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T_χ-Fibrations in Homotopy Theory for Topological Semigroups

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Abstract: The concept of T_χ-fibration map is introduced which generalized the notion of S_χ-fibrations in homotopy theory for topological semigroups. The composition property, pullback maps, and covering homotopy theorem are discussed for T_χ-fibrations. Furthermore, we extended the notion of approximate fibrations to topological semigroups and showed its relation with T_χ-fibrations.

Keywords: Topological semigroup, Fibration, Homotopy relation.

1 Introduction

The homotopy theory of topological spaces attempts to classify weak homotopy types of spaces and homotopy classes of maps. The classification of maps within a homotopy is a central problem in topology and several authors contributed in this area, see for example the related works in [6]. The concepts of Hurewicz fibrations, [7], in this theory have played very important roles for investigating the mutual relations of among the objects. For this purpose Coram and Duvall [3] introduced an approximate fibration as a map having the approximate homotopy lifting property for every space, which is a generalization of a Hurewicz fibration having valuable properties similar to the Hurewicz fibration and is widely applicable to the maps whose fibers are nontrivial shapes. A map $f : S \to B$ of compact metrizable spaces S and O is called an *approximate fibration* if for every space Z and for given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $g : Z \to S$ and $G : Z \times I \to O$ are maps with $d[G(z,0),(f \circ g)(z)] < \delta$, then there exists a homotopy $H : Z \times I \to S$ of $Z \times I$ into S such that $H_0 = g$ and $d[G(z,t),(f \circ H)(z,t)] < \varepsilon$ for all $z \in Z, t \in I$.

The concept of homotopy theory for topological semigroups and most of the backgrounds for this paper have been worked out previously by Zvonko in 2002, [8]. He introduced the concepts of $S-$homotopy relation, pathwise $S-$connectedness, $S-$homotopy domination, $S-$contractibility and $S_\chi-$fibration.

This paper is organized as follows: It consists of seven sections. Section 2 is devoted to some preliminaries. In Section 3, we start by giving the concepts of st-spaces and st-maps in homotopy theory for topological semigroups. Some properties for their are proved. In Section 4, we define an T_χ-fibration and study some its basic properties. In Section 5 we prove that the pullbacks of T_χ-fibrations are T_χ-fibrations. In Section 6, we give and prove the covering homotopy theorem for st-maps into T_χ-fibrations. In Section 7, we first define the $S_\chi-$approximate fibration property in homotopy theory for topological semigroups. Next we give and prove the relation between $S_\chi-$approximate fibrations and T_χ-fibrations.

2 Preliminaries

Every topological space in this paper will be assumed Hausdorff space and most of the backgrounds here have been worked out previously by Zvonko, [8].

A *topological semigroup* or an *S-space* is a pair (S,a) consisting a topological space S and a map (i.e., a continuous function) $a : S \times S \to S$ from the product space $S \times S$ into S such that $a(x,a(y,z)) = a(a(x,y),z)$ for all $x,y,z \in S$. That is, an $S-$space is a topological space with a continuous associative multiplication. We denote the class of all S-spaces by χ.

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An S-space (A,c) is called an *S-subspace* of (S,a) if A is a subspace of S and the map a takes the product $A\times A$ into A and $c(x,y)=a(x,y)$ for all $x,y\in A$. It is natural to denote the multiplication of an S-subspace with the same symbol used for the multiplication on the S-space under consideration.

For every space S, the *natural S-space* is an S-space (S,π_i), where π_i is a continuous associative multiplication on S given by $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for all $x,y\in S$. We denote the class of all natural S-spaces (S,π) by $\mathcal{N}_\pi$, where $\pi=\pi_1,\pi_2$.

Let (S,a) and (O,e) be S-spaces. The function $f:(S,a)\to(O,e)$ is called a *homomorphism* or *an S-map* if f is a map of a space S into O and $f(a(x,y))=e(f(x),f(y))$ for all $x,y\in S$. Recall [8] that the usual composition and the usual product of two S-maps are S-maps and that the function $f:S\to O$ of a natural S-space (S,π) into (O,π) is an S-map if and only if it is continuous.

