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Sufficient Conditions for Existence and Uniqueness of Solutions to Fractional Order Multi-Point Boundary Value Problems

*Salman Zeb** and Rahmat Ali Khan*

Department of Mathematics, University of Malakand, Chakdara, Dir (Lower), Khyber Pakhtunkhwa, Pakistan

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Abstract: In this article, the following fractional order multi-point boundary value problem

$$-{}^{c}D^{q}u(t) = f(t,u(t)); \quad t \in J = [0,1], 1 < q \le 2,$$

$$u(0) = g(u(\xi)), \quad {}^{c}D^{p}u(1) - \sum_{i=1}^{m-2} \delta_i u(\eta_i) = h(u(\eta)), \; 0 < p \le 1,$$

is considered, where $\xi, \eta, \delta_i, \eta_i \in (0,1)$ $g, h \in C(J,\mathbb{R})$ are given functions and $\sum_{i=1}^{m-2} \delta_i \eta_i < 1$; $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^{c}D^{q}$ is the Caputo derivative of fractional order q. The notation ${}^{c}D^{p}u(1)$ means the value of ${}^{c}D^{p}u(t)$ at $t = 1$. We use topological degree theory approach to establish sufficient conditions for existence and uniqueness of solutions. We provide an example to show the usefulness of our results.

Keywords: Fractional differential equations, boundary value problems, Caputo fractional derivative, Green's function, topological degree theory.

1 Introduction

The rapidly growing applications of fractional order differential equations in various fields of sciences have attracted the attentions of many researchers. This can be attributed largely due to rapid advances in the theory of fractional calculus and its wide range of applications in real life problems. These applications can be found in various scientific and engineering disciplines, for details, see [1,2,3,4]. Furthermore, its applicability in modeling real world phenomena have led the researchers to show great concern about the existence and uniqueness results. For details, the readers are referred to [5,6,7,8].

The existence and uniqueness of solutions of multi-point boundary value problems are studied quite recently by means of classical fixed point theorems such as Banach contraction principle, Schauder fixed point theorem, and Leray-Schauder degree etc. in [7,9,10,11,12,13,14,15]. The application of the above mentioned fixed point theorems require strong condition such as compactness of the corresponding operator. The non compact cases can not be covered under these results. For the generalization of the theory of existence and uniqueness to cover the case of non compact operators as well, the approach of coincidence degree theory for condensing maps has already been used, we refer to the recent work studied in [16,17,18,19,20].

Wang et al [18], considered some classes of nonlocal Cauchy problems via topological degree method and develop its existence and data dependence results. Chen et al [19] studied sufficient conditions for existence results via coincidence

* Corresponding author e-mail: zebsalman@gmail.com

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degree theory approach for the following p-Laplacian operator problem

$$
\begin{gathered}
{}^cD^\alpha \phi_p \left({}^cD^\beta u(t)\right) = f\left(t, u(t), {}^cD^\beta u(t)\right) \\
{}^cD^\beta u(0) = 0, {}^cD^\beta u(1) = 0,
\end{gathered}
$$

where ${}^cD^\alpha$ and ${}^cD^\beta$ represents caputo derivatives, $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2$. Tang et al [20] applied coincidence theory and established existence results for

$$
\begin{gathered}
{}^cD^\alpha \phi_p \left({}^cD^\beta u(t)\right) = f\left(t, u(t), {}^cD^\beta u(t)\right) \\
u(0) = 0, \quad {}^cD^\beta u(0) = {}^c D^\beta u(1),
\end{gathered}
$$

where ${}^cD^\alpha$ and ${}^cD^\beta$ are caputo derivatives, $0 < \alpha, \beta \leq 1, 1 < \alpha + \beta \leq 2$.

