-II Censored Data, *Statistical papers*, **54**, 287-307 (2013).
[14] C.T. Lin, B.S. Duran and T.O. Lewis, Inverted gamma as life distribution, *Microelectron Reliability*, **29**, 619-626 (1989).
[15] A.M. Abouammoh and A.M. Alshingiti, Reliability estimation of generalized inverted exponential distribution, *Journal of Statistical Computation and Simulation*, **79**, 1301-1315 (2009).
[16] H. Krishna and K. Kumar, Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample, *Journal of Statistical Computation and Simulation*, **83**, 1007-1019 (2013).
[17] R.A. Bakoban, Estimation in Step- Stress Partially Accelerated Life Tests for the Generalized Inverted Exponential Distribution using Type I Censoring, *American Journal of Scientific Research*, **3**, 25-35 (2012).
[18] A.A. Soliman, E.A. Ahmed, A.A. Farghal and A.A. AL-Shibany, Estimation of Generalized Inverted Exponential Distribution based on Adaptive Type-II Progressive Censoring Data, *Journal of Statistics Applications & Probability*, **9**, 215-230 (2020).
[19] M.H. DeGroot and P.K. Goel, Bayesian and optimal design in partially accelerated life testing, *Naval Research Logistic Quarterly*, **16**, 223-235 (1979).
[20] W.H. Greene, *Econometric Analysis*, 4th Ed. New York: Prentice Hall, (2000).
[21] N. Balakrishnan and R.A. Sandhu, A simple simulational algorithm for generating progressive Type-II censored samples, *The American Statistician*, **49**, 229-230 (1995).