For every a space S, by $P(S)$ we mean the space of all paths from the unit closed interval $I=[0,1]$ into S with the compact-open topology. Recall [8] that for every an S-space (S,a), $(P(S),\underline{a})$ is an S-space where $\underline{a}:P(S)\times P(S)\to P(S)$ is a map defined by $\underline{a}(\alpha,\beta)](t)=a(\alpha(t),\beta(t))$ for all $\alpha,\beta\in P(S)$, $t\in I$. The shorter notion for this S-space will be $P(S,a)$.

Definition 2.1. The S-maps $f,g:(S,a)\to(O,e)$ are called *S-homotopic* and write $f\simeq_s g$ provided there is an S-map $H:(S,a)\to P(O,e)$ called an *S-homotopy* such that $H(s)(0)=f(s)$ and $H(s)(1)=g(s)$ for all $s\in S$.

Throughout this paper, for every an S-homotopy $H:(S,a)\to P(O,e)$ and for every $t\in I$, by H_t (or $[H]_t$) we mean the S-map, [8], $H_t:(S,a)\to(O,e)$ which given by $H_t(s)=H(s)(t)$ for all $s\in S$. Also for every an S-homotopy $H:(S,a)\to P[P(O),\underline{e}]$ and for every $r,t\in I$, by H_{rt} (or $[H]_{rt}$) we mean the S-map $H_{rt}:(S,a)\to(O,e)$ which given by $H_{rt}(s)=[H(s)(r)](t)$ for all $s\in S$.

Theorem 2.2. The relation of S-homotopy $\simeq_s$ is an equivalence relation on the set of all S-maps of (S,a) into (O,e).

Theorem 2.3. If the S-maps $f,g:(S,a)\to(O,e)$ are S-homotopic then the relations $f\circ h\simeq_s g\circ h$ and $k\circ f\simeq_s k\circ g$ hold for all S-maps h into (S,a) and k from (O,e).

Theorem 2.4. If the S-maps $f,g:(S,a)\to(O,e)$ are S-homotopic then the maps $f,g:S\to O$ are homotopic.

Theorem 2.5. The S-maps $f,g:(S,\pi)\to(O,\pi)$ are S-homotopic if and only if the maps $f,g:S\to O$ are homotopic.

Definition 2.6. An $S-$map $f:(S,a)\to(O,e)$ is called an *S_χ-fibration* if for every an-space $(Z,u)\in\chi$, an $S-$map $g:(Z,u)\to(S,a)$ and an $S-$homotopy $G:(Z,u)\to P(O,e)$ with $G_0=f\circ g$, there is an $S-$homotopy $H:(Z,u)\to P(S,a)$ such that $H_0=g$ and $f\circ H_t=G_t$ for all $t\in I$.

Theorem 2.7. The map $f:S\to O$ is a Hurewicz fibration if and only if the $S-$map $f:(S,\pi)\to(O,\pi)$ is an $S_{\mathcal{N}_\pi}-$fibration.

Theorem 2.8. The composition of S_χ-fibrations is an S_χ-fibration.

3 The st-spaces and st-maps

By a *pair of two S-spaces* or an *st-space* we mean a triple $\{(S_1,a),(S_2,c),\gamma\}$ consisting of two S-spaces (S_1,a), (S_2,c) and an S-map $\gamma:(S_2,c)\to(S_1,a)$. The shorter notion for this st-space will be $S(ac\gamma)$.

There are many ways in which an S-space can be regarded as an st-space. In our work, we use an S-space (S,a) as $\{(S,a),(S,a),id\}$ where id is the identity S-map on S.

For any two S-spaces (S,a) and (O,e), one can easily to check that the product space $S\times O$ is an S-space with the usual multiplication product $a\times e$ of a and e. The product st-space $S(ac\gamma)\times Q(uv\mu)$ of two st-spaces

$$S(ac\gamma)=\{(S_1,a),(S_2,c),\gamma\}$$

and $Q(uv\mu)=\{(Q_1,u),(Q_2,v),\mu\}$ can be defined by

$$S(ac\gamma)\times Q(uv\mu)=\{(S_1\times Q_1,a\times u),(S_2\times Q_2,c\times v),\gamma\times\mu\}.$$

For every an S-map $f:(S,a)\to(O,e)$, A function $g:P(S,a)\to P(O,e)$ which is defined by $g(\alpha)=f\circ\alpha$ for all $\alpha\in P(S,a)$ is an S-map, [8]. An S-map g will be called an *S-map induced by* f and denoted by $\widehat{f}$. Then for an st-space $S(ac\gamma)$, the triple $\{P(S_1,a),P(S_2,c),\widehat{\gamma}\}$ is an st-space denoted by $PS(ac\widehat{\gamma})$.

