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Mathematical Analysis of the Global Properties of an SVEIR Epidemic Model

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Abstract: In this paper, an SVEIR epidemic model with waning preventive vaccine and the infection acquired following effective contact with infected population and exposed population is investigated. By analyzing the corresponding characteristic equations, the local stability of a disease-free equilibrium and an endemic equilibrium is discussed. By means of Lyapunov functional and LaSalle's invariance principle, it is shown that the global dynamics is almost determined by the basic reproduction number. It is proven that if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable. If the basic reproduction number is greater than unity, sufficient conditions are obtained for the global stability of the endemic equilibrium. Numerical simulations are carried out to illustrate the main theoretical results.

Keywords: SVEIR epidemic model, global stability, vaccination, latent period

1 Introduction

Mathematical models describing the population dynamics of infectious diseases is an invaluable epidemiological tool about foreseeing the transmission dynamics of infectious diseases. Over the past few decades, many compartmental mathematical models, such as SIS, SIR or SEIS (where S, I, E, and R denote the populations of susceptible, infectious, exposed and recovered), have been used to investigate the spread and control of infectious diseases (see, e.g. [1,2,3,4]). In [4], Gumel, McCluskey and Watmough considered the following infectious disease model:

$$\begin{cases} \dot{S}(t) = \Pi - \beta SI - \xi S - \mu S, \\ \dot{V}(t) = \xi - (1-\tau)\beta VI - \mu V, \\ \dot{E}(t) = \beta SI + (1-\tau)\beta VI - \alpha E - \mu E, \\ \dot{I}(t) = \alpha E - \delta I - dI - \mu I, \\ \dot{R}(t) = \delta I - \mu R. \end{cases} \tag{1}$$

In (1), the total population is subdivided into five compartments, which are the susceptible individuals $S(t)$, the vaccinated susceptible individuals $V(t)$, the exposed individuals but not yet infectious $E(t)$, the infectious individuals $I(t)$, and the recovered individuals with acquired full immunity $R(t)$. The parameters $\Pi, \beta, \xi, \mu, d, \alpha$, and δ are positive constants in which Π is the recruitment rate of susceptible human, β is the effective contact rate, ξ is the vaccination coverage rate, μ is the natural mortality rate, d is the disease-induced mortality rate, α is the rate at which exposed individuals become infectious, δ is the recovery rate. $0 \le \tau \le 1$ is the vaccine efficacy($\tau = 1$ represents a vaccine that offers 100% protection against infection, $\tau = 0$ models a vaccine that offers no protection at all). In [4], Gumel et al. considered the global stability of the disease free equilibrium and the endemic equilibrium of system (1) by Lyapunov function theory and the compound matrices theory.

In (1), the infection was acquired following effective contact with infected population. However, some patients with infectious diseases can discharge infectious pathogens at the end of the latent period, such as Hepatitis B, measles and Pertussis. Hence, the infection can also be acquired following effective contact with the latened population(see, e.g. [5]).

In addition, it was assumed that the vaccinees obtained the permanent immunity in system (1). However, some clinical studies have shown that the permanent immunity induced by the preventive vaccines may wane over time. Mossong et al. [6] pointed that the

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mean duration of vaccine-induced protection against Measles was 25 years. The additional cases of waning immunity in vaccines have been shown in [7,8].

Motivated by the works of Gumel et al.[4], Li et al.[5] and Mossong et al.[6], we consider the combined effects of a waning preventive vaccines and the infection acquired following effective contact with infected population and latened population. To this end, we study the following differential equations:

$$\begin{cases} \dot{S}(t) = \Pi - \beta SI - \eta\beta ES - \xi S - \mu S + \omega V, \\ \dot{V}(t) = \xi S - (1-\tau)\beta VI - \mu V - \omega V, \\ \dot{E}(t) = \beta SI + \eta\beta ES + (1-\tau)\beta VI - \alpha E - \mu E, \\ \dot{I}(t) = \alpha E - \delta I - dI - \mu I, \\ \dot{R}(t) = \delta I - \mu R, \end{cases} \quad (2)$$

where $0 < \eta \le 1$ is a constant describing the decrease in the relative infectiousness of population in the exposed individuals E in comparison to those in the infectious individuals I, the constant $\omega > 0$ is the rate at which vaccine wanes (that is $1/\omega$ is the duration of the loss of immunity acquired by preventive vaccine or by infection), and the other constants are the same as that defined in system (1).

