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Rough Topology Via Multifunction

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Abstract: The notion of rough set was originally proposed by Pawlak (1982) [6].Carmel Richard et al.[3] introduce a new topology called rough topology in terms of the lower and upper approximations of Pawlak approximation space.In this paper, we have introduced a new topology called rough topology in terms of lower and upper approximations of a rough set via multifunction.In addition we study and investigate some properties of rough generalized closed sets in rough topological spaces via multifunction (multi rough topological space).

Keywords: Rough Sets, Topological Space, Lower Approximation, Upper Approximation, Multifunction

1 Introduction

The main idea of rough sets corresponds to the lower and upper set approximations. These two approximations are exactly the interior and the closure of the set with respect to a certain topology τ on a collection U of imprecise data acquired from any real-life field.

The concept of multifunction has a great importance in both the theoretical and application fields, Maritz [5] started his generalization process by a surjective multifunction from a set to the class of proper subsets of the universe,i.e., Maritz used a new method via a catalyst set T to generate an approximation space $\Lambda^M = (U, \digamma(T))$.

The reference space in rough set theory is the approximation space whose topology is generated by the equivalence classes of R but, it is Maritz space in this paper.The topology constructed from this approximation space is a new topology called rough topology in terms of the lower and upper approximations via multifunction.

2 Preliminaries

By a multifunction $F : T \to U$, we mean a point-to-set correspondence from T into U , and always assume that $F(t) \neq \emptyset$ for all $t \in T$.

For a multifunction $F : T \to U$,we shall denote the upper and lower inverse of a subset B of U by $F^+(B)$ and $F^-(B)$,respectively,that is, $F^+(B) = \{t \in T : F(t) \subset B\}$ and $F^-(B) = \{t \in T : F(t) \cap B \neq \emptyset\}$.

In particular,$F^-(u) = \{t \in T : u \in F(t)\}$ for each point $u \in U$.For each $A \subset T, F(A) = \bigcup_{t \in A} F(t)$,and F is said to be a surjection if $F(T) = U$, or equivalently if for each $u \in U$ there exists an $t \in T$ such that $u \in F(t)$.

In this paper we assume that F is a surjection. The class $\digamma(T) = \{F(t) : t \in T\}$ is the collection of all atoms where $F(t)$ is called an atom.

Definition 2.1[5]. Let $F : T \to U$ be a surjective multifunction then the pair $(U, \digamma(T))$ is the Approximation space (Maritz space).

Definition 2.2[5]. Let $\Lambda^M = (U, \digamma(T))$ be a Maritz space and $A \subset U$ then

(i)The upper approximation of A in $(U, \digamma(T))$ is the set

$$\begin{aligned} F(F^-(A)) &= F(\{t \in T : F(t) \cap A \neq \phi\}) \\ &= \cup\{F(t) \in \digamma(T) : F(t) \cap A \neq \phi\}. \end{aligned} \tag{1}$$

(ii)The lower approximation of A in $\Lambda^M = (U, \digamma(T))$ is the set

$$\begin{aligned} F(F^+(A)) &= F(\{t \in T : F(t) \subset A\}) \\ &= \cup\{F(t) \in \digamma(T) : F(t) \subset A\}. \end{aligned} \tag{2}$$

We write $FF^-(A)$ instead of $F(F^-(A))$ and $FF^+(A)$ instead of $F(F^+(A))$.

(iii)The boundary region of A in $\Lambda^M = (U, \digamma(T))$ is the set $B_{\digamma(T)}(A) = FF^-(A) - FF^+(A)$

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The sets in $F(T)$ will be called atoms (or elementary sets) in $\Lambda^M = (U, F(T))$, every finite union of atoms is called a composed set and $com(\Lambda^M)$ is the collection of all composed sets in Λ^M.

Proposition 2.1 Let $\Lambda^M = (U, F(T))$ be a Maritz space and $A \subset U$ then A is $F-$rough set if and only if $FF^-(A) \neq FF^+(A)$.