The present article is a motivation from the above mentioned work. Here, we used the approach of the coincidence degree theory for condensing maps and carry out the investigations for the following BVP

$$
\begin{aligned}
&-{}^cD^q u(t) = f(t, u(t)); \quad t \in J, 1 < q \leq 2, \\
&u(0) = g(u(\xi)), \quad {}^cD^p u(1) - \sum_{i=1}^{m-2} \delta_i u(\eta_i) = h(u(\eta)), \; 0 < p \leq 1,
\end{aligned} \tag{1.1}
$$

where g, h are given functions and $\sum_{i=1}^{m-2} \delta_i \eta_i < 1$; f is a continuous function, $\xi, \eta, \delta_i, \eta_i \in (0,1)$ and the notation ${}^cD^p u(1)$ stands for the value of ${}^cD^p u(t)$ at $t = 1$. Finally, the results have been demonstrated with the help of an example.

2 Background Materials

Here, we represent the Banach spaces by X and the family of all its bounded sets will be denoted by $\mathbb{B} \in P(X)$. We state some important definitions and lemmas. For details, we refer to [2,3,4,21,22].

Definition 2.1 *Let $y \in L^1([a,b])$, then the integral of fractional order is defined by*

$$
I^q y(t) = \frac{1}{\Gamma(q)} \int_a^t \frac{y(s)}{(t-s)^{1-q}} ds, \; \text{where } q \in \mathbb{R}_+.
$$

Definition 2.2 *Let $y \in C^n[a,b]$ be a function, then its Caputo derivative is represented by*

$$
{}^cD^q y(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{y(s)}{(t-s)^{q-n+1}} ds, \; \text{where } n = [q]+1,
$$

and $[q]$ represents the integer part of q.

Lemma 2.3 *Let $q > 0$, then*

$$
y(t) = c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},
$$

is the solution of ${}^cD^q y(t) = 0$ for some $c_i \in R, i = 0, 1, 2, ..., n-1$.

Lemma 2.4 *For a fractional derivative and integral of order q, we have*

$$
I^q({}^cD^q) y(t) = y(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1} t^{n-1},
$$

where $c_i \in R, \quad i = 0, 1, 2, ..., n-1$.

We recall some important definitions, propositions and theorems from [22].

Definition 2.5 *We define the function $\alpha : \mathbb{B} \to \mathbb{R}_+$ as*

$$
\alpha(B) = inf \{d > 0\},
$$

where $B \in \mathbb{B}$ admits a finite cover by sets of diameter $\leq d$ and α is Kuratowski measure of noncompactness.

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Proposition 2.6 *The function* α *satisfy the following properties:*

(i) $\alpha(B) = 0$, *(if and only if* B *is relatively compact).*
(ii) $\alpha(B_1) + \alpha(B_2) \geq \alpha(B_1 + B_2)$.
(iii) *If* $B_1 \subset B_2$ *then* $\alpha(B_2) \geq \alpha(B_1)$.
(iv) $\alpha(convB) = \alpha(B)$.
(v) $\alpha(B) = \alpha(\bar{B})$.

Recall that for $K > 0$, the condition for the function $F : \Omega \to X$ to be Lipschitz is

$$\|F(x) - F(y)\| \leq K\|x - y\|,$$

and if $K < 1$, then F is a strict contraction.

Definition 2.7 *Let the function* $F : \Omega \to X$ *be a continuous bounded map, where* $\Omega \subset X$. *Then for* F *to be* α*-Lipschitz, we have*

$$\alpha(F(B)) \leq K\alpha(B), \text{ where } K \geq 0.$$

Further, F *will be strict* α*-contraction if* $K < 1$.

Definition 2.8 *Let* $\alpha(B) > 0$, *then for* F *to be* α*-condensing, we need*

$$\alpha(F(B)) < \alpha(B).$$

Moreover, $\alpha(B) = 0$, *if* $\alpha(B) \leq \alpha(F(B))$.

Here, we consider the following:
Let $\Theta C_\alpha(\Omega)$ be the class of all strict α-contractions F and $C_\alpha(\Omega)$ be the class of all α-condensing maps F, where $F : \Omega \to X$.

Remark 2.9 *For constant* $K = 1$, *every* $F \in C_\alpha(\Omega)$ *is* α*-Lipschitz and* $\Theta C_\alpha(\Omega) \subset C_\alpha(\Omega)$.