The initial conditions for system (2) take the form

$$S(0) > 0,\ V(0) > 0,\ E(0) > 0,\ I(0) > 0,\ R(0) > 0. \quad (3)$$

By the fundamental theory of functional differential equations [9], it is well known that system (2) has a unique solution $(S(t),V(t),E(t),I(t),R(t))$ satisfying initial conditions (3). Further, it is easy to show that all solutions of system (2) are defined on $[0,+\infty)$ and remain positive for all $t \ge 0$.

Notice that although the recovered population still make contacts with other members of the population, it does not contribute to the spread of the disease. Since the recovered population $R(t)$ does not feature in the first four equations of system (2), the model composed of the first four equations of system (2) will be discussed in the following:

$$\begin{cases} \dot{S}(t) = \Pi - \beta SI - \eta\beta ES - \xi S - \mu S + \omega V, \\ \dot{V}(t) = \xi S - (1-\tau)\beta VI - \mu V - \omega V, \\ \dot{E}(t) = \beta SI + \eta\beta ES + (1-\tau)\beta VI - \alpha E - \mu E, \\ \dot{I}(t) = \alpha E - \delta I - dI - \mu I. \end{cases} \quad (4)$$

The dynamic behaviors of $R(t)$ can be obtained from the last equation of system (2).

The paper is organized as follows. In the next section, the ultimate boundedness of the solutions for system (4) is presented. In Section 3, by analyzing the corresponding characteristic equations, the local stability of a disease-free equilibrium and an endemic equilibrium of system (4) is discussed. In Section 4, by constructing a suitable Lyapunov functional, it is shown that the disease-free equilibrium is globally asymptotically stable when the basic reproduction number is less than unity. By means of Lyapunov functional and LaSalle's invariance principle, sufficient conditions are obtained for the global asymptotic stability of the endemic equilibrium. Numerical simulations are carried out in Section 5 to illustrate the main theoretical results. A brief discussion is given in Section 6 to conclude this work.

2 Basic properties

In this section, we study the ultimate boundedness of the solutions for system (4).

Theorem 2.1. The arbitrary positive solution of system (4) with initial conditions in R_4^+ is ultimately bounded.

Proof. Let $(S(t),V(t),E(t),I(t))$ be any positive solution of system (4) with initial conditions in R_4^+. Define

$$L(t) = S(t) + V(t) + E(t) + I(t).$$

Calculating the derivative of $L(t)$ along the solutions of system (4), it follows that

$$\dot{L}(t) = \Pi - \mu L - dI - \delta I \le \Pi - \mu L,$$

a standard comparison argument shows that

$$\limsup_{t\to+\infty} L(t) \le \frac{\Pi}{\mu}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_1 > 0$ such that if $t > T_1$,

$$E(t) \le \frac{\Pi}{\mu} + \varepsilon, \quad I(t) \le \frac{\Pi}{\mu} + \varepsilon. \quad (5)$$

By the first two equations of system (4),

$$\begin{cases} \dot{S}(t) \le \Pi - (\xi+\mu)S + \omega V, \\ \dot{V}(t) \le \xi S - (\mu+\omega)V. \end{cases} \quad (6)$$

Consider the following auxiliary system

$$\begin{cases} \dot{z_1}(t) = \Pi - (\xi+\mu)z_1 + \omega z_2, \\ \dot{z_2}(t) = \xi z_1 - (\mu+\omega)z_2. \end{cases} \quad (7)$$

It is easy to prove that the positive equilibrium $z^*(\Pi(\mu+\omega)/(\mu(\xi+\mu+\omega)), \xi\Pi/(\mu(\xi+\mu+\omega)))$ of system (7) is globally asymptotically stable. By comparison, it follows that

$$\limsup_{t\to+\infty} S(t) \le \frac{\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)}, \limsup_{t\to+\infty} V(t) \le \frac{\xi\Pi}{\mu(\xi+\mu+\omega)}.$$

Hence, for $\varepsilon > 0$ sufficiently small, there exists a $T_2 > T_1$ such that if $t > T_2$,

$$S(t) \le \frac{\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)} + \varepsilon, V(t) \le \frac{\xi\Pi}{\mu(\xi+\mu+\omega)} + \varepsilon. \quad (8)$$

By (5) and (8), the arbitrary positive solution $(S(t),V(t),E(t),I(t))$ of system (4) is ultimately bounded. This completes the proof.□

Denote

$$D=\left\{(S,V,E,I)\in R^4_{+0}: S+V+E+I\le\frac{\Pi}{\mu},\right.$$
$$\left. S\le\frac{\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)}, V\le\frac{\xi\Pi}{\mu(\xi+\mu+\omega)}\right\}.$$

Theorem 2.1 implies that the set D is a positively invariant and the attracting region for the disease transmission model given by system (4) with initial conditions in R^4_+.