Proof. Obvious. ■

we list some of the properties of lower and upper approximations in the sense of Maritz.

let $\Lambda^M = (U, F(T))$ be a Maritz space and $A, B \subset U$,then

1.$FF^+(A) \subset A$.
2.$A \subset FF^-(A)$.
3.$FF^+(\emptyset) = \emptyset = FF^-(\emptyset)$.
4.$FF^+(U) = U = FF^-(U)$.
5.$FF^-(A) = FF^-(FF^-(A))$.
6.$FF^-(FF^-(A)) \supset FF^+(FF^-(A))$.
7.$FF^-(FF^+(A)) \supset FF^+(FF^+(A))$.
8.$FF^+(A \cap B) = FF^+(A) \cap FF^+(B)$.
9.$FF^-(A) \cap FF^+(A^c) \neq \emptyset$.
10.$FF^+(A) \cap FF^-(A^c) \neq \emptyset$.
11.$(FF^-(A^c))^c \subset FF^+(A)$.
12.$(FF^+(A^c))^c \subset FF^-(A)$.
13.$A \subset B \Rightarrow FF^+(A) \subset FF^+(B)$ and $FF^-(A) \subset FF^-(B)$.
14.$FF^-(A \cup B) = FF^-(A) \cup FF^-(B)$.
15.$FF^+(A) = FF^+(FF^+(A))$.

Remark 2.1 In [5] Maritz indicates that $FF^-(A)$ is the Kuratowski closure operator and $FF^+(A)$ is the Kuratowski interior operator.

3 Rough topology via multifunction

Carmel Richard et al.[3] introduce a new topology called rough topology in terms of the lower and upper approximations of Pawlak approximation space.In this section we introduce rough topology via multifunction(Meritz space).

Definition 3.1: Let U be the universe, $F : T \to U$ be a multifunction on U, $A \subseteq U$ and $\tau_{F(T)}(A) = \{U, \emptyset, FF^+_{F(T)}(A), FF^-_{F(T)}(A), B_{F(T)}(A)\}$.Then $\tau_{F(T)}(A)$ is a topology on U , called as the rough topology via multifunction with respect to A. Elements of the rough topology via multifunction are known as the rough-open sets in U and $(U, \tau_{F(T)}(A))$ is called the rough topological space via multifunction.

Definition 3.2:Let $(U, \tau_{F(T)}(A))$ be rough topological space via multifunction with respect to A where $A \subset U$ and if $X \subset U$, then the rough interior of X is defined as the union of all rough-open subsets of X and it is denoted by $RInt(X)$. That is, $RInt(X)$ is the largest rough-open subset of X. The rough closure of $X$is defined as the intersection of all rough closed subsets containing X and it is denoted by $RCl(X)$. That is, $RCl(X)$ is the smallest rough closed subset containing X.

Proposition 3.1:The basis for the rough topology via multifunction $\tau_{F(T)}(A)$ with respect to A is given by $\beta_F(A) = \{U, FF^+(A), B_{F(T)}(A)\}$.

Proof:

(1)$\bigcup_{A \in \beta_F} A = U$
(2)Consider U and $FF^+(A)$ from $\beta_F(A)$. Let $W = FF^+(A)$. Since $U \cap FF^+(A) = FF^+(A)$, $W \subset U \cap FF^+(A)$ and every x in $U \cap FF^+(A)$ belongs to W. If we consider U and $B_{F(T)}(A)$ from $\beta_F(A)$, taking $W = B_{F(T)}(A)$, $W \subset U \cap B_{F(T)}(A)$ and every x in $U \cap B_{F(T)}(A)$ belongs to W, since $U \cap B_{F(T)}(A) = B_{F(T)}(A)$, and when we consider $FF^+(A)$ and $B_{F(T)}(A)$, $FF^+(A) \cap B_{F(T)}(A) = \emptyset$. Thus $\beta_F(A)$ is a basis for $\tau_{F(T)}(A)$.

Example 3.1: Let $T = \{1,2,3\}$, $U = \{x,y,z,w\}$ and $F : T \to U$ defined by:
$F(1) = \{x\}$, $F(2) = \{x,y\}$ and $F(3) = \{y,z,w\}$, then $\Lambda^M = (U, F(T))$ be a Maritz space. If $A = \{x\}$, then $FF^+_{F(T)}(A) = \{x\}$, $FF^-_{F(T)}(A) = \{x,y\}$ and $B_{F(T)}(A) = \{y\}$, therefore
$\tau_{F(T)}(A) = \{U, \emptyset, \{x\}, \{x,y\}, \{y\}\}$ is a topology on U , called as the rough topology via multifunction with respect to A, and $\beta_{F(T)}(A) = \{U, \{x\}, \{y\}\}$ is a basis for the rough topology via multifunction $\tau_{F(T)}(A)$ with respect to A.

