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Growth Properties of Functions Analytic in the Unit Disc on the Basis of their Relative L^*-Orders and Relative L^*-Lower Orders

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Abstract: In this paper we introduce the idea of relative Nevanlinna L^* -order and relative Nevanlinna L^* -lower order of an analytic function with respect to an entire function in the unit disc $U = \{z : |z| < 1\}$. Hence we study some comparative growth properties of composition of two analytic functions in the unit disc U on the basis of their relative Nevanlinna L^* -orders and relative Nevanlinna L^* -lower orders.

Keywords: Growth, analytic function, composition, unit disc, relative Nevanlinna L^* -order, relative Nevanlinna L^* -lower order, slowly changing function in the unit disc.

1 Introduction, Definitions and Notations

A function f, analytic in the unit disc $U = \{z : |z| < 1\}$, is said to be of finite Nevanlinna order [2] if there exist a number μ such that the Nevanlinna characteristic function

$$T(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta$$

satisfies $T(r, f) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T(r, f)}{-\log(1-r)} .$$

Similarly, Nevanlinna lower order λ_f of f is given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T(r, f)}{-\log(1-r)} .$$

Datta et. al. [1] introduced the notion of Nevanlinna L-order for an analytic function f in the unit disc $U = \{z : |z| < 1\}$ where $L = L\left(\frac{1}{1-r}\right)$ is a positive continuous function in the unit disc U increasing slowly i.e., $L\left(\frac{a}{1-r}\right) \sim L\left(\frac{1}{1-r}\right)$ as $r \to 1$, for every positive constant 'a', in the following manner:

Definition 1.*If f be analytic in U, then the Nevanlinna L-order ρ_f^L and the Nevanlinna L-lower order λ_f^L of f are defined as*

$$\rho_f^L = \limsup_{r\to 1} \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} \quad \textit{and} \quad \lambda_f = \liminf_{r\to 1} \frac{\log T(r,f)}{\log\left(\frac{L\left(\frac{1}{1-r}\right)}{(1-r)}\right)} .$$

Now we introduce the concepts of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function f with respect to another analytic function g in the unit disc U which are as follows:

Definition 2.*If f be analytic in U and g be entire, then the relative Nevanlinna L^*-order of f with respect to g, denoted by $\rho_g^{L^*}(f)$ is defined by*

$$\rho_f^L = \inf\left\{\mu > 0 : T_f(r) < T_g\left[\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right]^\mu \textit{ for all } 0 < r_0(\mu) < r < 1\right\} .$$

Similarly, relative Nevanlinna L^*-order of f with respect

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to g denoted by $\lambda_g^{L^*}(f)$ *is given by*
$\lambda_g^{L^*}(f) = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}$.

When $g(z) = \exp z$, the definition coincides with the definition of the Nevanlinna L^*-order and the Nevanlinna L^*-lower order.

In this paper we study some growth properties of composition of two analytic functions in the unit disc $U = \{z : |z| < 1\}$ on the basis of their relative Nevanlinna L^*-orders (relative Nevanlinna L^*-lower orders). We do not explain the standard definitions and notations in the theory of entire functions as those are available in [3].

2 Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f, g be any two analytic functions in U and h be an entire function such that $0 < \lambda_h^{L^*}(f\circ g) \le \rho_h^{L^*}(f\circ g) < \infty$ and $0 < \lambda_h^{L^*}(f) \le \rho_h^{L^*}(f) < \infty$. If $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r \to 1$ then* $\frac{\lambda_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)} \le \liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \le \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)} \le \limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \le \frac{\rho_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}$.

Proof. From the definition of relative Nevanlinna L^*-order and relative Nevanlinna L^*-lower order of an analytic function in the unit disc U we have for arbitrary positive ε and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1} T_{f\circ g}(r) \ge \left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_{f\circ g}(r) \ge \left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\} \quad (1)$$

and

$$\log T_h^{-1} T_f(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_f(r) \le \left(\rho_h^{L^*}(f) + \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\}$$

i.e.,

$$\frac{\log T_h^{-1} T_f(r)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)} \le \log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right). \quad (2)$$

Now from (1) and (2), it follows for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1} T_{f\circ g}(r) \ge \frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)} \log T_h^{-1} T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \ge \frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \ge \frac{\frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)}}{1 + \frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1} T_f(r)}}. \quad (3)$$