3 Equilibria and local stability

In this section, we discuss the local stability of a disease-free equilibrium and an endemic equilibrium of system (4) by analyzing the corresponding characteristic equations, respectively.

System (4) always has a disease-free equilibrium

$$P_0=(S_0,V_0,E_0,I_0)=\left(\frac{\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)},\frac{\xi\Pi}{\mu(\xi+\mu+\omega)},0,0\right).$$

The characteristic equation of system (4) at the equilibrium P_0 is of the form

$$(\lambda+\mu)(\lambda+\xi+\mu+\omega)\left[\lambda^2+\left(\alpha+2\mu+\delta+d -\frac{\eta\beta\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)}\right)\lambda+(\alpha+\mu)(\delta+d+\mu)(1-R_0)\right]=0, \quad (9)$$

where R_0 is defined as

$$R_0=\frac{\eta\beta\Pi(\mu+\omega)(\delta+d+\mu)+\alpha\beta\Pi(\mu+\omega+(1-\tau)\xi)}{\mu(\alpha+\mu)(\delta+d+\mu)(\xi+\mu+\omega)}.$$

Clearly, Equation (9) always has two negative real roots $\lambda_1=-\mu$, $\lambda_2=-\xi-\mu-\omega$. The other roots λ_3,λ_4 of Equation (9) are determined by the following equation

$$\lambda^2+\left(\alpha+2\mu+\delta+d-\frac{\eta\beta\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)}\right)\lambda +(\alpha+\mu)(\delta+d+\mu)(1-R_0)=0. \quad (10)$$

If $R_0>1$, Equation (10) has one positive real part root. Hence, P_0 is unstable. If $R_0<1$,

$$\lambda_3\lambda_4=(\alpha+\mu)(\delta+d+\mu)(1-R_0)>0,$$
$$\begin{aligned}\lambda_3+\lambda_4&=\frac{\eta\beta\Pi(\mu+\omega)}{\mu(\xi+\mu+\omega)}-(\alpha+\mu)-(\delta+d+\mu)\\ &\le\left(1-\frac{\alpha\beta\Pi(\mu+\omega+(1-\tau)\xi)}{\mu(\alpha+\mu)(\delta+d+\mu)(\xi+\mu+\omega)}\right)(\alpha+\mu)\\ &\quad-(\alpha+\mu)-(\delta+d+\mu)\\ &=-\frac{\alpha\beta\Pi(\mu+\omega+(1-\tau)\xi)}{\mu(\delta+d+\mu)(\xi+\mu+\omega)}-(\delta+d+\mu)<0.\end{aligned}$$

Then, the characteristic roots of Equation (10) have negative real parts. Therefore, P_0 is locally asymptotically stable when $R_0<1$.

In conclusion, we have the following results. **Theorem 3.1.** The disease-free equilibrium P_0 is locally asymptotically stable if $R_0<1$ and unstable if $R_0>1$.

The threshold quantity R_0 is known as the basic reproduction number, which can also be derived by the method of next generation matrix by van den Driessche and Watmough [10].

To obtain the endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$ of system (4), we solve the following system of equations

$$\begin{cases}\Pi-\beta S^*I^*-\eta\beta E^*S^*-\xi S^*-\mu S^*+\omega V^*=0,\\ \xi S^*-(1-\tau)\beta V^*I^*-\mu V^*-\omega V^*=0,\\ \beta S^*I^*+\eta\beta E^*S^*+(1-\tau)\beta V^*I^*-\alpha E^*-\mu E^*=0,\\ \alpha E^*-\delta I^*-dI^*-\mu I^*=0.\end{cases} \quad (11)$$

From the first two equations and the last equation of system (11), we get

$$S^*=\frac{\Pi+\omega V^*}{\beta I^*+\eta\beta E^*+\xi+\mu},\quad V^*=\frac{\xi S^*}{(1-\tau)\beta I^*+\mu+\omega},$$
$$E^*=\frac{\delta+d+\mu}{\alpha}I^*. \quad (12)$$

From the third equation of system (11), we obtain

$$E^*=\frac{\beta S^*I^*+(1-\tau)\beta V^*I^*}{\alpha+\mu-\eta\beta S^*}. \quad (13)$$

