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A Generalized Class of Dual to Product-Cum-Dual to Ratio Type Estimators of Finite Population Mean In Sample Surveys

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Abstract: In this paper, we have suggested a general class of dual to product-cum-dual to ratio type estimators of finite population mean using an auxiliary variable *x* that is correlated with the variable *y* of interest. The proposed class of estimators includes several known estimators based on transformation in auxiliary variable *x*. The bias and mean squared error (*MSE*) expressions of the proposed class of estimators have been obtained to the first degree of approximation. We have compared the proposed class of dual to product-cum- dual to ratio type estimators of finite population mean to various existing ratio, product, and ratio-cum-product type estimators and shown that the suggested class of estimators is better than other existing estimators under some realistic conditions.

Keywords: Auxiliary variable, Dual to product-cum-dual to ratio type estimator, Finite population mean, Simple random sampling, Bias, Mean squared error.

1 Introduction

It is well established fact that in sample surveys, auxiliary information is often used to improve the precision of estimators of population parameters. The use of auxiliary information at the estimation stage appears to have started with the work of Cochran (1940). He envisaged the ratio estimator to estimate the population mean or total of the study variate y by using supplementary information on an auxiliary variate x, positively correlated with y. The ratio estimator is most effective when the relationship between study variate y and auxiliary variate x is linear passing through the origin and the mean square error of y is proportional to x. When the auxiliary variable x is negatively correlated with the study variate y, Robson (1957) and Murthy (1964) proposed the product estimator of the population mean or total. In fact, for the better utilization of information on an auxiliary variate x, Murthy (1964) has suggested the use of

- Ratio estimator $\bar{y}_R$ if, $\rho C_y / C_x > 1/2$,
- Product estimator $\bar{y}_P$ if, $\rho C_y / C_x < -1/2$,
- Unbiased estimator $\bar{y}$ if, $-1/2 \le \rho C_y / C_x \le 1/2$,

Where (C_y, C_x) are coefficients of variation of (y, x) respectively and ρ is the correlation coefficient between y and x respectively.

Consider a finite population $U = (U_1, U_2, ..., U_N)$ of N units. A sample of size $n(n < N)$ is drawn using simple random sampling without replacement (*SRSWOR*) method to estimate the population mean $\bar{Y} = N^{-1}\sum_{i=1}^{N} y_i$ of the study variate y. Let the sample means $(\bar{x}, \bar{y})$ be the unbiased estimators of the population means respectively $(\bar{X}, \bar{Y})$ based on n observations.

The classical ratio and product estimators of population mean $\bar{Y}$ are respectively given by

$$\bar{y}_R = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right), \quad (1.1)$$

$$\bar{y}_P = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right). \quad (1.2)$$

The biases and mean squared errors (*MSEs*) of $\bar{y}_R$ and $\bar{y}_P$ to the first degree of approximation, are respectively given by

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$$B(\bar{y}_R)=\frac{(1-f)}{n}\bar{Y}C_x^2[1-K], \tag{1.3}$$

$$B(\bar{y}_P)=\frac{(1-f)}{n}\bar{Y}C_x^2K, \tag{1.4}$$

$$MSE(\bar{y}_R)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+C_x^2(1-2K)\right], \tag{1.5}$$

$$MSE(\bar{y}_P)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+C_x^2(1+2K)\right], \tag{1.6}$$

where

$$f=\frac{n}{N}, K=\rho\frac{C_y}{C_x}, \rho=\frac{S_{xy}}{S_xS_y}, C_y=\frac{S_y}{\bar{Y}}, C_x=\frac{S_x}{\bar{X}}, S_{xy}=\frac{\sum_{i=1}^N(x_i-\bar{X})(y_i-\bar{Y})}{(N-1)},$$

$$S_x^2=\frac{\sum_{i=1}^N(x_i-\bar{X})^2}{(N-1)} \quad \text{and} \quad S_y^2=\frac{\sum_{i=1}^N(y_i-\bar{Y})^2}{(N-1)}.$$

Consider a transformation

$$x_i^*=\frac{N\bar{X}-nx_i}{N-n}=(1+g)\bar{X}-gx_i, i=1,2,\ldots,N,$$

where $g=n/(N-n)$. Then $\bar{x}^*=(1+g)\bar{X}-g\bar{x}$ is an unbiased estimator for $\bar{X}=N^{-1}\sum_{i=1}^N x_i$ and the correlation between $\bar{y}$ and $\bar{x}^*$ is negative. Using the transformation x_i^*, Srivenkataramana (1980) and Bandyopadhyaya (1980) obtained dual to ratio and dual to product estimators respectively as

$$\bar{y}_R^*=\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right), \tag{1.7}$$

$$\bar{y}_P^*=\bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right). \tag{1.8}$$

The biases and mean squared errors (*MSEs*) of $\bar{y}_R^*$ and $\bar{y}_P^*$ to the first degree of approximation are respectively given as

$$B(\bar{y}_R^*)=-\frac{(1-f)}{n}\bar{Y}C_x^2gK, \tag{1.9}$$

$$B(\bar{y}_P^*)=\frac{(1-f)}{n}\bar{Y}C_x^2g(g+K), \tag{1.10}$$

$$MSE(\bar{y}_R^*)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+C_x^2g(g-2K)\right], \tag{1.11}$$

$$MSE(\bar{y}_P^*)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+C_x^2g(g+2K)\right]. \tag{1.12}$$

