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Applications of New Time and Spatial Fractional Derivatives with Exponential Kernels

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Abstract: In the paper, we present some applications and features related with the new notions of fractional derivatives with a time exponential kernel and with spatial Gauss kernel for gradient and Laplacian operators. Specifically, for these new models we have proved the coherence with the thermodynamic laws. Hence, we have revised the standard linear solid of Zener within continuum mechanics and the model of Cole and Cole inside electromagnetism by these new fractional operators. Moreover, by the Gaussian fractional gradient and through numerical simulations, we have studied the bell shaped filtering effects comparing the results with exponential and Caputo kernel.

Keywords: Time fractional derivative, viscoelasticity, standard linear solid, spatial fractional derivative.

1 Introduction

Some classical constitutive equations used in the study of electromagnetism, continuum mechanics and thermodynamics are not always adequate to represent all materials occurring in these fields. Only as an example, we observe as in electromagnetism is not always convenient to assume unity parameters appearing in the constitutive equations mimicking the famous Cole and Cole (see [1], [2]) formula, or the standard linear solid model introduced by Zener [3] for anelastic solids.

More broadly, the evolution of the theoretical works occurred in recent time suggests that for obtaining convenient representation of the phenomena, it is reasonable to consider new or more general constitutive equations.

For these reasons, in this paper we use the definitions of fractional derivative studied in [4]. Especially, the representation of time fractional derivative by an exponential kernel and for non-local spatial dependence the new notion of fractional gradient and Laplacian by a Gaussian kernel function.

The inside of these new models, we have defined a new notion of fractional integral, which allowed us to obtain directly the classical Cattaneo-Maxwell equation.

Moreover, it is worth to note as these new derivatives with the exponential kernels (see [4]) maintain the pseudo-plastic feature of Caputo's derivative. Indeed, the response to a field that tends to a constant value, goes to zero. Besides, in the case of fatigue behavior, the fractional formalism can be used to change the order of the time derivative to take into account changes of the structural properties during the fatigue phenomena (see [5]).

Concerning constitutive equations in general, we note also the need that they are mathematically and physically rigorous, satisfying the thermodynamic restrictions (see [5] and [6]). Specifically, we have studied the consistency of the new constitutive equations with the dissipation law.

Finally, in the last part of the work, by numerical simulations, we have considered the bell shaped filtering effects comparing the results with exponential and Caputo kernel.

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2 Fractional Time Derivative with an Exponential Kernel

2.1 Definition and Properties

In the paper [4] a new notion of fractional derivative was proposed by means the following operator

$$D^{(\alpha)}f(t)=\frac{1}{(1-\alpha)}\int_a^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\frac{d}{d\tau}f(\tau)\,d\tau, \tag{1}$$

where $\alpha\in[0,1]$is the order of the derivative. While the function $f(t)$ is a subset of $C(a,\infty)$, such that $a\in[-\infty,0]$ and $f(a)=0$. We can prove that, when $\alpha=1$ we obtain the classical first derivative, whereas if $\alpha=0$, we have from (1) the function $f(t)$. Under these hypothesis, we can proceed through an integration by parts of the integral (1). So we obtain

$$D^{(\alpha)}f(t)=\frac{1}{(1-\alpha)}(f(t)-\frac{\alpha}{1-\alpha}\int_a^t f(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau\,,\,t>a \tag{2}$$

or if $\alpha>0$, in the equivalent form

$$D^{(\alpha)}f(t)=\frac{\alpha}{(1-\alpha)^2}\int_{-\infty}^t (f(t)-f_a(\tau))\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau\,,\,t>a, \tag{3}$$

where f_a denotes the extension of the function $f(t)$ by

$$f_a(t)=f(t),\quad t\geq a,$$
$$f_a(t)=0,\quad -\infty<t<a.$$

*Remark.*From definition (1), we can say that the operator k of the convolution (1) is given by

$$k=\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\frac{d}{d\tau}.$$

Moreover, by the use of representation (2) of (1), we obtain

$$D^{(\alpha)}f(t)=\frac{1}{(1-\alpha)}\frac{d}{dt}\int_a^t \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]f(\tau)\,d\tau.$$