Substituting the expressions of S^*,V^*,E^* in (12) into (13), which gives

$$Q(I^*)=AI^{*2}+BI^*+C=0, \quad (14)$$

where

$$A=\frac{(\alpha+\mu)(\delta+d+\mu)(1-\tau)\beta^2(\alpha+\eta(\delta+d+\mu))}{\alpha},$$
$$\begin{aligned}B=&\frac{\beta(\alpha+\mu)(\mu+\omega)(\delta+d+\mu)(\alpha+\eta(\delta+d+\mu))}{\alpha}\\ &\left[1+\frac{(1-\tau)\alpha\mu(\xi+\mu+\omega)}{(\mu+\omega)^2(\alpha+\eta(\delta+d+\mu))}\left(1-R_0\right.\right.\\ &\left.\left.+\frac{\xi\omega(\alpha+\mu)(\delta+d+\mu)+\alpha\beta\xi\Pi(1-\tau)}{\mu(\alpha+\mu)(\xi+\mu+\omega)(\delta+d+\mu)}\right)\right],\end{aligned}$$
$$C=\mu(\alpha+\mu)(\delta+d+\mu)(\xi+\mu+\omega)(1-R_0). \quad (15)$$

The endemic equilibrium of system (4) are given by (12) with the positive root I^* of Equation (14). Let I_1^*,I_2^* be the roots of Equation (14), and the conditions for Equation (14) to have positive roots are determined below.

Suppose $0\le\tau<1$, then $A>0$. If $R_0>1$, $C<0$. Then Equation (14) has a unique positive root for $I_1^*I_2^*=C/A<0$. If $R_0=1$, $B>0, C=0$. Here, $Q(I)=I(AI+B)$, with $I_1^*=0, I_2^*=-B/A<0$. Hence, Equation (14) has no positive root. If $R_0<1$, $A>0, B>0, C>0$. Thus, Equation (14) has no positive root.

Suppose $\tau = 1$, then $A = 0, B > 0$. Hence, $Q(I) = BI + C$, with the root $I = -C/B$. If $R_0 > 1, C < 0$. Then Equation (14) has a unique positive root. If $R_0 \le 1$, $C \ge 0$. Then Equation (14) has no positive root.

In conclusion, we have the following results. **Theorem 3.2.** System (4) has a unique endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$ when $R_0 > 1$ and no endemic equilibrium when $R_0 \le 1$.

By Theorem 3.2, system (4) has a unique endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$ when $R_0 > 1$. The characteristic equation of system (4) at the equilibrium P^* takes the form

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \qquad (16)$$

where

$$\begin{aligned}
a_1 &= (2-\tau)\beta I^* + \eta\beta E^* + 4\mu + \omega + \alpha + \xi + \delta + d - \eta\beta S^*,\\
a_2 &= \xi((1-\tau)\beta I^* + \mu)\\
&\quad + (\delta + d + \mu)(\beta I^* + \eta\beta E^* + \xi + \mu)\\
&\quad + (2\mu + \delta + d)((1-\tau)\beta I^* + \mu + \omega)\\
&\quad + (\beta I^* + \eta\beta E^*)((1-\tau)\beta I^* + 2\mu + \alpha + \omega)\\
&\quad + ((1-\tau)\beta I^* + 2\mu + \xi + \omega)(\alpha + \mu - \eta\beta S^*),\\
a_3 &= \xi(\alpha + \mu)(1-\tau)\beta I^*\\
&\quad + \omega(\beta I^* + \eta\beta E^* + \mu)(\delta + d + \mu)\\
&\quad + (\beta I^* + \eta\beta E^* + \xi + \mu)(\delta + d + \mu)((1-\tau)\beta I^* + \mu)\\
&\quad + (\beta I^* + \eta\beta E^*)((\alpha + \mu)((1-\tau)\beta I^* + \mu + \omega)\\
&\quad + \eta\beta S^*(\delta + d + \mu)) + \alpha\beta S^*(\tau\beta I^* + \eta\beta E^*)\\
&\quad + (2\mu + \delta + d)(1-\tau)\beta I^*(\alpha + \mu - \eta\beta S^*)\\
&\quad + \mu(\xi + \mu + \omega)(\alpha + \mu - \eta\beta S^*),\\
a_4 &= \eta\beta S^*((1-\tau)\beta I^* + \mu + \omega)((\delta + d + \mu)(\beta I^* + \eta\beta E^*)\\
&\quad + \alpha\beta E^*) + \alpha((1-\tau)\beta I^* + \omega)(1-\tau)\beta V^*\eta\beta E^*\\
&\quad + (\delta + d + \mu)(\alpha + \mu)\xi(1-\tau)\beta I^* + \tau\mu\alpha\beta S^*\beta I^*\\
&\quad + (\delta + d + \mu)((1-\tau)\beta I^*(\mu + \beta I^*)\\
&\quad + \omega\beta I^*)(\alpha + \mu - \eta\beta S^*). \qquad (17)
\end{aligned}$$

Equation (13) implies $\alpha + \mu > \eta\beta S^*$, then $a_1 > 0, a_2 > 0$, $a_3 > 0, a_4 > 0$. Hence, by Routh-Hurwitz criterion, all characteristic roots of Equation (16) have negative real parts for $H = a_3(a_1a_2 - a_3) - a_1^2a_4 > 0$.