Singh and Agnihotri (2008) defined a family of product-cum-ratio estimators of population mean $\bar{Y}$ in simple random sampling (*SRS*) as

$$\bar{y}_{PR}=\alpha\bar{y}\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)+(1-\alpha)\bar{y}\left(\frac{a\bar{X}+b}{a\bar{x}+b}\right), \tag{1.13}$$

where 'a' and 'b' are known characterizing positive scalars and α is a real constant to be determined such that the *MSE* of $\bar{y}_{PR}$ is minimum. The bias and *MSE* of $\bar{y}_{PR}$ to the first degree of approximation are respectively given as

$$B(\bar{y}_{PR})=-\frac{(1-f)}{n}\bar{Y}C_x^2\delta[K+\alpha(\delta-2K)-\delta], \tag{1.14}$$

$$MSE(\bar{y}_{PR})=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\delta(2\alpha-1)C_x^2\{\delta(2\alpha-1)+2K\}\right], \tag{1.15}$$

where $\delta=\dfrac{a\bar{X}}{a\bar{X}+b}$.

The aim of this paper is to suggest a generalised class of dual to product-cum-dual to ratio estimators for population mean $\bar{Y}$ in *SRSWOR* and their properties are studied under large sample approximation. It is shown that the proposed dual to product-cum dual to ratio estimator includes several known estimators based on transformation in auxiliary variable x. An empirical study is carried out to discuss the superiority of the proposed class of estimators.

2 A Generalized Class of Dual to Product-Cum- Dual to Ratio Type Estimators of Finite Population Mean

We suggest a class of dual to product-cum-dual to ratio type estimators in *SRSWOR* for population mean $\bar{Y}$ as

$$\bar{y}_{PR}^*=\eta\bar{y}\left(\frac{a\bar{X}+b}{a\bar{x}^*+b}\right)+(1-\eta)\bar{y}\left(\frac{a\bar{x}^*+b}{a\bar{X}+b}\right), \tag{2.1}$$

where (a,b) are same as defined earlier, η being a suitably chosen scalar and $\bar{x}^*=\dfrac{N\bar{X}-n\bar{x}}{N-n}=(1+g)\bar{X}-g\bar{x}$ with $g=\dfrac{n}{N-n}$.

To obtain the bias and *MSE* of $\bar{y}_{PR}^*$ to the first degree of approximation, we write

$$e_0=\frac{(\bar{y}-\bar{Y})}{\bar{Y}} \quad \text{and} \quad e_1=\frac{(\bar{x}-\bar{X})}{\bar{X}},$$

such that $E(e_0)=0, E(e_1)=0$ and

$$\left.\begin{aligned} E(e_0^2)&=\frac{(1-f)}{n}C_y^2\\ E(e_1^2)&=\frac{(1-f)}{n}C_x^2\\ E(e_0e_1)&=\frac{(1-f)}{n}KC_x^2\end{aligned}\right\}. \tag{2.2}$$

Table 2.1 shows the members of the proposed class of estimators $\bar{y}_{PR}^*$ for different choices of (a,b,δ,η).

In Table 2.1, C_x and $\beta_2(x)$ respectively are known coefficient of variation and coefficient of kurtosis respectively of an auxiliary variable x.

Table 2.1: Members of the estimator $\bar{y}_{PR}^{*}$ for different choices of (a,b,δ,η)

S. No.	Estimators	Values of constants (a,b,δ,η)			
		a	b	η	δ
1.	$\bar{y}_R^* = \bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)$ Dual to ratio estimator	1	0	0	1
2.	$\bar{y}_P^* = \bar{y}\left(\frac{\bar{X}}{\bar{x}^*}\right)$ Dual to product estimator	1	0	1	1
3.	$\bar{y}_{SD}^* = \bar{y}\left(\frac{\bar{x}^* + C_x}{\bar{X} + C_x}\right)$ Singh and Upadhyaya (1986) estimator	1	C_x	0	$\frac{\bar{X}}{\bar{X}+C_x}$
4.	$\bar{y}_{SK}^* = \bar{y}\left(\frac{\bar{x}^* + \beta_2(x)}{\bar{X} + \beta_2(x)}\right)$ Dual to Singh *et al.* (2004) estimator	1	$\beta_2(x)$	0	$\frac{\bar{X}}{\bar{X}+\beta_2(x)}$
5.	$\bar{y}_{UP1}^* = \bar{y}\left(\frac{\bar{x}^* C_x + \beta_2(x)}{\bar{X}C_x + \beta_2(x)}\right)$ Dual to Upadhyaya and Singh (1999) ratio-type estimator	C_x	$\beta_2(x)$	0	$\frac{\bar{X}C_x}{\bar{X}C_x+\beta_2(x)}$
6.	$\bar{y}_{UP2}^* = \bar{y}\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}^* C_x + \beta_2(x)}\right)$ Dual to Upadhyaya and Singh (1999) product-type estimator	C_x	$\beta_2(x)$	1	$\frac{\bar{X}C_x}{\bar{X}C_x+\beta_2(x)}$
7.	$\bar{y}_{UP3}^* = \bar{y}\left(\frac{\bar{x}^* \beta_2(x) + C_x}{\bar{X}\beta_2(x) + C_x}\right)$ Dual to Upadhyaya and Singh (1999) ratio-type estimator	$\beta_2(x)$	C_x	0	$\frac{\bar{X}C_x}{\bar{X}C_x+\beta_2(x)}$
8.	$\bar{y}_{UP4}^* = \bar{y}\left(\frac{\bar{X}\beta_2(x) + C_x}{\bar{x}^* \beta_2(x) + C_x}\right)$ Upadhyaya and Singh (1999) product-type estimator	$\beta_2(x)$	C_x	1	$\frac{\bar{X}\beta_2(x)}{\bar{X}\beta_2(x)+C_x}$
9.	$\bar{y}_{SE}^* = \bar{y}\left[\eta\left(\frac{\bar{x}^*}{\bar{X}}\right) + (1-\eta)\left(\frac{\bar{X}}{\bar{x}^*}\right)\right]$ Chaudhary and Singh (2012) estimator	1	0	η	1
13.	$\bar{y}_{SA1}^* = \bar{y}\left(\frac{a\bar{x}^* + b}{a\bar{X} + b}\right)$ Dual to Shah and Patel (1984) and Singh and Agnihotri (2008) ratio-type estimator	a	b	0	$\frac{a\bar{X}}{a\bar{X}+b}$
14.	$\bar{y}_{SA2}^* = \bar{y}\left(\frac{a\bar{X} + b}{a\bar{x}^* + b}\right)$ Dual to Singh and Agnihotri (2008) product-type estimator	a	b	1	$\frac{a\bar{X}}{a\bar{X}+b}$
15.	$\bar{y}_{TS}^* = \bar{y}\left[\eta\left(\frac{\bar{x}^* C_x + \beta_2(x)}{\bar{X}C_x + \beta_2(x)}\right) + (1-\eta)\left(\frac{\bar{X}C_x + \beta_2(x)}{\bar{x}^* C_x + \beta_2(x)}\right)\right]$ Dual to Tailor and Sharma (2009) estimator	C_x	$\beta_2(x)$	η	$\frac{\bar{X}C_x}{\bar{X}C_x+\beta_2(x)}$