Proposition 2.10 *Let for constants* K *and* $K^{'}$, F *and* G *are* α*-Lipschitz maps respectively, then* $F + G$ *are also* α*-Lipschitz with constant* $K + K^{'}$.

Proposition 2.11 *if function* F *is compact, then* F *will be* α*-Lipschitz having constant* $K = 0$.

Proposition 2.12 *The function* F *will be* α*-Lipschitz having same constant* K *provided* F *is a Lipschitz function with constant* K.

Theorem 2.13[23] *Consider* $\Theta = \{x \in X : \text{such that } x = \lambda F x \text{ where } \lambda \in [0,1]\}$ *where* $F : X \to X$ *is* α*-condensing, such that* $\Theta \subset B_r(0)$, *where* $r > 0$ *and* Θ *is a bounded set in* X, *then the degree is defined as*

$$D(I - \lambda F, B_r(0), 0) = 1, \forall \lambda \in [0,1].$$

This implies that F *has at least one fixed point and the set of these fixed points of* F *lies in* $B_r(0)$.

3 Main Results

To obtain the main results for BVP (1.1), we define $\Delta = \sum_{i=1}^{m-2} \delta_i \eta_i$ and assume throughout the paper that $\Delta < 1$.

Lemma 3.1 *For* $y \in L^1(J, \mathbb{R})$, *the BVP*

$$\begin{aligned} &{}^cD^q u(t) + y(t) = 0; \quad t \in J = [0,1], 1 < q \leq 2, \\ &u(0) = g(u(\xi)), \quad {}^cD^p u(1) - \sum_{i=1}^{m-2} \delta_i u(\eta_i) = h(u(\eta)), \ 0 < p \leq 1, \end{aligned} \tag{3.1}$$

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has a solution $u(t)=g(u(\xi))+\frac{\Gamma(2-p)\left(h(u(\eta))+\sum_{i=1}^{m-2}\delta_i g(u(\xi))\right)}{\Gamma(2)-\Gamma(2-p)\Delta}t+\int_0^1 G(t,s)y(s)\,ds$, *where*

$$G(t,s)=\begin{cases}\frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\left[\frac{(1-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{j=i}^{m-2}\delta_j}{\Gamma(q)}(\eta_j-s)^{q-1}\right]-\frac{1}{\Gamma(q)}(t-s)^{q-1}; & s\le t,\ \eta_{i-1}<s\le\eta_i,\\ & i=1,2,...,m-1,\\ \frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\left[\frac{(1-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{j=i}^{m-2}\delta_j}{\Gamma(q)}(\eta_j-s)^{q-1}\right]; & t\le s,\ \eta_{i-1}<s\le\eta_i,\\ & i=1,2,...,m-1.\end{cases}\tag{3.2}$$

Proof. We divide the boundary value problem (3.1) into two parts: (i) Non-homogeneous part of the equation with homogeneous conditions, and (ii) Homogeneous equation having non-homogeneous boundary conditions. Consider the first case, apply I^q on the fractional differential equation $-{}^cD^qu(t)=y(t)$ and from (2.4), we have

$$u(t)=-I^qy(t)+c_0+c_1t,\, c_0,c_1\in R.\tag{3.3}$$

Hence, it follows that

$${}^cD^pu(t)=-I^{q-p}y(t)+c_1\frac{\Gamma(2)}{\Gamma(2-p)}t^{1-p}.$$

Applying $u(0)=0$ gives $c_0=0$ and the boundary condition ${}^cD^pu(1)-\sum_{i=1}^{m-2}\delta_iu(\eta_i)=0$, implies

$$-I^{q-p}y(1)+c_1\frac{\Gamma(2)}{\Gamma(2-p)}=\sum_{i=1}^{m-2}\delta_i\Big[-I^qy(\eta_i)+c_0+c_1\eta_i\Big],$$

from which it follows that $c_1=\frac{\Gamma(2-p)}{\Gamma(2)-\Delta\Gamma(2-p)}\Big[I^{q-p}y(1)-\sum_{i=1}^{m-2}\delta_iI^qy(\eta_i)\Big]$. Hence, we obtain

$$u(t)=-I^qy(t)+\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big[I^{q-p}y(1)-\sum_{i=1}^{m-2}\delta_iI^qy(\eta_i)\Big].\tag{3.4}$$