From what has been discussed above, we have the following results.

Theorem 3.3. The unique positive endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$ of system (4) is locally asymptotically stable if $R_0 > 1$ and $H > 0$.

4 Global stability

In this section, we discuss the global stability of the disease-free equilibrium P_0 and the endemic equilibrium P^* of system (4) by Lyapunov functional and LaSalle's invariance principle, respectively.

Theorem 4.1. If $R_0 < 1$, then the disease-free equilibrium P_0 is globally asymptotically stable.

Proof. Let $(S(t),V(t),E(t),I(t))$ be any positive solution of system (4) with initial conditions in R_4^+.

System (4) can be rewritten as

$$\begin{cases}
\dot S(t) = -(\xi + \mu)(S - S_0) - \beta I(S - S_0) - \eta\beta E(S - S_0)\\
\qquad + \omega(V - V_0) - \beta I S_0 - \eta\beta E S_0,\\
\dot V(t) = \xi(S - S_0) - (1-\tau)\beta I(V - V_0)\\
\qquad - (\mu + \omega)(V - V_0) - (1-\tau)\beta I V_0,\\
\dot E(t) = \beta I(S - S_0) + (1-\tau)(V - V_0) + \beta I(S_0 + (1-\tau)V_0)\\
\qquad + \eta\beta E(S - S_0) + \eta\beta E S_0 - (\alpha + \mu)E,\\
\dot I(t) = \alpha E - (\delta + d + \mu)I.
\end{cases} \qquad (18)$$

Define a Lyapunov function

$$U_1(t) = \frac{c_1}{2}(S - S_0)^2 + \frac{c_2}{2}(V - V_0)^2 + c_3E + c_4I, \qquad (19)$$

where c_i are positive constants to be determined.

Calculating the derivative of $U_1(t)$ along solutions of system (18), it follows that

$$\begin{aligned}
\dot U_1(t) &= c_1(S - S_0)\dot S + c_2(V - V_0)\dot V + c_3\dot E + c_4\dot I\\
&= -c_1\beta I(S - S_0)^2 - c_1\eta\beta E(S - S_0)^2\\
&\quad - c_2(1-\tau)\beta I(V - V_0)^2 - c_1(\xi + \mu)(S - S_0)^2\\
&\quad + (c_1\omega + c_2\xi)(S - S_0)(V - V_0)\\
&\quad - c_2(\mu + \omega)(V - V_0)^2 + (c_3 - c_1S_0)\beta I(S - S_0)\\
&\quad + (c_3 - c_2V_0)(1-\tau)\beta I(V - V_0)\\
&\quad + (c_3 - c_1S_0)\eta\beta E(S - S_0)\\
&\quad + (c_3\beta(S_0 + (1-\tau)V_0) - c_4(\delta + d + \mu))I\\
&\quad + (c_3\eta\beta S_0 - (\alpha + \mu)c_3 + \alpha c_4)E. \qquad (20)
\end{aligned}$$

Set $c_1 = (\delta + d + \mu)/S_0, c_2 = (\delta + d + \mu)/V_0, c_3 = \delta + d + \mu, c_4 = \beta(S_0 + (1-\tau)V_0)$. We derive from (20) that

$$\begin{aligned}
\dot U_1(t) &= -\frac{(\delta + d + \mu)\beta I}{S_0}(S - S_0)^2\\
&\quad - \frac{(\delta + d + \mu)\eta\beta E}{S_0}(S - S_0)^2\\
&\quad - \frac{(\delta + d + \mu)(1-\tau)\beta I}{V_0}(V - V_0)^2\\
&\quad - \frac{(\delta + d + \mu)(\xi + \mu)}{S_0}(S - S_0)^2\\
&\quad + \left(\frac{(\delta + d + \mu)\omega}{S_0} + \frac{(\delta + d + \mu)\xi}{V_0}\right)(S - S_0)(V - V_0)\\
&\quad - \frac{(\delta + d + \mu)(\mu + \omega)}{V_0}(V - V_0)^2\\
&\quad + (\alpha + \mu)(\delta + d + \mu)(R_0 - 1)E
\end{aligned}$$