Expressing (2.1) in terms of $e's$, we have

$$\bar{y}_{PR}^* = \bar{Y}(1+e_0)[\eta(1-\delta g e_1)^{-1} + (1-\eta)(1-\delta g e_1)]. \qquad (2.3)$$

We assume that $|\delta g e_1| < 1$ so that $(1-\delta g e_1)^{-1}$ is expandable. From (2.3) we have

$$\bar{y}_{PR}^{*} = \bar{Y}(1+e_0)[\eta(1+\delta g e_1+\delta^2 g^2 e_1^2+...)+(1-\eta)(1-\delta g e_1)]$$
$$= \bar{Y}(1+e_0)\left[1+\delta g e_1(2\eta-1)+\eta\delta^2 g^2 e_1^2-...\right]$$
$$= \bar{Y}\left[1+e_0+\delta g e_1(2\eta-1)+\delta g e_1 e_0(2\eta-1)+\eta\delta^2 g^2 e_1^2-...\right]$$

Neglecting terms of e's having power greater than two we have

$$\bar{y}_{PR}^{*} \cong \bar{Y}[1+e_0+\delta g e_1(2\eta-1)+\delta g e_1 e_0(2\mu-1)+\eta\delta^2 g^2 e_1^2]$$

or

$$\left(\bar{y}_{PR}^{*}-\bar{Y}\right)=\bar{Y}[e_0+\delta g e_1(2\eta-1)+\delta g e_1 e_0(2\eta-1)+\eta\delta^2 g^2 e_1^2]. \quad (2.4)$$

Taking expectation of both the sides of (2.4), we obtain the bias of $\bar{y}_{PR}^{*}$ to the first degree of approximation, as

$$B\left(\bar{y}_{PR}^{*}\right)=\bar{Y}\frac{(1-f)}{n}\left[\delta g K C_x^2(2\eta-1)+\eta\delta^2 g^2 C_x^2\right]. \quad (2.5)$$
$$=\bar{Y}\frac{(1-f)}{n}\delta g C_x^2[K(2\eta-1)+\eta\delta g]$$

The bias of $\bar{y}_{PR}^{*}$ is almost unbiased when

$$B\left(\bar{y}_{PR}^{*}\right)=\bar{Y}\frac{(1-f)}{n}\delta g C_x^2[K(2\eta-1)+\eta\delta g]=0$$

i.e. $\eta=\dfrac{K}{2K+\delta g}$.

Squaring both sides of (2.4) and neglecting terms of e's having power greater than two we have

$$\left(\bar{y}_{PR}^{*}-\bar{Y}\right)^2=\bar{Y}^2\left[e_0^2+\delta^2 g^2 e_1^2(2\eta-1)^2+2\delta g(2\eta-1)e_0 e_1\right] \quad (2.6)$$

Taking expectation of both sides of (2.6) we get the *MSE* of $\bar{y}_{PR}^{*}$ to the first degree of approximation as

$$MSE\left(\bar{y}_{PR}^{*}\right)=\bar{Y}^2\frac{(1-f)}{n}\left[C_y^2+\delta g(2\eta-1)C_x^2\{\delta g(2\eta-1)+2K\}\right] \quad (2.7)$$

which is minimized when

$$\eta=\frac{1}{2}\left(1-\frac{K}{\delta g}\right)=\eta_{opt.}(say). \quad (2.8)$$