Definition 3.1. An *st-map* from an st-space $S(ac\gamma)$ into an st-space $Q(uv\mu)$ is a pair

$$\underline{h}=\{h_{au},h_{cv}\}:S(ac\gamma)\to Q(uv\mu)$$

of two S-maps $h_{au}:(S_1,a)\to(Q_1,u)$ and $h_{cv}:(S_2,c)\to(Q_2,v)$ such that $h_{au}\circ\gamma\simeq_s\mu\circ h_{cv}$.

In the last definition, if $h_{au}\circ\gamma=\mu\circ h_{cv}$, then $\underline{h}$ will be called an *t-map*. Trivially, if $\underline{f}=\{f_{ae},f_{ce}\}:S(ac\gamma)\to(O,e)$ is an t-map then $f_{ae}\circ\gamma=f_{ce}$.

We say that the st-maps $\underline{h},\underline{g}:S(ac\gamma)\to Q(uv\mu)$ are *equivalent st-maps*, we write $\underline{h}\equiv\underline{g}$, if $h_{au}\circ\gamma=g_{au}\circ\gamma$ and $\mu\circ h_{cv}=\mu\circ g_{cv}$. By $\underline{h}=\underline{g}$ we mean that $h_{au}=g_{au}$ and $h_{cv}=g_{cv}$. Trivially, if $\underline{h}=\underline{g}$ then $\underline{h}\equiv\underline{g}$.

Proposition 3.2. The product

$$\underline{h}\times\underline{g}:S(ac\gamma)\times Q(uv\mu)\to S'(a'c'\gamma')\times Q'(u'v'\mu')$$

of two st-maps $\underline{h}:S(ac\gamma)\to S'(a'c'\gamma')$ and $\underline{g}:Q(uv\mu)\to Q'(u'v'\mu')$ which is given by

$$\underline{h}\times\underline{g}=\{h_{aa'}\times g_{uu'},h_{cc'}\times g_{vv'}\}$$

is an st-map.

Proof. It's clear that $h_{aa'}\times g_{uu'}$ and $h_{cc'}\times g_{vv'}$ are S-maps. Since $\underline{h}$ and $\underline{g}$ are st-maps, then $h_{aa'}\circ\gamma\simeq_s\gamma'\circ h_{cc'}$ and $g_{uu'}\circ\mu\simeq_s\mu'\circ g_{vv'}$. Hence

$$\begin{aligned}(h_{aa'}\times g_{uu'})\circ(\gamma\times\mu) &= (h_{aa'}\circ\gamma)\times(g_{uu'}\circ\mu')\\ &\simeq_s (\gamma'\circ h_{cc'})\times(\mu'\circ g_{vv'})\\ &= (\gamma'\times\mu')\circ(h_{cc'}\times g_{vv'}).\end{aligned}$$

That is, $\underline{h}\times\underline{g}$ is an st-map. □

Easily to check that the composition

$$\underline{g}\circ\underline{h}=\{g_{a'u}\circ h_{aa'},g_{c'v}\circ h_{cc'}\}:S(ac\gamma)\to Q(uv\mu)$$

of two st-maps $\underline{h}:S(ac\gamma)\to S'(a'c'\gamma')$ and $\underline{g}:S'(a'c'\gamma')\to Q(uv\mu)$ is an st-map. For every an st-space, the pair $\underline{id}=\{id_{aa},id_{cc}\}:S(ac\gamma)\to S(ac\gamma)$ of the identity S-maps id_{aa} and id_{cc} is an st-map, will be called the *identity st-map* on $S(ac\gamma)$.

For every an st-map $\underline{h}:S(ac\gamma)\to Q(uv\mu)$, the pair

$$\{\widehat{h}_{au}:P(S_1,a)\to P(Q_1,u),\widehat{h}_{cv}:P(S_2,c)\to P(Q_2,v)\}$$

is an st-map from $PS(ac\widehat{\gamma})$ into $PQ(uv\widehat{\mu})$. The shorter notion for this st-map will be $\widehat{\underline{h}}$.