Since $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r \to 1$, it follows from (3) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \ge \frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\rho_h^{L^*}(f) + \varepsilon\right)}. \quad (4)$$

As $\varepsilon (> 0)$ is arbitrary, we get from (4) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \ge \frac{\lambda_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}. \quad (5)$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f\circ g}(r) \le \left(\lambda_h^{L^*}(f\circ g) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_{f\circ g}(r) \le \left(\lambda_h^{L^*}(f\circ g) + \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\} \quad (6)$$

and for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$\log T_h^{-1} T_f(r) \ge \left(\lambda_h^{L^*}(f) - \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_f(r) \geq \left(\lambda_h^{L^*}(f) - \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\}$$

i.e.,

$$\frac{\log T_h^{-1} T_f(r)}{\left(\lambda_h^{L^*}(f) - \varepsilon\right)} \geq \log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right) . \quad (7)$$

Combining (6) and (7) , we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\lambda_h^{L^*}(f\circ g) + \varepsilon\right)}{\left(\lambda_h^{L^*}(f) - \varepsilon\right)} \log T_h^{-1} T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\left(\lambda_h^{L^*}(f\circ g) + \varepsilon\right)}{\left(\lambda_h^{L^*}(f) - \varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\frac{\left(\lambda_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}{1 + \frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1} T_f(r)}} . \quad (8)$$

As $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r \to 1$ we get from (8) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\lambda_h^{L^*}(f\circ g) + \varepsilon}{\lambda_h^{L^*}(f) - \varepsilon} . \quad (9)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (9) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)} . \quad (10)$$

Also for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_f(r) \leq \left(\lambda_h^{L^*}(f) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_f(r) \leq \left(\lambda_h^{L^*}(f) + \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\}$$

i.e.,

$$\frac{\log T_h^{-1} T_f(r)}{\left(\lambda_h^{L^*}(f) + \varepsilon\right)} \leq \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\} . \quad (11)$$

Now from (1) and (11) , we obtain for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \geq \frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\lambda_h^{L^*}(f) + \varepsilon\right)} \log T_h^{-1} T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \geq \frac{\left(\lambda_h^{L^*}(f\circ g) - \varepsilon\right)}{\left(\lambda_h^{L^*}(f) + \varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \geq \frac{\frac{\left(\lambda_h^{L^*}(f\circ g)-\varepsilon\right)}{\left(\lambda_h^{L^*}(f)+\varepsilon\right)}}{1 + \frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1} T_f(r)}} . \quad (12)$$

In view of the condition $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r \to 1$, we obtain from (12) that

$$\limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \geq \frac{\lambda_h^{L^*}(f\circ g) - \varepsilon}{\lambda_h^{L^*}(f) + \varepsilon} . \quad (13)$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (13) that

$$\limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \geq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)} . \quad (14)$$

Also for all sufficiently large values of $\left(\frac{1}{1-r}\right)$,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \left(\rho_h^{L^*}(f\circ g) + \varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_{f\circ g}(r) \leq \left(\rho_h^{L^*}(f\circ g) + \varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\} . \quad (15)$$

So from (7) and (15), it follows for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\rho_h^{L^*}(f\circ g) + \varepsilon\right)}{\left(\lambda_h^{L^*}(f) - \varepsilon\right)} \log T_h^{-1} T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1} T_f(r)}}. \quad (16)$$

Using $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r\to 1$, we obtain from (16) that

$$\limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\lambda_h^{L^*}(f)-\varepsilon\right)}. \quad (17)$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (17) that

$$\limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)}. \quad (18)$$

Thus the theorem follows from (5), (10), (14) and (18).

Similarly in view of Theorem 1, we may state the following theorem without its proof for the right factor g of the composite function $f\circ g$:

Theorem 2. *Let f, g be any two analytic functions in U and h be an entire function with $0 < \lambda_h^{L^*}(f\circ g) \leq \rho_h^{L^*}(f\circ g) < \infty$ and $0 < \lambda_h^{L^*}(g) \leq \rho_h^{L^*}(g) < \infty$. If $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_g(r)\right\}$ as $r\to 1$ then* $\frac{\lambda_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)} \leq \liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(r)+L\left(\frac{1}{1-r}\right)} \leq \frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)} \leq \limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_g(r)+L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)}$.