Definition 3.3 Let U be the universe and $F : T \to U$ be a multifunction on U, $A \subseteq U$, $\tau_{F(T)}(A)$ be the rough topology via multifunction on$U$ and $\beta_{F(T)}(A)$is a basis for the rough topology via multifunction $\tau_{F(T)}(A)$ with respect to A. A subset M of A, the set of attributes is called the core of $F(T)$ if $\beta_{F(T)M} \neq \beta_{F(T)-r}$ for every r in M. That is, a core of $F(T)$ is a subset of attributes which is such that none of its elements can be removed without affecting the classification power of attributes.

Example 3.2 Consider Table(1) which contains information about patients suffering from a heart disease Dyspnea (D),Gasometry (G),Heart rate (Hr)and Pulmonary states (P)respectively .

Table 1: ???????

U	D	G	Hr	P	Result
P_1	1	20	70	1	Fitness
P_2	0	20	88	0	Fitness
P_3	0	30	88	1	Infect
P_4	0	30	70	0	Fitness
P_5	1	32	70	1	Infect
P_6	0	30	70	0	Infect

The columns of the table represent the attributes (the heart disease) where $T = \{D, G, Hr, P\}$ and the rows represent the objects (the patients)where
$U = \{P_1, P_2, P_3, P_4, P_5, P_6\}$. The entries in the table are the

attribute values,and the multifunction $F : T \to U$ defined by:
$F(D) = \{\{P_1,P_5\},\{P_2,P_3,P_4,P_6\}\}$,
$F(G) = \{\{P_1,P_2\},\{P_3,P_4,P_6\},\{P_5\}\}$,
$F(Hr) = \{\{P_1,P_4,P_5,P_6\},\{P_2,P_3\}\}$,and
$F(P) = \{\{P_1,P_3,P_5\},\{P_2,P_4,P_6\}\}$,
then $\Lambda^M = (U,F(T))$ be a Maritz space.
The class $F(T) = \{\{P_4,P_6\},\{P_1\},\{P_2\},\{P_3\},\{P_5\}\}$.
Let $A = \{P_3,P_5,P_6\}$, be the set of patients suffering from a heart disease,
then $FF^+_{F(T)}(A) = \{P_3,P_5\}, FF^-_{F(T)}(A) = \{P_3,P_4,P_5,P_6\}$ and $B_{F(T)}(A)\} = \{P_4,P_6\}$,therefore
$\tau_{F(T)}(A) = \{U,\emptyset,\{P_3,P_5\},\{P_3,P_4,P_5,P_6\},\{P_4,P_6\}\}$ is a topology on U , called as the rough topology via multifunction with respect to A, and $\beta_{F(T)}(A) = \{U,\{P_3,P_5\},\{P_4,P_6\}\}$ is a basis for the rough topology via multifunction $\tau_{F(T)}(A)$ with respect to A.

If we remove the attribute Dyspnea from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-D) = \{\{P_4,P_6\},\{P_1\},\{P_2\},\{P_3\},\{P_5\}\}$. The corresponding lower and upper approximations are given by: $FF^+_{F(T)}(A) = \{P_3,P_5\}, FF^-_{F(T)}(A) = \{P_3,P_4,P_5,P_6\}$ and $B_{F(T)}(A)\} = \{P_4,P_6\}$, therefore
$\beta_{F(T-D)}(A) = \{U,\{P_3,P_5\},\{P_4,P_6\}\} = \beta_{F(T)}(A)$.

If we remove the attribute Gasometry from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-G) = \{\{P_1,P_5\},\{P_2,P_6\},\{P_2\},\{P_3\}\}$.The corresponding lower and upper approximations are given by:
$FF^+_{F(T-G)}(A) = \{P_3\}$, $FF^-_{F(T-G)}(A) = \{P_1,P_3,P_4,P_5,P_6\}$ and $B_{F(T-G)}(A)\} = \{P_1,P_4,P_5,P_6\}$, therefore
$\beta_{F(T-G)}(A) = \{U,\{P_3\},\{P_1,P_4,P_5,P_6\}\} \neq \beta_{F(T)}(A)$.

If we remove the attribute heart rate from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-Hr) = \{\{P_4,P_6\},\{P_1\},\{P_2\},\{P_3\},\{P_5\}\}$. The corresponding lower and upper approximations are given by: $FF^+_{F(T)}(A) = \{P_3,P_5\}, FF^-_{F(T)}(A) = \{P_3,P_4,P_5,P_6\}$ and $B_{F(T)}(A)\} = \{P_4,P_6\}$,therefore
$\beta_{F(T-D)}(A) = \{U,\{P_3,P_5\},\{P_4,P_6\}\} = \beta_{F(T)}(A)$.