Substituting the value of $\eta_{opt.}$ in (2.1) yields the asymptotically optimum estimator (*AOE*) as

$$\bar{y}_{PR,opt.}^{*}=\eta_{opt.}\bar{y}\left(\frac{a\bar{X}+b}{a\bar{x}^{*}+b}\right)+(1-\eta_{opt.})\bar{y}\left(\frac{a\bar{x}^{*}+b}{a\bar{X}+b}\right), \quad (2.9)$$

Thus, the resulting bias and *MSE* of $\bar{y}_{PR,opt.}^{*}$ respectively are respectively given by

$$B\left(\bar{y}_{PR,opt.}^{*}\right)=\bar{Y}\frac{(1-f)}{n}\delta g C_x^2\left[K(2\eta_{opt.}-1)+\eta_{opt.}\delta g\right], \quad (2.10)$$

$$\min\ MSE\left(\bar{y}_{PR}^{*}\right)=MSE\left(\bar{y}_{PR,opt.}^{*}\right)=\bar{Y}^2\frac{(1-f)}{n}C_y^2(1-\rho^2). \quad (2.11)$$

3 Efficiency Comparison

(i)Under *SRSWOR*, the variance of sample mean $\bar{y}$ is

$$Var(\bar{y})=\frac{(1-f)}{n}\bar{Y}^2 C_y^2. \quad (3.1)$$

From (2.7) and (3.1), it is found that the proposed dual to product-cum-dual to ratio type estimator $\bar{y}_{PR}^{*}$ is more efficient than $\bar{y}$ if

$$Var(\bar{y})-MSE\left(\bar{y}_{PR}^{*}\right)=\frac{(1-f)}{n}\bar{Y}^2 C_y^2-\frac{(1-f)}{n}\bar{Y}^2\left[\begin{matrix}C_y^2+\delta g(2\eta-1)C_x^2\\ \{\delta g(2\eta-1)+2K\}\end{matrix}\right]>0$$
$$=-\frac{(1-f)}{n}\bar{Y}^2\left[\delta g(2\eta-1)C_x^2\{\delta g(2\eta-1)+2K\}\right]>0.$$

This condition holds if

either $\dfrac{1}{2}>\eta$ and $\dfrac{1}{2}-\dfrac{K}{\delta g}<\eta$,

or $\dfrac{1}{2}<\eta$ and $\dfrac{1}{2}-\dfrac{K}{\delta g}>\eta$. (3.2)

Therefore, the range of η under which the proposed dual to product-cum-dual to ratio type estimator $\bar{y}_{PR}^{*}$ is more efficient than $\bar{y}$ is

$$\eta\in\left[\min\left\{\frac{1}{2},\frac{1}{2}-\frac{K}{\delta g}\right\},\max\left\{\frac{1}{2},\frac{1}{2}-\frac{K}{\delta g}\right\}\right]. \quad (3.3)$$

(ii) The *MSEs* of dual to Shah and Patel (1984) and Singh and Agnihotri (2008) ratio(product) type estimator $\bar{y}_{SA1}^{*}\left(\bar{y}_{SA2}^{*}\right)$ is obtained by putting $\eta=0(1)$ as

$$MSE\left(\bar{y}_{SA1}^{*}\right)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\delta g C_x^2\{\delta g-2K\}\right], \quad (3.4)$$

$$MSE\left(\bar{y}_{SA2}^{*}\right)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\delta g C_x^2\{\delta g+2K\}\right]. \quad (3.5)$$

From (2.7) and (3.4), it is found that the proposed dual to product-cum-dual to ratio type estimator $\bar{y}_{PR}^{*}$ is more efficient than dual to Shah and Patel (1984) and Singh and Agnihotri (2008) class of ratio estimators $\bar{y}_{SA1}^{*}$ if

$$MSE\left(\bar{y}_{SA1}^{*}\right)-MSE\left(\bar{y}_{PR}^{*}\right)=\frac{(1-f)}{n}\bar{Y}^2\left[C_y^2+\delta g C_x^2\{\delta g-2K\}\right]$$
$$-\frac{(1-f)}{n}\bar{Y}^2\left[\begin{matrix}C_y^2+\delta g(2\eta-1)C_x^2\\ \{\delta g(2\eta-1)+2K\}\end{matrix}\right]>0$$
$$=\left[4\delta g\eta C_x^2\{\delta g(1-\eta)-K\}\right]>0.$$

This condition holds if

either $0<\eta<\left(1-\dfrac{K}{\delta g}\right)$, or $\left(1-\dfrac{K}{\delta g}\right)<\eta<0$. (3.6)

Therefore, the range of η under which the proposed dual to product-cum-dual to ratio type estimator $\bar{y}_{PR}^{*}$ is more efficient than dual to Shah and Patel (1984) and Singh and Agnihotri (2008) class of ratio estimators $\bar{y}_{SA1}^{*}$ if

$$\eta\in\left[\min\left\{0,1-\frac{K}{\delta g}\right\},\max\left\{0,1-\frac{K}{\delta g}\right\}\right]. \quad (3.7)$$