Theorem 3. *Let f, g be any two analytic functions in U and h be an entire function such that $0 < \rho_h^{L^*}(f\circ g) < \infty$ and $0 < \rho_h^{L^*}(f) < \infty$. If $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r\to 1$ then* $\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r)+L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)} \leq \limsup_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r)+L\left(\frac{1}{1-r}\right)}$.

Proof. From the definition of $\rho_h^{L^*}(f)$, the relative Nevanlinna L^*-order of an analytic function f in the unit disc U with respect to an entire function h we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_f(r) \geq \left(\rho_h^{L^*}(f)-\varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_f(r) \geq \left(\rho_h^{L^*}(f)-\varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\}$$

i.e.,

$$\frac{\log T_h^{-1} T_f(r)}{\left(\rho_h^{L^*}(f)-\varepsilon\right)} \geq \log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right). \quad (19)$$

Now from (15) and (19), it follows for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1} T_{f\circ g}(r) \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\rho_h^{L^*}(f)-\varepsilon\right)} \log T_h^{-1} T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\rho_h^{L^*}(f)-\varepsilon\right)} \cdot \frac{\log T_h^{-1} T_f(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\frac{\left(\rho_h^{L^*}(f\circ g)+\varepsilon\right)}{\left(\rho_h^{L^*}(f)-\varepsilon\right)}}{1+\frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1} T_f(r)}}. \quad (20)$$

Using $L\left(\frac{1}{1-r}\right) = o\left\{\log T_h^{-1} T_f(r)\right\}$ as $r\to 1$, we obtain from (20) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)+\varepsilon}{\rho_h^{L^*}(f)-\varepsilon}. \quad (21)$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from (21) that

$$\liminf_{r\to 1} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_h^{-1} T_f(r) + L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}. \quad (22)$$

Again for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity,

$$\log T_h^{-1} T_{f\circ g}(r) \geq \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right) \log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\log T_h^{-1} T_{f\circ g}(r) \geq \left(\rho_h^{L^*}(f\circ g)-\varepsilon\right) \left\{\log\left(\frac{1}{1-r}\right) + L\left(\frac{1}{1-r}\right)\right\}. \quad (23)$$

So combining (2) and (23), we get for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1}T_{f\circ g}(r) \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{\left(\rho_h^{L^*}(f)+\varepsilon\right)}\log T_h^{-1}T_f(r)$$

i.e.,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)} \geq \frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{\left(\rho_h^{L^*}(f)+\varepsilon\right)}\cdot\frac{\log T_h^{-1}T_f(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)}$$

i.e.,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)} \geq \frac{\frac{\left(\rho_h^{L^*}(f\circ g)-\varepsilon\right)}{\left(\rho_h^{L^*}(f)+\varepsilon\right)}}{1+\frac{L\left(\frac{1}{1-r}\right)}{\log T_h^{-1}T_f(r)}}. \tag{24}$$

Since $L\left(\frac{1}{1-r}\right)=o\left\{\log T_h^{-1}T_f(r)\right\}$ as $r\to 1$, it follows from (24) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)} \geq \frac{\rho_h^{L^*}(f\circ g)-\varepsilon}{\rho_h^{L^*}(f)+\varepsilon}. \tag{25}$$

As $\varepsilon\,(>0)$ is arbitrary, we get from (25) that

$$\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)} \geq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}. \tag{26}$$

Thus the theorem follows from (22) and (26).

Theorem 4. *Let f, g be any two analytic functions in U and h be an entire function with $0<\rho_h^{L^*}(f\circ g)<\infty$ and $0<\rho_h^{L^*}(g)<\infty$. If $L\left(\frac{1}{1-r}\right)=o\left\{\log T_h^{-1}T_g(r)\right\}$ as $r\to 1$ then*

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)+L\left(\frac{1}{1-r}\right)} \leq \frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)+L\left(\frac{1}{1-r}\right)}.$$

The proof is omitted .

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5. *Let f, g be any two analytic functions in U and h be an entire function such that $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(f)\leq\rho_h^{L^*}(f)<\infty$. If $L\left(\frac{1}{1-r}\right)=o\left\{\log T_h^{-1}T_f(r)\right\}$ as $r\to 1$ then*

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)} \leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\} \leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(f)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(f)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)+L\left(\frac{1}{1-r}\right)}.$$

The proof is omitted.