The definition (1) and the consequence representations (2) and (3) allow us to identify the domain on which the operator (1) is well defined by the set

$$\mathscr{W}^{\alpha,1}(a,\infty)=\left\{f(t)\in L^1(a,\infty);(f(t)-f_a(\tau))\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\in L^1(a,t)\times L^1(a,\infty)\right\},$$

whose norm is given for $\alpha\neq 1$ by

$$\|f(t)\|_{\mathscr{W}^{\alpha,1}(a,\infty)}=\int_a^\infty |f(t)|\,dt+\frac{\alpha}{(1-\alpha)}\int_a^\infty\int_{-\infty}^t |f_a(\tau)|\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau dt.$$

2.2 Some Applications

A natural application of the definition (1) is given in the study of visco-plastic materials related to a body $\mathscr{B}$. So that, the stress $\sigma(x,t)$, by the use of the representation (1) of this new fractional derivative, is denoted by

$$\begin{aligned}\sigma(x,t)&=\frac{1}{1-\alpha}\mathbf{A}(x)\int_{-\infty}^t \dot{\varepsilon}_a(x,\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\\&=\frac{1}{1-\alpha}\mathbf{A}(x)\int_{-\infty}^t(\varepsilon(x,t)-\varepsilon_a(x,\tau))\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau\\&=-\frac{1}{1-\alpha}\mathbf{A}(x)\int_0^\infty(\varepsilon(x,t)-\varepsilon_a(x,t-s))\exp\left[-\frac{\alpha s}{1-\alpha}\right]ds,\end{aligned} \tag{4}$$

where the coefficient $\mathbf{A}(x)$ is a positive define fourth order tensor. The state for this problem in a point $x \in \mathscr{B}$ is given by the pair $(\varepsilon(x,t), \varepsilon^t(x,s))$, where $\varepsilon_a^t(x,s) = \varepsilon_a(x,t-s)$ (with $s \in (a,t]$) is the history of the strain tensor.

Now, we can test if the equation (4) is compatible with the dissipation principle, which we state through the free energy $\psi(\varepsilon(x,t), \varepsilon^t(x,s))$

$$\psi(\varepsilon(x,t),\varepsilon^t(x,s)) = \frac{1}{2\rho(1-\alpha)}\int_0^\infty \mathbf{A}(\varepsilon(x,t)-\varepsilon_a(x,t-s))\cdot(\varepsilon(x,t)-\varepsilon_a(x,t-s))\exp\left[-\frac{\alpha s}{1-\alpha}\right]ds$$

by the inequality

$$\rho\dot{\psi}(\varepsilon^t(x,s)) \le \sigma(x,t)\cdot\dot{\varepsilon}(x,t), \tag{5}$$

so that

$$\begin{aligned}\sigma(t)\cdot\dot{\varepsilon}(t) =& \frac{1}{1-\alpha}\int_0^\infty \mathbf{A}(\varepsilon(t)-\varepsilon_a(t-s))\exp\left[-\frac{\alpha s}{1-\alpha}\right]\cdot\frac{d}{dt}(\varepsilon(t)-\varepsilon +_a(t-s))ds\\ &+\int_0^\infty \mathbf{A}(\varepsilon(t)-\varepsilon_a(t-s))\exp\left[-\frac{\alpha s}{1-\alpha}\right]\cdot\frac{d}{ds}(\varepsilon(t)-\varepsilon_a(t-s)))ds\\ =&\rho\dot{\psi}(\varepsilon(t),\varepsilon^t(s))\\ &+\frac{1}{2(1-\alpha)}\int_0^\infty\frac{d}{ds}[\mathbf{A}(\varepsilon(t)-\varepsilon_a(t-s))\cdot(\varepsilon(t)-\varepsilon_a(t-s))]\exp\left[-\frac{\alpha s}{1-\alpha}\right]ds.\end{aligned} \tag{6}$$

Then, we have

$$\begin{aligned}\rho\dot{\psi}(\varepsilon(t),\varepsilon^t(s)) =& \sigma(t)\cdot\dot{\varepsilon}(t)\\ &-\frac{1}{2(1-\alpha)}\int_0^\infty\frac{d}{ds}[\mathbf{A}(\varepsilon(t)-\varepsilon_a(t-s))\cdot(\varepsilon(t)-\varepsilon_a(t-s))]\exp\left[-\frac{\alpha s}{1-\alpha}\right]ds\\ =&\sigma(t)\cdot\dot{\varepsilon}(t)-D(t),\end{aligned} \tag{7}$$

where the dissipation $D(t) \ge 0$ is defined by

$$D(t) = \frac{\alpha}{2(1-\alpha)^2}\int_0^\infty \mathbf{A}(\varepsilon(t)-\varepsilon_a(t-s))\cdot(\varepsilon(t)-\varepsilon_a(t-s))\exp\left[-\frac{\alpha s}{1-\alpha}\right]ds.$$

Then, from (7) we obtain the inequality (5).