For $0\le t\le\eta_1$, equation (3.4) can be rewritten as

$$\begin{aligned}u(t)=&\int_0^t\Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{j=1}^{m-2}\delta_j}{\Gamma(q)}(\eta_j-s)^{q-1}\Big)-\frac{(t-s)^{q-1}}{\Gamma(q)}\Big]y(s)\,ds\\&+\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\int_t^{\eta_1}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{j=1}^{m-2}\delta_j}{\Gamma(q)}(\eta_j-s)^{q-1}\Big)y(s)\,ds\\&+\sum_{i=2}^{m-2}\int_{\eta_{i-1}}^{\eta_i}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{j=1}^{m-2}\delta_j}{\Gamma(q)}(\eta_j-s)^{q-1}\Big)y(s)\,ds+\frac{1}{\Gamma(q-p)}\int_{\eta_{m-2}}^1\frac{y(s)}{(1-s)^{1+p-q}}ds,\end{aligned}$$

for $\eta_{l-1} \le t \le \eta_l$, $2 \le l \le m-2$, we obtain

$$
\begin{aligned}
u(t) = & \int_0^{\eta_1} \Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=1}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) - \frac{(t-s)^{q-1}}{\Gamma(q)}\Big] y(s)\,ds \\
& + \sum_{i=2}^{m-2}\int_{\eta_{i-1}}^{\eta_i} \Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=i}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) + (1-s)^{q-p-1} - \frac{(t-s)^{q-p-1}}{\Gamma(q)}\Big] y(s)\,ds \\
& + \int_{\eta_{l-1}}^{t} \Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=l}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) - \frac{(t-s)^{q-1}}{\Gamma(q)}\Big] y(s)\,ds \\
& + \frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big[\int_t^{\eta_l}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=l}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big)\Big] y(s)\,ds \\
& + \sum_{i=l+1}^{m-2}\int_{\eta_{i-1}}^{\eta_i}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=i}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) y(s)\,ds + \frac{1}{\Gamma(q-p)}\int_{\eta_{m-2}}^{1}\frac{y(s)}{(1-s)^{1+p-q}}\,ds.
\end{aligned}
$$

Further, for $\eta_{m-2} \le t \le 1$, we have

$$
\begin{aligned}
u(t) = & \int_0^{\eta_1} \Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=1}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) - \frac{(t-s)^{q-1}}{\Gamma(q)}\Big] y(s)\,ds \\
& + \sum_{i=2}^{m-2}\int_{\eta_{i-1}}^{\eta_i} \Big[\frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big(\frac{(1-s)^{q-p-1}}{\Gamma(q-p)} - \frac{\sum\limits_{j=i}^{m-2}\delta_j}{\Gamma(q)}(\eta_j - s)^{q-1}\Big) - \frac{(t-s)^{q-1}}{\Gamma(q)}\Big] y(s)\,ds \\
& + \frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(q)\Gamma(2-p)\Delta}\times \\
& \Big[\int_{\eta_{m-2}}^{t}\big(\frac{\Gamma(2)-\Gamma(2-p)\Delta}{\Gamma(2-p)t}(t-s)^{q-1} + \Gamma(q)(1-s)^{q-p-1}\big) y(s)\,ds + \frac{1}{\Gamma(q-p)}\int_t^1 \frac{y(s)}{(1-s)^{1+p-q}}\,ds\Big].
\end{aligned}
$$