$$\leq -\frac{(\delta+d+\mu)(\xi+\mu)}{S_0}(S-S_0)^2 + \left(\frac{(\delta+d+\mu)\omega}{S_0}+\frac{(\delta+d+\mu)\xi}{V_0}\right)(S-S_0)(V-V_0) - \frac{(\delta+d+\mu)(\mu+\omega)}{V_0}(V-V_0)^2 + (\alpha+\mu)(\delta+d+\mu)(R_0-1)E$$
$$= -\frac{(\delta+d+\mu)(\xi+\mu)}{S_0}\left\{\left[(S-S_0)-\frac{\omega V_0+\xi S_0}{2V_0(\xi+\mu)}(V-V_0)\right]^2 + \frac{4S_0V_0(\xi+\mu)(\mu+\omega)-(\omega V_0+\xi S_0)^2}{4V_0^2(\xi+\mu)^2}(V-V_0)^2\right\} + (\alpha+\mu)(\delta+d+\mu)(R_0-1)E. \quad (21)$$

On substituting $V_0=\xi S_0/(\mu+\omega)$ into (21), we obtain that if $R_0<1$,

$$\dot{U}_1(t) \leq -\frac{(\delta+d+\mu)(\xi+\mu)}{S_0}\left\{\left[(S-S_0) - \frac{\omega V_0+\xi S_0}{2V_0(\xi+\mu)}(V-V_0)\right]^2 + \frac{\mu[\xi(3\mu+4\omega)+4(\mu+\omega)^2]}{4\xi(\xi+\mu)^2}(V-V_0)^2\right\} + (\alpha+\mu)(\delta+d+\mu)(R_0-1)E$$
$$\leq (\alpha+\mu)(\delta+d+\mu)(R_0-1)E < 0. \quad (22)$$

Hence, by the Lyapunov Theorem, the disease-free equilibrium P_0 of system (4) is globally asymptotically stable. This completes the proof.□

Theorem 4.2. If $R_0>1$ and $H>0$, then the endemic equilibrium $P^*(S^*,V^*,E^*,I^*)$ is globally asymptotically stable.

Proof. Let $(S(t),V(t),E(t),I(t))$ be any positive solution of system (4) with initial conditions in R_4^+. By Theorem 3.3, the endemic equilibrium P^* of system (4) is locally asymptotically stable if $R_0>1$ and $H>0$.

System (4) can be rewritten as

$$\begin{cases} \dot{S}(t) = S\left[\Pi\left(\frac{1}{S}-\frac{1}{S^*}\right)+\omega\left(\frac{V}{S}-\frac{V^*}{S^*}\right)-\beta(I-I^*) - \eta\beta(E-E^*)\right], \\ \dot{V}(t) = V\left[\xi\left(\frac{S}{V}-\frac{S^*}{V^*}\right)-(1-\tau)\beta(I-I^*)\right], \\ \dot{E}(t) = \beta E\left[\left(\frac{SI}{E}-\frac{S^*I^*}{E^*}\right)+(1-\tau)\left(\frac{VI}{E}-\frac{V^*I^*}{E^*}\right) + \eta(S-S^*)\right], \\ \dot{I}(t) = \alpha I\left(\frac{E}{I}-\frac{E^*}{I^*}\right). \end{cases} \quad (23)$$

Define a Lyapunov function

$$U_2(t) = \left(S-S^*-S^*\ln\frac{S}{S^*}\right)+\left(V-V^*-V^*\ln\frac{V}{V^*}\right) + \left(E-E^*-E^*\ln\frac{E}{E^*}\right)+c_5\left(I-I^*-I^*\ln\frac{I}{I^*}\right), \quad (24)$$

where c_5 is a positive constant to be determined.

Calculating the derivative of $U_2(t)$ along solutions of system (23), it follows that