Combining Theorem 2 and Theorem 4, we may state the following theorem:

Theorem 6. *Let f, g be any two analytic functions in U and h be an entire function with $0<\lambda_h^{L^*}(f\circ g)\leq\rho_h^{L^*}(f\circ g)<\infty$ and $0<\lambda_h^{L^*}(g)\leq\rho_h^{L^*}(g)<\infty$. If $L\left(\frac{1}{1-r}\right)=o\left\{\log T_h^{-1}T_g(r)\right\}$ as $r\to 1$ then*

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)+L\left(\frac{1}{1-r}\right)} \leq \min\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\} \leq \max\left\{\frac{\lambda_h^{L^*}(f\circ g)}{\lambda_h^{L^*}(g)},\frac{\rho_h^{L^*}(f\circ g)}{\rho_h^{L^*}(g)}\right\} \leq \limsup_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)+L\left(\frac{1}{1-r}\right)}.$$

Theorem 7. *Let f be an analytic function in U and h be entire such that $\rho_h^{L^*}(f)<\infty$. Also let g be analytic in U. If $\lambda_h^{L^*}(f\circ g)=\infty$ then*

$$\lim_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_f(r)}=\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta>0$ such that for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity

$$\log T_h^{-1}T_{f\circ g}(r)\leq\beta\log T_h^{-1}T_f(r). \tag{27}$$

Again from the definition of $\rho_h^{L^*}(f)$, it follows that for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ that

$$\log T_h^{-1}T_f(r)\leq\left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right). \tag{28}$$

Thus from (27) and (28), we have for a sequence of values of $\left(\frac{1}{1-r}\right)$ tending to infinity that

$$\log T_h^{-1}T_{f\circ g}(r)\leq\beta\left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)$$

i.e.,

$$\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}\leq\frac{\beta\left(\rho_h^{L^*}(f)+\varepsilon\right)\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}$$

$$\liminf_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log\left(\frac{\exp\left\{L\left(\frac{1}{1-r}\right)\right\}}{(1-r)}\right)}=\lambda_h^{L^*}(f\circ g)<\infty.$$

This is a contradiction.

This proves the theorem.

Remark. Theorem 7 is also valid with “limit superior” instead of “limit” if $\lambda_h^{L^*}(f\circ g)=\infty$ is replaced by $\rho_h^{L^*}(f\circ g)=\infty$ and the other conditions remaining the same.

Corollary 1. *Under the assumptions of Theorem 7 or Remark 2,*
$$\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_f(r)}=\infty\,.$$

Proof. From Theorem 7 or Remark 2, we obtain for all sufficiently large values of $\left(\frac{1}{1-r}\right)$ and for $K>1$ that

$$\log T_h^{-1}T_{f\circ g}(r)>K\log T_h^{-1}T_f(r)$$

i.e.,

$$T_h^{-1}T_{f\circ g}(r)>\left\{T_h^{-1}T_f(r)\right\}^K,$$

from which the corollary follows.

Theorem 8. *Let f, g be any two analytic functions in U and h be any entire function such that $\rho_h^{L^*}(g)<\infty$ and $\lambda_h^{L^*}(f\circ g)=\infty$. Then*
$$\lim_{r\to 1}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_h^{-1}T_g(r)}=\infty\,.$$

We omit the proof of Theorem 8 because it can be carried out in the line of Theorem 7.

Remark. Theorem 8 is also valid with "limit superior" instead of "limit" if $\lambda_h^{L^*}(f\circ g)=\infty$ is replaced by $\rho_h^{L^*}(f\circ g)=\infty$ and the other conditions remaining the same.

In the line of Corollary 1, we may easily verify the following:

Corollary 2. *Under the assumptions of Theorem 8 or Remark 2,*
$$\limsup_{r\to 1}\frac{T_h^{-1}T_{f\circ g}(r)}{T_h^{-1}T_g(r)}=\infty\,.$$

3 Conclusion

The main aim of this paper is to extend the notion of Nevanlinna order to relative Nevanlinna L^*-order in case of the growth properties of functions analytic in unit disc. In fact, the relative Nevanlinna L^*-order of growth gives a quantitative assessment of how different functions scale each other and until what extent they are self-similar in growth. Actually, in this paper we have established some theorems in this connection. Here, we are trying to extend the notion of the growth properties of functions analytic in unit disc on the basis of the relative Nevanlinna L^*-order and the relative Nevanlinna L^*-lower order. But still there are some problems to be investigated further. One of such problems is the study of the growth properties of the same in some poly disc. These type of studies can be regarded as open problems for the future workers in this branch.

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