Hence, the solution having homogeneous boundary conditions and non-homogeneous part of (3.1) is given as

$$
\begin{aligned}
u(t) = & \frac{\Gamma(2-p)t}{\Gamma(2)-\Delta\Gamma(2-p)}\Big[\frac{1}{\Gamma(q-p)}\int_0^1 (1-s)^{q-p-1} y(s)\,ds - \frac{\sum\limits_{j=i}^{m-2}\delta_i}{\Gamma(q)}\int_0^{\eta_j}(\eta_j - s)^{q-1} y(s)\,ds\Big] \\
& - \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} y(s)\,ds = \int_0^1 G(t,s) y(s)\,ds.
\end{aligned} \tag{3.5}
$$

Now, consider the homogeneous part of the equation (3.1) with non-homogeneous boundary conditions $u(0) = g(u(\xi))$, and ${}^cD^p u(1) - \sum\limits_{i=1}^{m-2}\delta_i u(\eta_i) = h(u(\eta))$. Apply I^q on the fractional differential equation ${}^cD^q u(t) = 0$, we obtain

$$
u(t) = c_2 + c_3 t,\; c_2, c_3 \in R. \tag{3.6}
$$

Applying $u(0) = g\big(u(\xi)\big)$ gives $c_2 = g\big(u(\xi)\big)$. Also, the second boundary condition ${}^cD^p u(1) - \sum\limits_{i=1}^{m-2}\delta_i u(\eta_i) = h\big(u(\eta)\big)$ yields

$$
c_3 = \frac{\Gamma(2-p)}{\Gamma(2)-\Gamma(2-p)\Delta}\Big[h\big(u(\eta)\big) + \sum_{i=1}^{m-2}\delta_i g\big(u(\xi)\big)\Big].
$$

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Hence, we get

$$u(t)=g\big(u(\xi)\big)+\frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\Big[h\big(u(\eta)\big)+\sum_{i=1}^{m-2}\delta_i\big(u(\xi)\big)\Big]. \tag{3.7}$$

Combining (3.5) and (3.7), we obtain

$$u(t)=\int_0^1 G(t,s)y(s)\,ds+g\big(u(\xi)\big)+\frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\Big[h\big(u(\eta)\big)+\sum_{i=1}^{m-2}\delta_i\big(u(\xi)\big)\Big]. \tag{3.8}$$

Now, from (3.1), the BVP (1.1) has a solution of the form

$$u(t)=g\big(u(\xi)\big)+\frac{\Gamma(2-p)\big(h\big(u(\eta)\big)+\sum\limits_{i=1}^{m-2}\delta_i g\big(u(\xi)\big)\big)}{\Gamma(2)-\Delta\Gamma(2-p)}t+\int_0^1 G(t,s)f\big(s,u(s)\big)ds \tag{3.9}$$

which is an integral representation of the BVP (1.1). All we need to show that the equation (3.9) has a solution. Define the following operators $F,G,T:X\to X$ as follows:

$$(Fu)(t)=g\left(u(\xi)\right)+\frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\Big[h\big(u(\eta)\big)+\sum_{i=1}^{m-2}\delta_i\big(u(\xi)\big)\Big]$$

$$(Gu)(t)=\int_0^1 G(t,s)f\left(s,u(s)\right)ds,\ \text{and } T=F+G$$

The operator T is well defined as f is continuous. We write the integral equation (3.9) as an operator equation

$$u=Tu=Fu+Gu. \tag{3.10}$$

The solution of the equation (3.9) will be the fixed point of T. From now onward, we assume that there exists constants K_g,C_g,q_1,K_h,C_h,q_2, and $C_f,q_3\in[0,1)$ and $u,v\in X$ such that the following holds:

$(A1)$ $|g(u)-g(v)|\le\frac{K_g}{E}\|u-v\|$, where $E=\Big(1+\frac{\Gamma(2-p)t\sum_{i=1}^{m-2}\delta_i}{\Gamma(2)-\Gamma(2-p)\Delta}\Big)$.
$(A2)$ $|g(u)|\le C_g\|u\|^{q_1}$.
$(A3)$ $|h(u)-h(v)|\le K_h\|u-v\|$.
$(A4)$ $|h(u)|\le C_h\|u\|^{q_2}$.
$(A5)$ $|f(t,u(s))|\le C_f\|u\|^{q_3}$.