(iii) From (2.7) and (3.5), it is found that the proposed dual to product-cum-dual to ratio type estimator $\bar{y}_{PR}^{*}$ is more efficient than dual to Singh and Agnihotri (2008) class of product estimators $\bar{y}^*_{SA2}$ if

$$MSE(\bar{y}^*_{SA2}) - MSE(\bar{y}^*_{PR}) = \frac{(1-f)}{n}\bar{Y}^2\left[C_y^2 + \delta g C_x^2\{\delta g + 2K\}\right] - \frac{(1-f)}{n}\bar{Y}^2\left[\begin{matrix} C_y^2 + \delta g(2\eta-1) \\ C_x^2\{\delta g(2\eta-1) + 2K\}\end{matrix}\right] > 0$$

$$= 4\delta g(1-\eta)C_x^2\{\delta g\eta + K\} > 0.$$

This condition holds if

Either $1 < \eta < \left(-\frac{K}{\delta g}\right)$, or $\left(-\frac{K}{\delta g}\right) < \eta < 1$. (3.8)

Therefore, the range of η under which the proposed dual to product-cum-dual to ratio type estimator $\bar{y}^*_{PR}$ is more efficient than dual to Singh and Agnihotri (2008) class of product estimators $\bar{y}^*_{SA2}$ if

$$\eta \in \left[\min\left\{-\frac{K}{\delta g}, 1\right\}, \max\left\{-\frac{K}{\delta g}, 1\right\}\right]. \quad (3.9)$$

Remark 3.1 The conditions in which suggested dual to product-cum-dual to ratio type estimator $\bar{y}^*_{PR}$ is better than other existing estimators (discussed in Table 2.1) can be easily obtained by putting different values of (a, b, η, δ).

4 Illustration

(i) To illustrate the general results we consider $(a,b) = (1, C_x)$. For $(a,b) = (1, C_x)$ in (2.1), we get a class of estimators for population mean $\bar{Y}$ as

$$\bar{y}^*_{PR(1)} = \eta\bar{y}\left(\frac{\bar{X} + C_x}{\bar{x}^* + C_x}\right) + (1-\eta)\bar{y}\left(\frac{\bar{x}^* + C_x}{\bar{X} + C_x}\right), \quad (4.1)$$

where η and $\bar{x}^*$ are same as defined earlier.

The bias and *MSE* of $\bar{y}^*_{PR(1)}$ to the large sample approximation are respectively given as

$$B(\bar{y}^*_{PR(1)}) = \bar{Y}\frac{(1-f)}{n}\left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2\left[K(2\eta-1) + \eta\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g\right], \quad (4.2)$$

$$MSE(\bar{y}^*_{PR(1)}) = \bar{Y}^2\frac{(1-f)}{n}\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)C_x^2\left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1) + 2K\right\}\right]. \quad (4.3)$$

For $\eta = 0$, we get an estimator for $\bar{Y}$ as

$$\bar{y}^*_{PR(1)1} = \bar{y}\left(\frac{\bar{x}^* + C_x}{\bar{X} + C_x}\right), \quad (4.4)$$

which is due to Shah and Patel (1984) and Singh and Upadhyaya (1986). We note that the estimator $\bar{y}^*_{PR(1)1}$ is dual to Sisodia and Dwivedi's (1981) ratio-type estimator.

For $\eta = 1$, $\bar{y}^*_{PR(1)}$ reduces to the estimator for $\bar{Y}$ as

$$\bar{y}^*_{PR(1)2} = \bar{y}\left(\frac{\bar{X} + C_x}{\bar{x}^* + C_x}\right), \quad (4.5)$$

which is dual to product type estimator.

The biases and *MSEs* of $\bar{y}^*_{PR(1)1}$ and $\bar{y}^*_{PR(1)2}$ to the first degree of approximation are respectively given as

$$B(\bar{y}^*_{PR(1)1}) = -\bar{Y}\frac{(1-f)}{n}\left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2K, \quad (4.6)$$

$$B(\bar{y}^*_{PR(1)2}) = \bar{Y}\frac{(1-f)}{n}\left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2\left[\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g - K\right], \quad (4.7)$$

$$MSE(\bar{y}^*_{PR(1)1}) = \bar{Y}^2\frac{(1-f)}{n}\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2\left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g - 2K\right\}\right], \quad (4.8)$$

$$MSE(\bar{y}^*_{PR(1)2}) = \bar{Y}^2\frac{(1-f)}{n}\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2\left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g + 2K\right\}\right]. \quad (4.9)$$

It is observed from (4.3) that $MSE(\bar{y}^*_{PR(1)}) < Var(\bar{y})$ if

$$\left.\begin{matrix} either\ \frac{1}{2} < \eta < \left\{\frac{1}{2} - \frac{(\bar{X}+C_x)K}{g\bar{X}}\right\} \\ or\ \left\{\frac{1}{2} - \frac{(\bar{X}+C_x)K}{g\bar{X}}\right\} < \eta < \frac{1}{2}\end{matrix}\right\}. \quad (4.10)$$

From (1.11) and (4.3) we see that the proposed class of estimator $MSE(\bar{y}^*_{PR(1)})$ will dominate over Srivenkataramana (1980) estimator $\bar{y}^*_R$ if

$$\left.\begin{matrix} either\ -\frac{C_x}{2\bar{X}} < \eta < \frac{1}{2}\left\{1 + \left(1 + \frac{C_x}{\bar{X}}\right)\left(1 - \frac{2K}{g}\right)\right\} \\ or\ \frac{1}{2}\left\{1 + \left(1 + \frac{C_x}{\bar{X}}\right)\left(1 - \frac{2K}{g}\right)\right\} < \eta < -\frac{C_x}{2\bar{X}}\end{matrix}\right\}. \quad (4.11)$$