3 Constitutive Equations by the New Fractional Derivative

In order to solve some specific initial condition problems concerning the heat diffusion, and diffusion in general, we then suggest the following new fractional constitutive equation using the new fractional derivatives with different exponential kernels

$$au(t)+\frac{b}{(1-\alpha)}\int_0^t u'(\tau)\exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau = cw(t)+\frac{d}{(1-\gamma)}\int_0^t w'(\tau)\exp\left[-\frac{\gamma(t-\tau)}{1-\gamma}\right]d\tau, \tag{8}$$

or in a formal sense

$$(a+bD^{(\alpha)})u(t) = (c+dD^{(\gamma)})w(t), \tag{9}$$

where a,b,c,d are parameters, D^α and D^γ are fractional derivatives with the exponential kernel. This equation is a natural generalization mimicking the time domain of the constitutive equation of Zener [3], for the standard linear solid, and the frequency domain of Cole and Cole formula [1] for dielectrics.

Depending on the values and dimensions of the parameters the functions $w(t)$ and $u(t)$ could alternatively be the heat flux and the temperature gradient in the case of heat transmission or the flux and the concentration in the case of a diffusion of substance in a porous medium, or the induction and the applied field in the case of electromagnetic phenomena.

In order to solve equation (9) we take its Laplace Transform (LT), with p variable, indicating with U and W the LT of u and w, respectively and assume zero initial conditions $u(0)=0$ and $w(0)=0$. Then, from (9) we have

$$U(a+\frac{\frac{b}{1-\alpha}p}{p+\frac{\alpha}{1-\alpha}})=W(c+\frac{\frac{d}{1-\gamma}p}{p+\frac{\gamma}{1-\gamma}}), \tag{10}$$

or equivalently

$$W=U\frac{(a+B)(p+aV)}{(c+D)(p+cZ)}\frac{p+Z}{p+V}. \tag{11}$$

where

$$V=\frac{\alpha}{1-\alpha}, \quad Z=\frac{\gamma}{1-\gamma}, \quad B=\frac{b}{1-\alpha}, \quad D=\frac{d}{1-\gamma}.$$

The form (10) and (11) for the system functions is the most simple for the computation of the time domain expressions as we will see later. However, if we put

$$\begin{aligned} &L=a+B, \quad H=c+D, \quad M=a\frac{V}{L}, \\ &N=c\frac{Z}{H}=c\frac{\gamma}{1-\gamma}(c+\frac{d}{1-\gamma})=c\gamma\frac{c(1-\gamma)+d}{(1-\gamma)^2}, \end{aligned} \tag{12}$$

then, for the computation of the wave velocity, the equation (11) may be more simply written

$$\begin{aligned} W&=U\frac{L}{H}\frac{p+M}{p+N}\frac{p+Z}{p+V}, \\ U&=W\frac{H}{L}\frac{p+N}{p+M}\frac{p+V}{p+Z}. \end{aligned} \tag{13}$$

The system functions in the LT domain are then

$$\frac{1}{R}=\frac{H}{L}\frac{p+c\frac{Z}{H}}{p+a\frac{V}{L}}\frac{p+V}{p+Z}. \tag{14}$$

In the Fourier domain, the system function have the following expression to be used for the computation of the wave velocity.

$$\begin{aligned} R=&\frac{L}{H}\{((f^2+MN)(f^2+ZV)-(N-M)(V-Z))+ \\ &+if((N-M)(f^2+ZV)+(V-Z)(f^2+MN))\}\frac{1}{(M^2+f^2)(f^2+V^2)}. \end{aligned} \tag{15}$$