Lemma 3.2 *The operator $F:X\to X$ satisfies the Lipschitz and α-Lipschitz conditions with same constant K_F. Moreover, the operator F satisfies*

$$\|Fu\|\le C_g\|u\|^{q_1}+C_h\|u\|^{q_2},\ \textit{for every}\quad u\in X. \tag{3.11}$$

*Proof.*From $(A1)$ and $(A3)$, and for every $u,v\in X$, we obtain

$$\Big|Fu-Fv\Big|\le K_F\|u-v\|,\quad \text{where}\quad K_F=K_g+K_h.$$

Thus, from proposition (2.12), we conclude that F is also α-Lipschitz with same constant K_F .

For growth condition, we consider $(Fu)(t)=g(u(\xi))+\frac{\Gamma(2-p)\Big(h(u(\eta))+\sum_{i=1}^{m-2}\delta_i g(u(\xi))\Big)}{\Gamma(2)-\Gamma(2-p)\Delta}t$, and use the assumptions $(A2)$ and $(A4)$, we get

$$\|Fu\|=C_g\|u\|^{q_1}+C_h\|u\|^{q_2}.$$

Lemma 3.3 *The operator $G:C(J,\mathbb{R})\to C(J,\mathbb{R})$ is continuous. It also satisfies*

$$\|Gu\|\le\frac{\Gamma(2-p)\left(\Gamma(q+1)+\Gamma(q-p+1)\right)+\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}{\Gamma(q+1)\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}C_f\|u\|^{q_3}, \tag{3.12}$$

for every $u\in X$.

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Proof. Let us consider $u_n \to u$ in $\overline{B}$, where $\{u_n\}$ is a sequence of bounded set $\overline{B} = \{\|u\| \leq \kappa : u \in X\}$. To show that $\|Gu_n - Gu\| \to 0$ as $n \to \infty$, we consider the following:
Since f is continuous and $u_n \to u$, it follows that $f(s, u_n(s)) \to f(s, u(s))$ as $n \to \infty$. Now consider the difference $|(Gu_n)(t) - (Gu)(t)|$, and by using assumption $(A5)$ and the Lebesgue Dominated Convergence theorem, we get

$$\|(Gu_n)(t) - (Gu)(t)\| \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that G is continuous. Furthermore, we obtain the growth condition (3.12) by using assumption $(A5)$.

Lemma 3.4 *The operator $G : X \to X$ is compact. It further implies that G is α-Lipschitz having zero constant.*

Proof. Take a bounded set $\mathscr{D} \subset \overline{B} \subseteq X$ and a sequence $\{u_n\}$ in $\mathscr{D} \subset \overline{B}$, then in view of (3.12), it follows that

$$\|Gu_n\| \leq \frac{\Gamma(2-p)\left(\Gamma(q+1)+\Gamma(q-p+1)\right)+\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}{\Gamma(q+1)\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}\left(C_f \kappa^{q_3}\right),$$

which means that $G(\mathscr{D})$ is bounded in X. Now, we need to show that $\{Gu_n\}$ is equicontinuous, For this, consider $0 \leq t_1 < t_2 \leq T$, then

$$|(Gu_n)(t_1) - (Gu)(t_2)| \leq \left[\frac{1}{\Gamma(q+1)} + \frac{2(t_2-t_1)^q}{\Gamma(q+1)} + \frac{\Gamma(2-p)(t_2-t_1)}{\Gamma(2)-\Gamma(2-p)\Delta}\left(\frac{1}{\Gamma(q-p+1)}\right)\right]\left(C_f \kappa^{q_3}\right).$$

The above relation approaches zero when $t_2 \to t_1$, which implies $\{Gu_n\}$ is equicontinuous. From Arzela-Ascoli theorem, we safely concluded that $G(\mathscr{D})$ is relatively compact in X which in view of the proposition (2.11) implies that G is α-Lipschitz wtih constant zero.