From (4.3) and (4.8) we note that

$MSE(\bar{y}^*_{PR(1)}) < MSE(\bar{y}^*_{PR(1)1})$ if

$$\left.\begin{matrix} either\ 0 < \eta < \left\{1 - \frac{K(\bar{X}+C_x)}{g\bar{X}}\right\} \\ or\ \left\{1 - \frac{K(\bar{X}+C_x)}{g\bar{X}}\right\} < \eta < 0\end{matrix}\right\}. \quad (4.12)$$

Further, from (4.3) and (4.9) we have that

$MSE(\bar{y}^*_{PR(1)}) < MSE(\bar{y}^*_{PR(1)2})$ if

$$\left.\begin{matrix} either\ -\frac{K(\bar{X}+C_x)}{g\bar{X}} < \eta < 1 \\ or\ 1 < \eta < -\frac{K(\bar{X}+C_x)}{g\bar{X}}\end{matrix}\right\}. \quad (4.13)$$

(ii) To illustrate the general results we consider $(a,b) = (1, S_x = \bar{X}C_x)$. For $(a,b) = (1, S_x)$ in (2.1), we get a class of estimators for population mean $\bar{Y}$ as

$$\bar{y}^*_{PR(2)} = \eta\bar{y}\left(\frac{\bar{X}+S_x}{\bar{x}^*+S_x}\right)+(1-\eta)\bar{y}\left(\frac{\bar{x}^*+S_x}{\bar{X}+S_x}\right), \qquad (4.14)$$

where η and $\bar{x}^*$ are same as defined earlier.

The bias and MSE of $\bar{y}^*_{PR(2)}$ to the first degree of approximation are respectively given as

$$B\left(\bar{y}^*_{PR(2)}\right) = \bar{Y}\frac{(1-f)}{n}\left(\frac{1}{1+C_x}\right)gC_x^2\left[K(2\eta-1)+\eta\left(\frac{1}{1+C_x}\right)g\right]. \qquad (4.15)$$

$$MSE\left(\bar{y}^*_{PR(2)}\right) = \bar{Y}^2\frac{(1-f)}{n}\left[\begin{array}{l} C_y^2+\left(\frac{1}{1+C_x}\right)g(2\eta-1) \\ C_x^2\left\{\left(\frac{1}{1+C_x}\right)g(2\eta-1)+2K\right\}\end{array}\right]. \qquad (4.16)$$

For $\eta = 0$, we get an estimator for $\bar{Y}$ as

$$\bar{y}^*_{PR(2)1} = \bar{y}\left(\frac{\bar{x}^*+S_x}{\bar{X}+S_x}\right), \qquad (4.17)$$

which is due to Shah and Gupta (1986). We note that the estimator $\bar{y}^*_{PR(2)1}$ is dual to Shah and Patel (1984) ratio type estimator. For $\eta = 1$, $\bar{y}^*_{PR(2)}$ reduces to the estimator for $\bar{Y}$ as

$$\bar{y}^*_{PR(2)2} = \bar{y}\left(\frac{\bar{X}+S_x}{\bar{x}^*+S_x}\right), \qquad (4.18)$$

which is dual to product-type estimator. The biases and $MSEs$ of $\bar{y}^*_{PR(2)1}$ and $\bar{y}^*_{PR(2)2}$ to the first degree of approximation are respectively given as

$$B\left(\bar{y}^*_{PR(2)1}\right) = -\bar{Y}\frac{(1-f)}{n}\left(\frac{1}{1+C_x}\right)gC_x^2K \qquad (4.19)$$

$$B\left(\bar{y}^*_{PR(2)2}\right) = \bar{Y}\frac{(1-f)}{n}\left(\frac{1}{1+C_x}\right)gC_x^2\left[\left(\frac{1}{1+C_x}\right)g-K\right], \qquad (4.20)$$

$$MSE\left(\bar{y}^*_{PR(2)1}\right) = \bar{Y}^2\frac{(1-f)}{n}\left[C_y^2+\left(\frac{1}{1+C_x}\right)gC_x^2\left\{\left(\frac{1}{1+C_x}\right)g-2K\right\}\right],$$

(4.21)

$$MSE\left(\bar{y}^*_{PR(2)2}\right) = \bar{Y}^2\frac{(1-f)}{n}\left[C_y^2+\left(\frac{1}{1+C_x}\right)gC_x^2\left\{\left(\frac{1}{1+C_x}\right)g+2K\right\}\right]. \qquad (4.22)$$

We note from (4.16) that the proposed class of estimators $\bar{y}^*_{PR(2)}$ is more efficient than the usual unbiased estimator $\bar{y}$ if

$$\left.\begin{array}{ll} either & \frac{1}{2} < \eta < \left\{\frac{1}{2}-\frac{(1+C_x)K}{g}\right\} \\ or & \left\{\frac{1}{2}-\frac{(1+C_x)K}{g}\right\} < \eta < \frac{1}{2}\end{array}\right\}. \qquad (4.23)$$

It follows from (1.11) and (4.16) that the suggested class of estimators $\bar{y}^*_{PR(2)}$ is better than Srivenkataramana (1980) estimator $\bar{y}^*_R$ if

$$\left.\begin{array}{ll} either & -\frac{C_x}{2} < \eta < \frac{1}{2}\left\{1+(1+C_x)\left(1-\frac{2K}{g}\right)\right\} \\ or & \frac{1}{2}\left\{1+(1+C_x)\left(1-\frac{2K}{g}\right)\right\} < \eta < -\frac{C_x}{2}\end{array}\right\}. \qquad (4.24)$$