We set now

$$\begin{aligned} &R(\omega)=\alpha(\omega)+i\beta(\omega)=Re(R(\omega)=Im(R(\omega)), \\ &\alpha(\omega)=\frac{L}{H}\frac{(\omega^2+MN)(\omega^2+ZV)-(N-M)(V-Z)}{(M^2+\omega^2)(\omega^2+V^2)}, \\ &\beta(\omega)=\omega\frac{L}{H}\frac{(N-M)(\omega^2+ZV)+(V-Z)(\omega^2+MN)}{(M^2+\omega^2)(\omega^2+V^2)} \end{aligned} \tag{16}$$

which provides the function

$$\frac{1}{R(\omega)}=\frac{\alpha(\omega)-i\beta(\omega}{\alpha(\omega)^2+\beta(\omega)^2}. \tag{17}$$

4 Case When $w(x,t)$ Is Flux and $u(x,t)$ Is Gradient

An interesting case occurs when $w(x,t)$ is the flux of a fluid and $u(x,t)$ is the gradient of concentration of a solute $s(x,t)$ carried by the fluid. In this case, we may associate to the system function in the LT domain the continuity equation of the solute obtaining

$$\begin{aligned} &W_x + pS = 0, \\ &W = \frac{L}{H} S_x (1 + \frac{M-N}{p+N})(1 + \frac{Z-V}{p+V}), \end{aligned} \tag{18}$$

where $U = gradS = S_x$.

So differentiating with respect to x we find

$$\begin{aligned} &W_x + pS = 0, \\ &W_x = S_{xx} (1 + \frac{M-N}{p+N})(1 + \frac{Z-V}{p+V}) \frac{L}{H}, \end{aligned}$$

and finally obtain

$$\begin{aligned} &R(p) = \frac{L}{H}(1 + \frac{M-N}{p+N})(1 + \frac{Z-V}{p+V}) = \frac{L}{H} \frac{(p+N)(p+V)}{(p+M)(p+Z)}, \\ &\frac{R(p)}{p} S_{xx} = S, \\ &S(x,p) = J(p) \exp\left[-x(\frac{R(p)}{p})^{0.5}\right]. \end{aligned} \tag{19}$$

The solution $s(x,t)$ is found with the LT^{-1} integrating along the Bromwich path

$$s(x,t) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} J(p) \exp\left[pt - px(\frac{d}{R(p)})^{0.5}\right] dp. \tag{20}$$

The fitting to the data depends on the choice of the parameters $a, b, c, d, \alpha, \gamma$ defining $R(p)$ which imply the limits to the wave velocity. For instance setting in equations (10) and (11): $d = a = 0$ one obtains the classic case of diffusion.

5 The Case of the Equations of Elasticity

In the case when W is LT stress and U is deformation, we associate the system equations to equilibrium condition Eq. in the LT domain

$$W_x(x,p) = \rho p^2 S, \tag{21}$$

$$W_x = U_x \frac{L}{H} \frac{(p+N)(p+V)}{(p+M)(p+Z)}, \tag{22}$$

where ρ is the density of the medium. Hence, we obtain

$$S_{xx} = \rho p^2 \frac{(p+N)(p+V)}{(p+M)(p+Z)} \frac{H}{L} S, \tag{23}$$

$$S_{xx} = (p^2 \frac{\rho}{R(p)}) S,$$

whose solution is

$$S(x,p) = J(p) \exp\left[-x(p(\frac{\rho}{R(p)})^{0.5})\right].$$

The solution $s(x,t)$ is obtained with the LT^{-1} integrating along the Bromwich path , the straight line parallel to the imaginary axis on the real portion of the complex plane

$$s(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} J(p)\exp\left[pt - x(p(\frac{\rho}{R(p)})^{0.5}\right]dp. \tag{24}$$

It all depends on the choice of the parameters a,b,c,d,α,γ defining $R(p)$ which imply the limits to the wave velocity.

For instance the case $d = 0$ which would mimic the Kelvin-Voigt classic case gives the following time domain form of the system function, that is the response to a unit delta function of strain

$$h(t) = \frac{L}{c}\{1 + (M - V)\exp(-Vt)\},$$

or more explicitly

$$w(t) = \frac{a + \frac{b}{1-a}}{c}\{1 + (M - V))\exp[-Vt]\}, \tag{25}$$

$$h(t) = \frac{a + \frac{b}{1-a}}{c}\{1 + \frac{a\alpha(a(1-\alpha)+b)}{1-\alpha}\exp\left[-\frac{\alpha}{1-\alpha}t\right]\}, \tag{26}$$

which implies a simple relaxation to the initial form governed by an exponential at a different rate than in as in the classic Kelvin Voigt model with the first order derivative or as in the case when the fractional derivative is of the Caputo type.