Proposition 3.3. Let $S(ac\gamma)$ and $Q(uv\mu)$ be two st-spaces and $\underline{h}:S(ac\gamma)\to PQ(uv\widehat{\mu})$ be an st-map. Then for every $t\in I$, the pair

$$[\underline{h}]_t=\{[h_{a\underline{u}}]_t,[h_{c\underline{v}}]_t\}:S(ac\gamma)\to Q(uv\mu)$$

is an st-map.

Proof. Consider an S-map $h_{a\underline{u}}:(S_1,a)\to P(Q_1,u)$. Recall [8] that for every $t\in I$, there is a natural evaluation S-map $\mathscr{E}_t:P(Q_1,u)\to(Q_1,u)$ given by $\mathscr{E}_t(\alpha)=\alpha(t)$ for all $\alpha\in P(Q_1)$. Then for every $t\in I$, the the composition $\mathscr{E}_t\circ h_{a\underline{u}}$ is an S-map; thus

$$[h_{a\underline{u}}]_t(x)=h_{u\underline{a}}(x)(t)=(\mathscr{E}_t\circ h_{u\underline{a}})(x)$$

for every $x\in Q_1$, that is, $[h_{a\underline{u}}]_t$ is an S-map. Similarly, for every $t\in I$, $[h_{c\underline{v}}]_t$ is an S-map.

Now since $h_{a\underline{u}}\circ\gamma\simeq_s\widehat{\mu}\circ h_{c\underline{v}}$, then for every $t\in I$,

$$\begin{aligned}[h_{a\underline{u}}]_t\circ\gamma&=(\mathscr{E}_t\circ h_{u\underline{a}})\circ\gamma=\mathscr{E}_t\circ(h_{u\underline{a}}\circ\gamma)\simeq_s\mathscr{E}_t\circ(\widehat{\mu}\circ h_{c\underline{v}})\\&=\mu\circ[h_{c\underline{v}}]_t.\end{aligned}$$

That is, for every $t\in I$, $[\underline{h}]_t$ is an st-map. □

We shall say that an st-space $S'(a'c'\gamma')$ is an *st-subspace* of an st-space $S(ac\gamma)$ provided (S'_1,a') is an S-subspace of (S_1,a), (S'_2,c') is an S-subspace of (S_2,c), and $\gamma'=\gamma|S'_2$ where $\gamma|S'_2$ the restriction S-map γ on an S-subspace (S'_2,c').

Let $\underline{h}:S(ac\gamma)\to Q(uv\mu)$ be an st-map. One easily to check that for an st-subspace $S'(a'c'\gamma')$ of $S(ac\gamma)$, the pair

$$\{h_{au}|S'_1:(S'_1,a')\to(Q_1,u),h_{cv}|S'_2:(S'_2,c')\to(Q_2,v)\}$$

is an st-map from $S'(a'c'\gamma')$ into $Q(uv\mu)$. This pair is called the *restriction st-map* of $\underline{h}$ on $S'(a'c'\gamma')$, denoted by $\underline{h}|S'(a'c'\gamma')$.

Theorem 3.4. Let $\underline{f}:S(ac\gamma)\to(O,e)$ be an t-map and (E,e) be an S-subspace of (O,e). Then the triple

$$\underline{f}^{-1}(E)=\{(f_{ae}^{-1}(E),a),(f_{ce}^{-1}(E),c),\gamma|f_{ce}^{-1}(E)\}$$

is an st-subspace of $S(ac\gamma)$ and $\underline{f}|\underline{f}^{-1}(E)$ is an t-map from $\underline{f}^{-1}(E)$ into (E,e).