From (4.16) and (4.21) that

$MSE(\bar{y}^*_{PR(2)}) < MSE(\bar{y}^*_{PR(2)1})$ if

$$\left.\begin{array}{ll} either & 0 < \eta < \left\{1-\frac{K(1+C_x)}{g}\right\} \\ or & \left\{1-\frac{K(1+C_x)}{g}\right\} < \eta < 0\end{array}\right\}. \qquad (4.25)$$

Further, it observed from (4.16) and (4.22) that $MSE(\bar{y}^*_{PR(2)}) < MSE(\bar{y}^*_{PR(2)2})$ if

$$\left.\begin{array}{ll} either & -\frac{K(1+C_x)}{g} < \eta < 1 \\ or & 1 < \eta < -\frac{K(1+C_x)}{g}\end{array}\right\}. \qquad (4.26)$$

5 Empirical Study

To illustrate the performance of the proposed class of estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ over usual unbiased estimator $\bar{y}$, ratio estimator $\bar{y}_R$, product estimator $\bar{y}_P$, $\bar{y}^*_{PR(1)1}$, $\bar{y}^*_{PR(1)2}$, $\bar{y}^*_{PR(2)1}$ and $\bar{y}^*_{PR(2)2}$, we consider two natural population data sets.

Population I: (Italian bureau for the environmental protection-APAT Waste 2004)

Y =Total amount (tons) of recyclable-waste collection in Italy in 2003 and

X = amount (tons) of recyclable-waste collection in Italy in 2002

$N = 103$, $n = 40$, $\bar{Y} = 626.2123$, $\bar{X} = 557.1909$, $\rho = 0.9936$, $C_y = 1.4588$, $C_x = 1.4683$.

Population II: (Italian bureau for the environmental protection- APAT Waste 2004)

Y =Total amount (tons) of recyclable-waste collection in Italy in 2003 and

X = Number of inhabitants in 2003

$N = 103$, $n = 40$, $\bar{Y} = 62.6212$, $\bar{X} = 556.5541$, $\rho = 0.7298$, $C_y = 1.4588$, $C_x = 1.0963$.

We have computed the percent relative efficiencies (*PREs*) of the estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ with respect to $\bar{y}$, $\bar{y}^*_R$, $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ by using the following formulae:

$$E_{11} = PRE(\bar{y}^*_{PR(1)}, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.1)$$

$$E_{12} = PRE(\bar{y}^*_{PR(1)}, \bar{y}^*_R) = \frac{[C_y^2 + C_x^2 g(g-2K)]}{\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.2)$$

$$E_{13} = PRE(\bar{y}^*_{PR(1)}, \bar{y}^*_{PR(1)1}) = \frac{\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)gC_x^2 \left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g-2K\right\}\right]}{\left[C_y^2 + \left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{\bar{X}}{\bar{X}+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.3)$$

$$E_{21} = PRE(\bar{y}^*_{PR(2)}, \bar{y}) = \frac{C_y^2}{\left[C_y^2 + \left(\frac{1}{1+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{1}{1+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.4)$$

$$E_{22} = PRE(\bar{y}^*_{PR(2)}, \bar{y}^*_R) = \frac{[C_y^2 + C_x^2 g(g-2K)]}{\left[C_y^2 + \left(\frac{1}{1+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{1}{1+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.5)$$

$$E_{23} = PRE(\bar{y}^*_{PR(2)}, \bar{y}^*_{PR(2)1}) = \frac{\left[C_y^2 + \left(\frac{1}{1+C_x}\right)gC_x^2 \left\{\left(\frac{1}{1+C_x}\right)g-2K\right\}\right]}{\left[C_y^2 + \left(\frac{1}{1+C_x}\right)g(2\eta-1)C_x^2 \left\{\left(\frac{1}{1+C_x}\right)g(2\eta-1)+2K\right\}\right]} \times 100 \quad (5.6)$$

Table 5.1 shows the percent relative efficiencies (*PREs*) of $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ with respect to the usual unbiased estimator $\bar{y}$ for different value of η.

Table 5.1: Range of η in which the proposed class of estimators $\bar{y}^*_{PR(1)}$ is better than $\bar{y}$, $\bar{y}^*_R$ and $\bar{y}^*_{PR(1)1}$.

Estimator	Range of η	
	Population I	**Population II**
$\bar{y}$	(-1.727, 0.500)	(-1.689, 0.500)
$\bar{y}^*_R$	(-1.226, -0.001)	(-1.188, -0.001)
$\bar{y}^*_{PR(1)1}$	(-1.227, 0.000)	(-1.189, 0.000)

We note that the correlation between study variable y and x is positive for both the population data sets I and II. Owing to this the proposed classes of estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ are comparable with usual unbiased estimator $\bar{y}$, dual to ratio estimator $\bar{y}^*_R$, Shah and Patel (1984) and Singh and Upadhyaya (1986) estimator $\bar{y}^*_{PR(1)1}$ and Shah and Gupta (1986) estimator $\bar{y}^*_{PR(2)1}$.