6 Equivalent Representations of Constitutive Equations with Different Fractional Time Derivatives

Now we consider a particular case of the equation (8) with $\alpha = \gamma$, then

$$au^{(\alpha)}(t) + bu(t) = cw^{(\alpha)}(t) + dw(t), \tag{27}$$

where we suppose, $u(0) = w(0) = 0$. So, from definition of fractional derivative (1), we obtain

$$u^{(\alpha)}(t) = \frac{1}{1-\alpha}(u(t) - \frac{\alpha}{1-\alpha}\int_0^t u(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau) \tag{28}$$

by a time derivation of (28) we have

$$\begin{aligned}\frac{d}{dt}u^{(\alpha)}(t) =& \frac{1}{1-\alpha}(\dot{u}(t) - \frac{\alpha}{1-\alpha}(u(t) - \frac{\alpha}{1-\alpha}\int_0^t u(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]d\tau) \\ =& \frac{1}{1-\alpha}(\dot{u}(t) - \alpha u^{(\alpha)}(t)),\end{aligned} \tag{29}$$

a similar expression also applies to $w^{(\alpha)}$. Hence, if we derive the equation (27) with respect to time t, we have by (29)

$$\frac{a}{1-\alpha}(\dot{u}(t) - \alpha u^{(\alpha)}(t)) + b\dot{u}(t) = \frac{c}{1-\alpha}(\dot{w}(t) - \alpha w^{(\alpha)}(t)) + d\dot{w}(t)$$

then, using again the equation (27) we obtain the equivalence of the equation (27) with the following

$$(\frac{a}{1-\alpha} + b)\dot{u}(t) + \frac{\alpha}{1-\alpha}bu(t) = (\frac{c}{1-\alpha} + d)\dot{w} + \frac{\alpha}{1-\alpha}dw(t). \tag{30}$$

Finally, by a suitable choice of the coefficients a,b,c,d, we can obtain other particular cases of equivalence between different fractional derivatives.

7 Fractional Integral

It is worth to associate to the fractional derivative (1) the following fractional integral

$$ {}_0\mathscr{I}^{\alpha} f(t) = \frac{1}{\alpha}\int_0^t f(\tau)\exp\left[-\frac{1-\alpha}{\alpha}(t-\tau)\right]d\tau, \quad \alpha\in[0,1] \tag{31} $$

for $\alpha = 0$, we obtain the function $f(t)$. Otherwise, when $\alpha = 1$, we have

$$ {}_0\mathscr{I}^{1} f(t) = \int_0^t f(\tau)d\tau $$

while

$$ \frac{d}{dt}{}_0\mathscr{I}^{\alpha} f(t) = \frac{1}{\alpha}f(t) - \frac{1-\alpha}{\alpha^2}\int_0^t f(\tau)\exp\left[-\frac{1-\alpha}{\alpha}(t-\tau)\right]d\tau = \frac{1}{\alpha}f(t) - \frac{1-\alpha}{\alpha}{}_0\mathscr{I}^{\alpha} f(t). $$

Now, if we suggest for the heat flux $\mathbf{q}$ the following constitutive equation

$$ \mathbf{q}(t) = -k_0\, {}_0I^{\alpha}\nabla\theta(t) = -\frac{k_0}{\alpha}\int_0^t \nabla\theta(\tau)\exp(-\frac{1-\alpha}{\alpha}(t-\tau))d\tau \tag{32} $$

we have

$$ \dot{\mathbf{q}}(t) = -\frac{k_0}{\alpha}\nabla\theta(t) + k_0\frac{1-\alpha}{\alpha^2}\int_0^t \nabla\theta(\tau)\exp(-\frac{1-\alpha}{\alpha}(t-\tau))d\tau $$

from which

$$ \dot{\mathbf{q}}(t) = -\frac{k_0}{\alpha}\nabla\theta(t) - \frac{(1-\alpha)}{\alpha}\mathbf{q}(t), \tag{33} $$

that coincides with the Cattaneo-Maxwell equation

$$ \frac{\alpha}{(1-\alpha)}\dot{\mathbf{q}}(t) = -\mathbf{q}(t) - \frac{k_0}{(1-\alpha)}\nabla\theta(t). \tag{34} $$

Of course, when $\alpha = 0$ we obtain the Fourier law,

$$ \mathbf{q}(t) = -k_0\nabla\theta(t). \tag{35} $$

8 Fractional Time Derivative by a Gaussian Kernel

In this section, by the use of Gaussian kernels, we introduce a new of fractional time derivative. In the next section, by the definition given in [4], we study some features and behaviors of the gradient and Laplacian Gaussian fractional operators.