If we remove the attribute pulmonary states from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-P) = \{\{P_4,P_6\},\{P_1\},\{P_2\},\{P_3\},\{P_5\}\}$. The corresponding lower and upper approximations are given by: $FF^+_{F(T)}(A) = \{P_3,P_5\}$, $FF^-_{F(T)}(A) = \{P_3,P_4,P_5,P_6\}$ and $B_{F(T)}(A)\} = \{P_4,P_6\}$, therefore
$\beta_{F(T-D)}(A) = \{U,\{P_3,P_5\},\{P_4,P_6\}\} = \beta_{F(T)}(A)$.
Therefore, CORE $(F(T)) = \{G\}$.

Example 3.3 Consider Table(2) which is giving information about five students in a school having an exam in three different language (English(E), French(F), German(G)) respectively.

Table 2: ?????????

U	English	French	German	Result
S_1	true	false	false	fail
S_2	false	true	true	fail
S_3	true	true	true	pass
S_4	true	false	false	pass
S_5	false	false	true	fail

The columns of the table represent the attributes (the langues)where $T = \{E,F,G\}$ and the rows represent the objects (the students)
where $U = \{S_1,S_2,S_3,S_4,S_5\}$. The entries in the table are the attribute values,and the multifunction $F : T \to U$ defined by:
$F(E) = \{S_1,S_3,S_4\}$, $F(F) = \{S_2,S_3\}$ and $F(G) = \{S_2,S_3,S_5\}$, then $\Lambda^M = (U,F(T))$ be a Maritz space. The class $F(T) = \{\{S_2\},\{S_1,S_4\},\{S_3\},\{S_5\}\}$. Let $A = \{S_3,S_4\}$, the set of students pass the exam , then $FF^+_{F(T)}(A) = \{S_3\}$, $FF^-_{F(T)}(A) = \{S_1,S_3,S_4\}$ and $B_{F(T)}(A)\} = \{S_1,S_4\}$, therefore
$\tau_{F(T)}(A) = \{U,\emptyset,\{S_3\},\{S_1,S_3,S_4\},\{S_1,S_4\}\}$ is a topology on U , called as the rough topology via multifunction with respect to A, and $\beta_{F(T)}(A) = \{U,\{S_3\},\{S_1,S_4\}\}$ is a basis for the rough topology via multifunction $\tau_{F(T)}(A)$ with respect to A.

If we remove the attribute English from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-E) = \{\{S_1,S_4\},\{S_2,S_3\},\{S_5\}\}$. The corresponding lower and upper approximations are given by: $FF^+_{F(T-E)}(A) = \emptyset$, $FF^-_{F(T-E)}(A) = \{S_1,S_2,S_3,S_4\}$ and $B_{F(T-E)}(A)\} = \{S_1,S_2,S_3,S_4\}$, therefore
$\beta_{F(T-E)}(A) = \{U,\emptyset,\{S_1,S_2,S_3,S_4\}\} \neq \beta_{F(T)}(A)$.

If we remove the attribute French from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-F) = \{\{S_3\},\{S_1,S_4\},\{S_2,S_5\}\}$. The corresponding lower and upper approximations are given by: $FF^+_{F(T-F)}(A) = \{S_3\}, FF^-_{F(T-F)}(A) = \{S_1,S_3,S_4\}$ and $B_{F(T-F)}(A)\} = \{S_1,S_4\}$,
therefore $\beta_{F(T-F)}(A) = \{U,\{S_3\},\{S_1,S_4\}\} = \beta_{F(T)}(A)$.

If we remove the attribute German from the set of condition attributes, the family of classes corresponding to the resulting set of attributes is given by:$F(T-G) = \{\{S_3\},\{S_1,S_4\},\{S_5\},\{S_2\}\}$.The corresponding lower and upper approximations are given by: $FF^+_{F(T-G)}(A) = \{S_3\}, FF^-_{F(T-G)}(A) = \{S_1,S_3,S_4\}$ and $B_{F(T-G)}(A)\} = \{S_1,S_4\}$,
therefore $\beta_{F(T-G)}(A) = \{U,\{S_3\},\{S_1,S_4\}\} = \beta_{F(T)}(A)$.
Therefore, CORE $(F(T)) = E$.