Proof. Note that for

$$x,y\in f_{ae}^{-1}(E),\ f_{ae}(xay)=f_{ae}(x)ef_{ae}(y)\in E;$$

thus $xay\in f_{ae}^{-1}(E)$. Hence $(f_{ae}^{-1}(E),a)$ is an S-subspace of (S_1,a). Similarly, $(f_{ce}^{-1}(E),c)$ is an S-subspace of (S_2,c). Since $\underline{f}$ is an t-map then for $x\in f_{ce}^{-1}(E)$,

$$f_{ae}[\gamma|f_{ce}^{-1}(E)(x)]=f_{ae}[\gamma(x)]=f_{ce}(x)\in f_{ce}[f_{ce}^{-1}(E)]\subseteq E.$$

That is, $\gamma|f_{ce}^{-1}(E)(x)\in f_{ae}^{-1}(E)$. Then $\gamma|f_{ce}^{-1}(E)$ takes $f_{ce}^{-1}(E)$ into $f_{ae}^{-1}(E)$ and sice γ is an S-map, then $\gamma|f_{ce}^{-1}(E)$ is also an S-map. Hence The triple $\underline{f}^{-1}(E)$ is an st-space.

Similarly, $f_{ae}|f_{ae}^{-1}(E)$ and $f_{ce}|f_{ce}^{-1}(E)$ are S-maps take $f_{ae}^{-1}(E)$ and $f_{ce}^{-1}(E)$ into E, respectively. Since $f_{ae}\circ\gamma=f_{ce}$, then

$$f_{ae}|f_{ae}^{-1}(E)\circ\gamma|f_{ce}^{-1}(E)=(f_{ae}\circ\gamma)|f_{ce}^{-1}(E)=f_{ce}|f_{ce}^{-1}(E).$$

That is, $\underline{f}|\underline{f}^{-1}(E)\}$ is an t-map from $\underline{f}^{-1}(E)$ into (E,e). □

4 T_χ-fibrations

In this section, we introduce the concept of T_χ-fibration and study some its basic properties.

Definition 4.1. An t-map $\underline{f}:S(ac\gamma)\to(O,e)$ is called an T_χ*-fibration* if for every an S-space $(Z,u)\in\chi$, an S-map $g:(Z,u)\to(S_2,c)$ and an S-homotopy $G:(Z,u)\to P(O,e)$ with $G_0=f_{ce}\circ g$, there exists an S-homotopy $H:(Z,u)\to P(S_1,a)$ such that $H_0=\gamma\circ g$ and $f_{ae}\circ H_t=G_t$ for all $t\in I$.

For every two S-spaces (S,a) and (O,e), throughout this paper by $\mathscr{P}_1$ we mean the usual first projection map of $S\times O$ onto S which is also S-map of $(S\times O,a\times e)$ onto (S,a). Similarly, we mean by $\mathscr{P}_2$ the usual second projection map of $S\times O$ onto O.

Example 4.2. For every an st-space $S(ac\gamma)$ and an S-space (O,e), the t-map $\underline{f}:S(ac\gamma)\times(O,e)\to(O,e)$ which is given by

$$\underline{f}=\{f_1:(S_1\times O,a\times e)\to(O,e),$$
$$f_2:(S_2\times O,c\times e)\to(O,e)\}$$

is an T_χ-fibration, where $f_1(x,r)=r$ and $f_2(y,r)=r$ for all $x\in S_1$, $y\in S_2$, $r\in O$. Note that If $(Z,u)\in\chi$, $g:(Z,u)\to(S_2\times O,c\times e)$ is an S-map, and $G:(Z,u)\to P(O,e)$ is an S-homotopy with $G_0=f_2\circ g$, define the desired S-homotopy H from (Z,u) into $P(S_1\times O,a\times e)$ by

$$H(z)(t)=[\gamma[\mathscr{P}_1(g(z))],G(z)(t)]$$

for all $z \in Z$, $t \in I$.

The following result shows that the the composition of an T_χ-fibration and S_χ-fibration will be an T_χ-fibration.

Theorem 4.3. The composition t-map $f \circ \underline{f}$ of an T_χ-fibration $\underline{f} : S(ac\gamma) \to (O,e)$ and an S_χ-fibration $f : (O,e) \to (O',e')$ is an T_χ-fibration.