The definitions (9) and (31) of fractional derivative and integral show interesting connections with the temporal derivatives, as point out in the Sections 2 and 3. However, accordingly to these remarks and similarly to that proposed and defined in [4] for the gradient and the Laplacian fractional, the introduction of a new temporal fractional derivative by an error function kernel appears of interest.

Then, we consider a smooth function $f(t):[a,T]$ such that $a\in[-\infty,T)$ and $f(a)=0$. If α is a fractional coefficient such that $0\le\alpha\le 1$, we define the new time fractional derivative by

$$ \frac{D^{\alpha}}{Dt^{\alpha}}\ f(t) = \frac{1+\alpha^2}{\sqrt{\pi^{\alpha}(1-\alpha)}}\int_a^t \dot{f}(\tau)\exp\left[(-\frac{(t-\tau)^2}{(1-\alpha)}\alpha\right]d\tau. $$

It is well known that

$$ \lim_{\alpha\to 1}\frac{2}{\sqrt{\pi^{\alpha}(1-\alpha)}}\exp\left[-\frac{(t-\tau)^2}{(1-\alpha)}\alpha\right] = \delta(t-\tau) $$

so that we have

$$ \frac{D^{\alpha=1}}{Dt^{\alpha=1}}\ f(t) = \lim_{\alpha\to 1}\frac{1+\alpha^2}{\sqrt{\pi^{\alpha}(1-\alpha)}}\int_a^t \dot{f}(\tau)\exp\left[-\left(\frac{(t-\tau)^2}{(1-\alpha)}\alpha\right)\right]\ d\tau = \dot{f}(t). $$

While, for $\alpha = 0$ we obtain

$$\frac{D^{\alpha=0}}{Dt^{\alpha=0}} f(t) = \int_a^t \dot{f}(\tau)\, d\tau = f(t).$$

If we denote with $f_a(t)$ the extension of $f(t)$ by

$$f_a(t) = \begin{cases} f(t) & \text{for } t \geq a, \\ 0 & \text{for } -\infty < t < a, \end{cases}$$

then for $t \geq a$ we have

$$\frac{D^\alpha}{Dt^\alpha} f(t) = \frac{2\alpha(1+\alpha^2)}{\sqrt{\pi^\alpha(1-\alpha)^3}} \int_{-\infty}^t (f_a(\tau) - f(t))(\tau - t) \exp\left[-\frac{(t-\tau)^2}{(1-\alpha)}\alpha\right] d\tau.$$

A natural application of it we have in the study of viscoelastic materials related to a body $\mathscr{B}$. So that, the stress $\sigma(x,t)$ becomes

$$\sigma(x,t) = \frac{2(1+\alpha^2)}{\sqrt{\pi^\alpha(1-\alpha)^3}} \mathbf{A}(\alpha,x) \int_0^\infty (\varepsilon(x,t) - \varepsilon_a(x,t-s)) \frac{d}{ds}\varphi(\alpha,s) ds,$$

where the coefficient $\mathbf{A}(\alpha,x)$ is a positive define fourth order tensor. The state for this problem in a point $x \in \mathscr{B}$ is given by the pair $(\varepsilon(x,t), \varepsilon^t(x,s))$, where the history of the strain tensor $\varepsilon^t(x,s) := \varepsilon(x,t-s)$, with $s \in (a,t]$.