$$\dot{U}_2(t) = \left[\Pi\left(\frac{1}{S}-\frac{1}{S^*}\right)+\omega\left(\frac{V}{S}-\frac{V^*}{S^*}\right)-\beta(I-I^*) - \eta\beta(E-E^*)\right](S-S^*)$$
$$+\beta\left[\left(\frac{SI}{E}-\frac{S^*I^*}{E^*}\right)+(1-\tau)\left(\frac{VI}{E}-\frac{V^*I^*}{E^*}\right) + \eta(S-S^*)\right](E-E^*)$$
$$+\left[\xi\left(\frac{S}{V}-\frac{S^*}{V^*}\right)-(1-\tau)\beta(I-I^*)\right](V-V^*) + c_5\alpha\left(\frac{E}{I}-\frac{E^*}{I^*}\right)(I-I^*)$$
$$= \Pi\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\omega\left(V-\frac{V}{S}S^*-\frac{S^*}{V^*}V+V^*\right) + \beta(S^*I+SI^*)-\beta\left(\frac{SI}{E}E^*+\frac{S^*I^*}{E^*}E\right) - (1-\tau)\beta\left(\frac{VI}{E}E^*+\frac{V^*I^*}{E^*}E\right)$$
$$+\xi\left(S-\frac{S}{V}V^*-\frac{S^*}{V^*}V+S^*\right)+(1-\tau)\beta(V^*I+VI^*) + c_5\alpha\left(E-\frac{E}{I}I^*-\frac{E^*}{I^*}I+E^*\right)$$
$$= \Pi\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\frac{S}{S^*}(\beta S^*I^*-\omega V^*+\xi S^*) + \frac{V}{V^*}(\omega V^*-\xi S^*+(1-\tau)\beta V^*I^*) + \frac{E}{E^*}(-\beta S^*I^*-(1-\tau)\beta V^*I^*+c_5\alpha E^*) + \frac{I}{I^*}(\beta S^*I^*+(1-\tau)\beta V^*I^*-c_5\alpha E^*)$$
$$-\omega\frac{V}{S}S^*+\omega V^*-\xi\frac{S}{V}V^*+\xi S^*-\beta E^*\frac{SI}{E} - (1-\tau)\beta\frac{VI}{E}E^*-c_5\alpha I^*\frac{E}{I}+c_5\alpha E^*. \quad (25)$$

Set $c_5 = \beta I^*(S^* + (1-\tau)V^*)/(\alpha E^*)$. We derive from (25) that

$$\begin{aligned}\dot{U}_2(t) &= \Pi\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\frac{S}{S^*}(\beta S^*I^*-\omega V^*+\xi S^*)\\&+\frac{V}{V^*}(\omega V^*-\xi S^*+(1-\tau)\beta V^*I^*)\\&-\omega\frac{V}{S}S^*+\omega V^*-\xi\frac{S}{V}V^*+\xi S^*-\beta E^*\frac{SI}{E}\\&-(1-\tau)\beta\frac{VI}{E}E^*-\frac{\beta I^*(S^*+(1-\tau)V^*)}{\alpha E^*}\alpha I^*\frac{E}{I}\\&+\frac{\beta I^*(S^*+(1-\tau)V^*)}{\alpha E^*}\alpha E^*. \qquad (26)\end{aligned}$$

From the first two equations of system (11), we obtain

$$\begin{cases}\Pi &= \beta I^*(S^*+(1-\tau)V^*)+\mu S^*+\mu V^*+\eta\beta S^*E^*,\\ \xi S^* &= (1-\tau)\beta V^*I^*+\mu V^*+\omega V^*.\end{cases} \qquad (27)$$

Substituting the expressions of Π and ξS^* in (27) into (26), which gives

$$\begin{aligned}\dot{U}_2(t) &= \eta\beta E^*S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\mu S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)\\&+\omega V^*\left(2-\frac{S^*}{S}\frac{V}{V^*}-\frac{S}{S^*}\frac{V^*}{V}\right)\\&+\beta S^*I^*\left(3-\frac{S^*}{S}-\frac{E}{E^*}\frac{I^*}{I}-\frac{S}{S^*}\frac{I}{I^*}\frac{E^*}{E}\right)\\&+\mu V^*\left(3-\frac{S^*}{S}-\frac{V}{V^*}-\frac{S}{S^*}\frac{V^*}{V}\right)\\&+(1-\tau)\beta V^*I^*\left(4-\frac{S^*}{S}-\frac{S}{S^*}\frac{V^*}{V}\right.\\&\left.-\frac{E}{E^*}\frac{I^*}{I}-\frac{V}{V^*}\frac{E^*}{E}\frac{I}{I^*}\right). \qquad (28)\end{aligned}$$

Since the arithmetic mean is greater than or equal to the geometric mean, it is shown that

$$\begin{aligned}&2-\frac{S^*}{S}-\frac{S}{S^*}\le 0,\\&2-\frac{S^*}{S}\frac{V}{V^*}-\frac{S}{S^*}\frac{V^*}{V}\le 0,\\&3-\frac{S^*}{S}-\frac{V}{V^*}-\frac{S}{S^*}\frac{V^*}{V}\le 0,\\&3-\frac{S^*}{S}-\frac{E}{E^*}\frac{I^*}{I}-\frac{S}{S^*}\frac{I}{I^*}\frac{E^*}{E}\le 0,\\&4-\frac{S^*}{S}-\frac{S}{S^*}\frac{V^*}{V}-\frac{E}{E^*}\frac{I^*}{I}-\frac{V}{V^*}\frac{E^*}{E}\frac{I}{I^*}\le 0,\end{aligned}$$