Observation: From the above example we conclude that

”English” is the key attributes necessary to decide whether a student has pass the exam or not.

4 Rough Generalized closed sets in multi rough topological space

Levine [4] introduced the class of g- closed sets , a super class of closed sets .This concept was introduced as a generalization of closed sets in topological spaces through which new results in general topology were introduced.The basic objective of this article is to introduce and investigate some properties of rough generalized closed sets in rough topological spaces via multifunction (multi rough topological space).

Definition 4.1 Let U be the universe and $F : T \to U$ be a multifunction on U, $A \subseteq U$, $\tau_{F(T)}(A)$ be the rough topology via multifunction on U. A subset X of $\tau_{F(T)}(A)$ is called rough generalized closed set (briefly Rg- closed) if $RCl(X) \subseteq V$ where $X \subseteq V$ and V is rough open.

Example 4.1: Let $T = \{1,2,3\}$,$U = \{a,b,c,d\}$ and $F : T \to U$ defined by:
$F(1) = \{d\}, F(2) = \{a,b\}$ and $F(3) = \{c\}$, then $\Lambda^M = (U,F(T))$ be a Maritz space. If $A = \{b,c,d\}$, then $FF^{+}_{F(T)}(A) = \{c,d\}$, $FF^{-}_{F(T)}(A) = U$ and $B_{F(T)}(A)\} = \{a,b\}$, therefore
$\tau_{F(T)}(A) = \{U,\emptyset,\{c,d\},\{a,b\}\}$ is the rough topology via multifunction with respect to A. Let $V = \{a,b\}$ and $X = \{a\}$,then $Rcl(X) = \{a,b\} \subseteq V$,then X is Rg- closed set in $(U,\tau_{F(T)}(A))$.

Theorem 4.1 If X is Rg-closed and $X \subseteq Y \subseteq RCl(X)$, then Y is Rg-closed.

Proof.Let V is rough open in $\tau_{F(T)}(A), Y \subseteq V$ and $X \subseteq Y \Rightarrow X \subseteq V$. since X is Rg-closed then, $RCl(X) \subseteq V$,also $Y \subseteq RCL(X) \Rightarrow RCl(Y) \subseteq RCl(X)$.Thus $RCl(Y) \subseteq V$ and so Y is Rg-closed.

Example 4.2: Suppose $Y = \{a,b\}$ in Example (4.1), we have $X \subseteq Y \subseteq RCl(X)$,and $Rcl(Y) = \{a,b\} \subseteq V$,therefor Y is Rg- closed set in $(U,\tau_{F(T)}(A))$.

Theorem 4.2 Every rough closed set is rough generalized closed set.

Proof. Let X be rough closed set,$X \subseteq V$ and V be any rough open set, then $RCl(X) \subseteq X \subseteq V$,therefor X is rough generalized closed set.

Theorem 4.3

(i)The union of two Rg-closed sets is again Rg-closed set.
(ii)The Intersection of two Rg-closed sets is again Rg-closed set.

Proof.

(i)Suppose X and Y are Rg- closed sets,we have $RCl(X) \subseteq V$ and $Rcl(Y) \subseteq V$ where $X,Y \subseteq V$ and V is rough open set.Since $X,Y \subset V \Rightarrow (X \cup Y) \subset V$,therefor $RCl(X \cup Y) = RCl(X) \cup RCl(Y) \subseteq V \Rightarrow (X \cup Y)$ is Rg-closed.
(ii)similar to (i).

The following example illustrates the above theorem.

Example 4.3: Suppose $V = \{a,b\}$ and $X = \{a\}, Y = \{a,b\}$ are Rg-closed sets in Example (4.2), we have $RCl(X \cup Y) = \{a,b\} \subseteq V$.
In the same manner $RCl(X \cap Y) = \{a,b\} \subseteq V$.

5 Conclusion

The rough set model is based on the original data only and does not need any external information, unlike probability in statistics or grade of membership in the fuzzy set theory. It is also a tool suitable for analyzing not only quantitative at-tributes but also qualitative ones. The results of the rough set model are easy to understand, while the results from other methods need an interpretation of the technical parameters. Thus it is advantageous to use rough topology via multifunction in real life situations.In this paper, we have introduced a new topology called rough topology in terms of lower and upper approximations of a rough set via multifunction.In addition we study and investigate some properties of rough generalized closed sets in rough topological spaces via multifunction (multi rough topological space).

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