For this system the free energy $\psi(\varepsilon^t(s))$ is defined by

$$\psi(\varepsilon(x,t), \varepsilon^t(s)) = \frac{\alpha(1+\alpha^2)}{\sqrt{\pi^\alpha(1-\alpha)^3}} \int_0^\infty (\mathbf{A}(\varepsilon(x,t) - \varepsilon_a(x,t-s)) \cdot (\varepsilon(x,t) - \varepsilon_a(x,t-s)) s \exp\left[\frac{-s^2\alpha}{(1-\alpha)}\right] ds.$$

9 Fractional Derivative by a Spatial Gaussian Kernel

In order to study an equation of the type (9), when the fractional derivative is given by a Gaussian kernel, we take its Fourier Transform (FT) with variable ω, indicating here with U and W the FT of u and w respectively.

Let us first remember the definition of fractional gradient and of its FT proposed in [4]

$$\begin{aligned} \nabla^\alpha u(x) &= \frac{\alpha}{1-\alpha} \pi^{\frac{\alpha}{2}} \int_\Omega \nabla u(y) \exp\left[-\frac{\alpha^2(x-y)^2}{1-\alpha^2}\right] dy, \\ FT(\nabla^\alpha u(x))(\xi) &= \pi^{\frac{1-\alpha}{2}} FT(\nabla u)(\xi) \exp\left[-\frac{\pi^2(1-\alpha)^2}{\alpha^2}\xi^2)\right], \end{aligned} \tag{36}$$

then applying these definitions to equation of type (9) we find

$$(a + b\pi^{\frac{1-\alpha}{2}}) \exp(-\pi\omega^2(\frac{1-\alpha}{\alpha})^2) U = (c + d\pi^{\frac{1-\gamma}{2}}) \exp(-\pi\omega^2(\frac{1-\gamma}{\gamma})^2) W, \tag{37}$$

or

$$UH = W, \tag{38}$$

with

$$H = \frac{(a + b\pi^{\frac{1-\alpha}{2}}) \exp(-\pi\omega^2(\frac{1-\alpha}{\alpha})^2)}{(c + d\pi^{\frac{1-\gamma}{2}}) \exp(-\pi\omega^2(\frac{1-\gamma}{\gamma})^2)}.$$

The system function $H(f)$ is an interesting real function, which seems to imply no phase change, but amplitude change and frequency dependent wave velocity. Its time domain expression is obtained by integrating on the Bromwich path.

Concerning the possibility of the various models obtained eliminating one of the parameters a, b, c, d, we note that:

$a = 0$ →presence of both Gaussian exponential, $-\pi\omega^2(\frac{1-\alpha}{\alpha})^2$, $-\pi\omega^2(\frac{1-\gamma}{\gamma})^2$.

$b = 0$ → presence of only the exponential with exponent, $-\pi\omega^2(\frac{1-\gamma}{\gamma})^2$.

$c = 0$ → presence of both Gaussian exponents, $-\pi\omega^2(\frac{1-\alpha}{\alpha})^2$, $-\pi\omega^2(\frac{1-\gamma}{\gamma})^2$.

$d = 0 \rightarrow$ presence of only the Gaussian exponential, $-\pi\omega^2(\frac{1-\alpha}{\alpha})^2$.

Concerning the models with $a = 0$ or $c = 0$ we note that because of the presence of two different exponential they could seem more capable to fit a larger variety of phenomena. However again we note that each phenomenon is probably associate to a particular constitutive equation and the flexibility of the constitutive equation models considered here is not a real issue; the first issue is the search for the most appropriate constitutive equation.

We note the interesting case when $c = 0$ which mimics the Maxwell model and has an increasing and eventually asymptotically diverging system function. Is of interest also the case when d = 0 which mimics the Voigt model when the system function rapidly converges to the constant value c/a, models with $a = 0$ or $c = 0$:

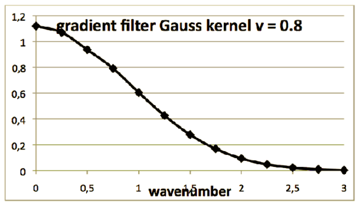

Fig. 1: Bell shaped filtering effect of fractional derivative with Gaussian kernel.

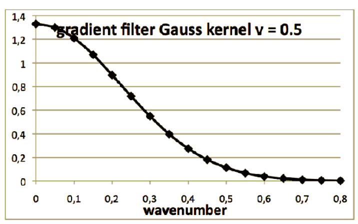

Fig. 2: Bell shaped filtering effect of fractional derivative with Gaussian kernel.