Table 5.2: Range of η in which the proposed class of estimators $\bar{y}^*_{PR(2)}$ is better than $\bar{y}$, $\bar{y}^*_R$ and $\bar{y}^*_{PR(2)1}$.

Estimator	Range of η	
	Population I	**Population II**
$\bar{y}$	(-4.982, 0.500)	(-4.080, 0.500)
$\bar{y}^*_R$	(-3.014, -0.734)	(-3.032, -0.548)
$\bar{y}^*_{PR(2)1}$	(-4.482, 0.000)	(-3.580, 0.000)

Tables 5.1 and 5.2 show that there is enough scope of selecting the value of scalar η in obtaining estimators better than $\bar{y}$, $\bar{y}^*_R$ $\bar{y}^*_{PR(1)1}$ and $\bar{y}^*_{PR(2)1}$. That is, even if the scalar η deviates from its true optimum value η_{opt}, considerable gain in efficiency by using proposed classes of estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ over other existing estimators can be obtained. Largest gain in efficiency is observed at the optimum value η_{opt} of η. Tables 5.3 and 5.4 exhibit that there is appreciable gain in efficiency by using the proposed class of estimators $\bar{y}^*_{PR(i)}$, i=1, 2; over usual unbiased estimator $\bar{y}$, Srivenkataramana's (1980) dual to ratio estimator $\bar{y}^*_R$, Shah and Patel (1984) and Singh and Upadhyaya (1986) estimator $\bar{y}^*_{PR(1)1}$ and Shah and Gupta (1986) estimator $\bar{y}^*_{PR(2)1}$. Findings closed in Tables 5.1 to 5.4 are in the favour of proposed classes of estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$. Thus we recommend the proposed classes of estimators $\bar{y}^*_{PR(1)}$ and $\bar{y}^*_{PR(2)}$ for their use in practice.

Table 5.3: *PREs* of the proposed class of estimators $\bar{y}^*_{PR(1)}$ with respect to $\bar{y}$, $\bar{y}^*_R$ and $\bar{y}^*_{PR(1)1}$ for different values of η.

Population I				Population II			
η	E_{11}	E_{12}	E_{13}	η	E_{11}	E_{12}	E_{13}
-1.500	156.61	*	*	-1.887	*	*	*
-1.276	276.06	*	167.56	-1.686	100.32	*	*
-1.250	298.20	*	181.00	-1.500	120.23	*	*
-1.225	322.05	100.21	195.48	-1.250	151.91	*	119.12
-1.000	759.26	236.25	460.86	-1.000	185.04	115.47	145.10
-0.750	3623.70	1127.55	2199.53	-0.750	209.15	130.52	164.01
-0.500	4344.91	1351.96	2637.29	-0.500	212.15	132.39	166.36
-0.250	847.71	263.77	514.54	-0.250	192.24	119.97	150.75
0.000	320.06	*	194.27	0.000	160.11	*	125.56
0.250	164.89	*	100.09	0.250	127.47	*	*
0.500	100.00	*	*	0.500	100.00	*	*
-0.613	**7837.46**	**2438.71**	**4757.22**	**-0.595**	**213.95**	**133.52**	**167.78**

**Stands for PREs less than 100%.*

Table 5.4: *PREs* of the proposed class of estimators $\bar{y}^*_{PR(2)}$ with respect to $\bar{y}$, $\bar{y}^*_R$ and $\bar{y}^*_{PR(2)1}$ for different values of η.

Population I				Population II			
η	E_{21}	E_{22}	E_{23}	η	E_{21}	E_{22}	E_{23}
-5.000	*	*	*	-4.050	101.43	*	*
-4.890	107.00	*	*	-4.000	103.82	*	*
-4.500	146.39	*	*	-3.750	116.63	*	*
-4.480	148.97	*	100.21	-3.560	127.32	*	100.94
-3.010	1106.07	344.17	744.05	-3.500	130.85	*	103.74
-3.000	1131.18	351.98	760.94	-3.030	160.39	100.09	127.16
-2.000	4901.13	1525.04	3296.98	-3.000	162.33	101.30	128.70
-1.750	2249.19	699.86	1513.02	-2.000	211.93	132.25	168.02
-1.500	1177.30	366.33	791.97	-1.000	188.39	117.57	149.36
-1.000	464.75	144.61	312.64	-0.750	173.23	108.10	137.34
-0.750	327.96	102.05	220.62	-0.546	160.11	*	126.94
-0.500	243.26	*	163.64	-0.500	157.13	*	124.57
-0.250	187.38	*	126.05	-0.250	141.19	*	111.94
0.000	148.66	*	100.00	0.000	126.13	*	100.00
0.250	120.75	*	81.23	0.250	112.35	*	*
0.500	100.00	*	67.27	0.500	100.00	*	*
-2.241	**7837.58**	**2438.74**	**5272.32**	**-1.790**	**213.95**	**133.52**	**169.63**

**Stands for PREs less than 100%.*

6 Conclusion

This paper proposed a general class of dual to product-cum-dual to ratio type estimator $\bar{y}^*_{PR}$ estimator based on the transformation in an auxiliary variable x in simple random sampling. The envisaged class of estimators includes several known estimators based on transformation in auxiliary variable x. The bias and *MSE* expressions of the proposed class of estimators have been obtained under large sample approximation. It is interesting to note that this study unifies several estimators with their properties at one place. Empirical study also shows the superiority of the proposed class of estimators over other existing estimators.

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