Theorem 3.5 *The equation* (1.1) *has at least one solution $u \in X$ if $(A2), (A4)$, and $(A5)$ holds. Moreover, the solutions set of the BVP* (1.1) *is bounded.*

*Proof.*By proposition (2.10), the operator T is strict α-contraction. Now setting

$$S_0 = \{u \in X : \text{ such that } u = \lambda Tu, \text{ where } \lambda \in [0,1]\}.$$

In order to show that S_0 is bounded in X, Let $u \in S_0$, then we have

$$|u| \leq \lambda\left(\|Fu\| + \|Gu\|\right),$$

which in view of (3.11) and (3.12), implies that

$$\|u\| \leq \lambda\left[C_g\|u\|^{q_1} + C_h\|u\|^{q_2} + \frac{\Gamma(2-p)(\Gamma(q+1)+\Gamma(q-p+1))+\Gamma(q-p+1)(\Gamma(2)-\Gamma(2-p)\Delta)}{\Gamma(q+1)\Gamma(q-p+1)(\Gamma(2)-\Gamma(2-p)\Delta)}\left(C_f\|u\|^{q_3}\right)\right]. \tag{3.13}$$

Hence, the inequality (3.13) together with $q_1 < 1$, $q_2 < 1$ and $q_3 < 1$ implies that S_0 is bounded in X. By theorem (2.13), we obtain the required result.

Define

$$H = \left\{\frac{\Gamma(2-p)\left(\Gamma(q+1)+\Gamma(q-p+1)\right)+\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}{\Gamma(q+1)\Gamma(q-p+1)\left(\Gamma(2)-\Gamma(2-p)\Delta\right)}\right\},$$

and assume that

$(A6)$ $|f(t,u) - f(t,v)| \leq \frac{L_f}{H}|u-v|$, where $L_f \in [0,1)$.

Theorem 3.6 *Under the assumptions $(A1), (A3)$, and $(A6)$, the BVP* (1.1) *has unique solution in $C(J,\mathbb{R})$.*

Proof. We use Banach contraction principle. From (3.10), we have

$$|(Tu)(t)-(Tv)(t)| = \left|\left(1+\frac{\Gamma(2-p)t\sum_{i=1}^{m-2}\delta_i}{\Gamma(2)-\Gamma(2-p)\Delta}\right)\left(g(u(\xi))-g(v(\xi))\right)+\frac{\Gamma(2-p)t}{\Gamma(2)-\Gamma(2-p)\Delta}\left(h(u(\eta))-h(v(\eta))\right)+\int_0^1 G(t,s)\left(f(s,u(s))-f(s,v(s))\right)ds\right|.$$

Using the assumptions $(A1)$, $(A3)$, and $(A6)$, we have

$$|(Tu)(t)-(Tv)(t)| \leq M\|u-v\|,$$

where $M = K_g + K_h + L_f < 1$. Hence, we get the required result.

4 Example

Consider the BVP

$$\begin{aligned} &{}^cD^q u(t) = \frac{u(t)}{(1+8\cos t)}, \\ &u(0) = g(u(\xi)) = \frac{1}{4}, \quad {}^cD^p u(1) = \sum_{i=1}^{m-2} \delta_i u(\eta_i) + h(u(\eta)) = \frac{1}{5}. \end{aligned} \tag{4.1}$$

Here, we take $q = \frac{2}{3}$, $q_1 = q_2 = q_3 = \frac{1}{2}$, $L_f = C_f = \frac{1}{9}$, $C_g = K_g = K_h = C_h = \frac{1}{5}$, $p = \frac{1}{3}$, $\Delta = \frac{1}{3}$ and $\sum_{i=1}^{m-2} \delta_i = \frac{1}{3}$, $h(u(\eta)) = \frac{-2}{15}$. Then the BVP (4.1) satisfies the assumptions $(A1) - (A6)$. Also, the fixed point u is bounded as

$$\|u\| \leq \left(\frac{23}{90}\right)^2.$$

Hence, by Theorem (3.5) their exists at least one solution for (4.1). Also, it satisfies the condition for uniqueness given in (3.6).

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