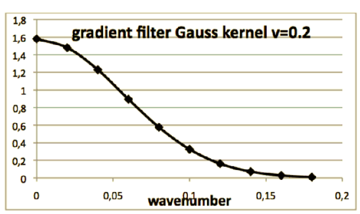

Fig. 3: Bell shaped altering effect of fractional derivative with Gaussian kernel.

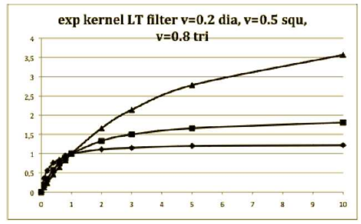

Fig. 4: High pass converging filtering effect of fractional derivative with exponential kernel.

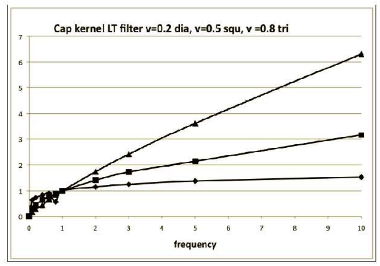

Fig. 5: High pass diverging filtering effect of fractional derivative with Caputo kernel.

10 Appendix. The Time Domain Expression of the System Functions

It is of interest, to the practitioner of application of fractional calculus, the time domain expression of the system functions associated to the constitutive equation defined in Section 3, by means of formulae (8) and (9). Namely, we consider equations (15), (16) and (17) with $f = i\omega$, that the system functions,

$$W = U\frac{L}{H}\frac{p+M}{p}\frac{p+Z}{p+V},$$
$$U = W\frac{H}{L}\frac{p+N}{p+M}\frac{p+V}{p+Z} \tag{39}$$

and obtain their time domain expression. For instance concerning $w(t)$ we find

$$w(t) = \frac{L}{H}(\delta(t) + (M-N)exp(-Nt)) * (\delta(t) + (Z-V)exp((V)t)), \tag{40}$$

where $*$ denotes the convolution. Finally we have

$$\begin{aligned} w(t) =& \frac{L}{H}\{1 + (M-N)exp(-Nt) + (Z-V)exp(-Vt) \\ &- \frac{(M-N)(Z-V)}{N-V}(exp(-Nt) - (exp(-Vt))\}. \end{aligned} \tag{41}$$

We note that the solution $w(t)$ in the general case is formed by the function and its correlation with a linear combination of two different exponents and that in order to be physically admissible it implies that $N \geq 0, V \geq 0$.

Simplifying further equation (41) we find

$$\begin{aligned} w(t) =& \frac{L}{H}\{1 + \frac{(M-N)(N-Z)}{(N-V)}exp(-Nt) \\ &+ \frac{(Z-V)(M-V)}{N-V}exp(-Vt)\}. \end{aligned} \tag{42}$$

Concerning the presence of the two exponentials appearing in equation (40) we note that in three special cases only the exponent $-V$ is present as shown in the following table

$$\begin{aligned}
&d = 0 \to D = \gamma = Z = N = 0, H = c; N = 0,\\
&\quad \text{only the exponential with exponent } -Vt.\\
&c = 0 \to H = D, N = 0; N = 0,\\
&\quad \text{only the exponential with exponent } -Vt.\\
&a = 0 \to M = 0, L = B,\\
&\quad \text{the exponential with exponent} -Vt \text{ and } -Nt.\\
&b = 0 \to B = 0, L = a, M = V; M = V,\\
&\quad \text{only the exponential with exponent } -Vt.
\end{aligned} \tag{43}$$

It is seen in the equations (43) that, with the exception of the case when $a = 0$, the time domain expression of the system function contains only the exponential with the negative exponent $-Vt = -\alpha/(1-\alpha)t$.

Note that the exponent $-Nt$ implies only the presence of the parameters of the operator applied to $w(t)$ and the exponent $-Vt$ implies only the parameters of the operator applied to $u(t)$; the two exponential are then independent operators.

One may note that the constitutive equation model with $a = 0$, which includes the exponential with negative exponents : $-Vt = -\alpha/(1-\alpha)t$ and $-Nt = -c\gamma t(c(1-\gamma)+d)/(1-\gamma)^2$ could seem more capable to fit a larger variety of phenomena. However each phenomenon is probably associate to a particular constitutive equation and the flexibility of the constitutive equation models considered here is not a real issue; the first issue is the search for the most appropriate constitutive equation.

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