with equality if and only if

$$\frac{S^*}{S}=\frac{S}{S^*},\quad \frac{S^*}{S}\frac{V}{V^*}=\frac{S}{S^*}\frac{V^*}{V},\quad \frac{S^*}{S}=\frac{E}{E^*}\frac{I^*}{I}=\frac{S}{S^*}\frac{I}{I^*}\frac{E^*}{E},$$
$$\frac{S^*}{S}=\frac{V}{V^*}=\frac{S}{S^*}\frac{V^*}{V},\quad \frac{S^*}{S}=\frac{S}{S^*}\frac{V^*}{V}=\frac{E}{E^*}\frac{I^*}{I}=\frac{V}{V^*}\frac{E^*}{E}\frac{I}{I^*},$$

that is, $S=S^*, V=V^*, EI^*=E^*I$. Together with (28), it follows that $\dot{U}_2\le 0$, with equality if and only if $S=S^*, V=V^*, EI^*=E^*I$. Hence, we now look for the invariant subset $\mathscr{M}$ within the set

$$N=\{(S,V,E,I): S=S^*, V=V^*, EI^*=E^*I\}.$$

Since $V=V^*$ and $S=S^*$ on $\mathscr{M}$, then $0=\dot{V}(t)=\xi S^*-(1-\tau)\beta V^*I-\mu V^*-\omega V^*$, which yields $I=I^*$. It follows from the fourth Equation of system (4) that $0=\dot{I}(t)=\alpha E-(\delta+d+\mu)I^*$, which yields $E=E^*$. Thus, the only invariant set in N is $\mathscr{M}=(S^*,V^*,E^*,I^*)$. Hence, by Lasalle's invariance principle, P^* is globally asymptotically stable. The proof is complete.□

5 Numerical simulations

In this section, we show the feasibility of the conditions of Theorem 4.2.

Example In system (2), let $\Pi=4.1920, \beta=3.4843, \eta=0.7623, \xi=1.0135, \mu=0.9550, \omega=4.6852, \tau=0.9966, \alpha=1.4990, \delta=2.2815, d=1.0379$. System (2) with above coefficients has an endemic equilibrium $P^*(0.6327, 0.1136, 1.4178, 0.4972, 1.18786)$. A direct calculation show that $R_0=5.8812>1$, $H=5.05\times10^4>0$. By Theorem 4.2, we see that the endemic equilibrium P^* is globally asymptotically stable. Numerical simulation illustrates our result (see Fig.1).

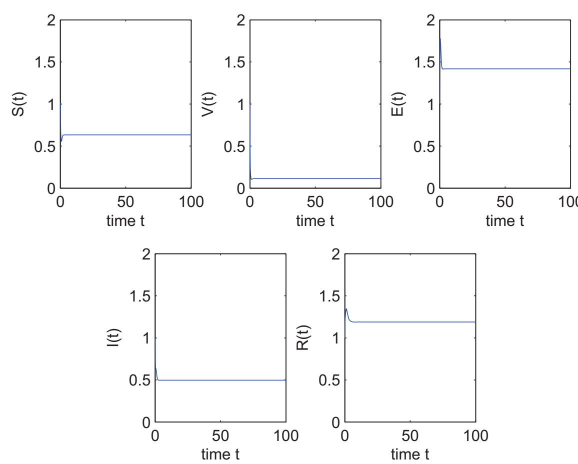

Fig.1 The temporal solution found by numerical integration of system (2) with $\Pi=4.1920, \beta=3.4843, \eta=0.7623, \xi=1.0135, \mu=0.9550, \omega=4.6852, \tau=0.9966, \alpha=1.4990, \delta=2.2815, d=1.0379$ and initial conditions $S(0)=1, V(0)=1, E(0)=1, I(0)=1, R(0)=1$.

6 Discussion

In this paper, the dynamics of a SVEIR epidemic model with waning preventive vaccine and the infection acquired following effective contact with infected population and exposed population is investigated. We have shown that the dynamics of the system are almost completely determined by the basic reproductive number R_0. If $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable while the endemic equilibrium is not feasible. In this case, the disease dies out. If $R_0 > 1$ and $H > 0$, the endemic equilibrium is globally asymptotically stable. To control the disease, a strategy should reduce the basic reproduction number to below unity. From the expression of R_0, we see that the rate τ describing the vaccine efficacy, the rate ξ measuring the vaccination coverage rate and $1/\omega$ describing the duration of the loss of immunity acquired by preventive vaccine do affect the value of the basic reproduction number. Clearly, if τ, ξ or $1/\omega$ increase, the basic reproduction number decreases. Hence, it is necessary and important for public health management to control an epidemic by increasing τ, ξ or $1/\omega$, which reduces the the basic reproduction number.

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