Proof. Let $(Z,u) \in \chi$, $g : (Z,u) \to (S_2,c)$ be an S-map and $G : (Z,u) \to P(O',e')$ be an S-homotopy with $G_0 = (f \circ f_{ce}) \circ g = f \circ (f_{ce} \circ g)$. Since $f_{ce} \circ g$ is an S-map and f is an S_χ-fibration, then there is an S-homotopy $F : (Z,u) \to P(O,e)$ such that $F_0 = f_{ce} \circ g$ and $f \circ F_t = G_t$ for all $t \in I$. Now since $\underline{f}$ is an T_χ-fibration, then there is an S-homotopy $H : (Z,u) \to P(S_1,a)$ such that $H_0 = \gamma \circ g$ and $f_{ae} \circ H_t = F_t$ for all $t \in I$. Then $(f \circ f_{ae}) \circ H_t = f \circ (f_{ae} \circ H_t) = f \circ F_t = G_t$ for all $t \in I$. Hence $f \circ \underline{f} : S(ac\gamma) \to (O',e')$ is an T_χ-fibration. □

Theorem 4.4. The product

$$\underline{f} \times \underline{f'} : S(ac\gamma) \times S'(a'c'\gamma') \to (O \times O', e \times e')$$

of two T_χ-fibrations $\underline{f} : S(ac\gamma) \to (O,e)$ and $\underline{f'} : S'(a'c'\gamma') \to (O',e')$ is an T_χ-fibration.

Proof. Let $(Z,u) \in \chi$, $g : (Z,u) \to (S_2 \times S'_2, c \times c')$ be an S-map, and $G : (Z,u) \to P(O \times O', e \times e')$ be an S-homotopy with $G_0 = (f_{ce} \times f'_{c'e'}) \circ g$. Define S-homotopies $G^1 : (Z,u) \to P(O,e)$ and $G^2 : (Z,u) \to P(O',e')$ by $G^1_t = \mathscr{P}_1 \circ G_t$ and $G^2_t = \mathscr{P}_2 \circ G_t$ for all $t \in I$, respectively.

For an T_χ-fibration $\underline{f}$, consider an S-map $\mathscr{P}_1 \circ g : (Z,u) \to (S_2,c)$ and an S-homotopy G^1 with

$$G^1_0 = \mathscr{P}_1 \circ G_0 = \mathscr{P}_1 \circ [(f_{ce} \times f'_{c'e'}) \circ g] = f_{ce} \circ (\mathscr{P}_1 \circ g).$$

Then there is an S-homotopy $F : (Z,u) \to P(S_1,a)$ such that $F_0 = \gamma \circ (\mathscr{P}_1 \circ g)$ and $f_{ae} \circ F_t = G^1_t$ for all $t \in I$. For an T_χ-fibration $\underline{f'}$, similarly, there is an S-homotopy $F' : (Z,u) \to P(S'_1,a')$ such that $F'_0 = \gamma \circ (\mathscr{P}_2 \circ g)$ and $f'_{ae} \circ F'_t = G^2_t$ for all $t \in I$.

Define an S-homotopy $H : (Z,u) \to P(S_1 \times S'_1, a \times a')$ by $H_t = F_t \times F'_t$ for all $t \in I$. Note that

$$\begin{aligned} H_0 &= [\gamma \circ (\mathscr{P}_1 \circ g)] \times [\gamma \circ (\mathscr{P}_2 \circ g)] \\ &= \gamma \circ [(\mathscr{P}_1 \circ g) \times (\mathscr{P}_2 \circ g)] = \gamma \circ g \end{aligned}$$

and

$$\begin{aligned} f_{ae} \times f'_{a'e'}) \circ H_t &= (f_{ae} \times f'_{a'e'}) \circ (F_t \times F'_t) \\ &= (f_{ae} \circ F_t) \times (f'_{a'e'} \circ F'_t) \\ &= G^1_t \times G^2_t = G_t \end{aligned}$$

for all $t \in I$. Hence $\underline{f} \times \underline{f'}$ is an T_χ-fibration. □

In the following theorem, we show that the restriction t-map $\underline{f}|\underline{f}^{-1}(E)$ of any T_χ-fibration $\underline{f} : S(ac\gamma) \to (O,e)$ on $\underline{f}^{-1}(E)$ is an T_χ-fibration, for every S-subspace (E,e) of (O,e).

Theorem 4.5. Let $\underline{f} : S(ac\gamma) \to (O,e)$ be an T_χ-fibration and let (E,e) be an S-subspace of (O,e). Then the restriction t-map $\underline{f}|\underline{f}^{-1}(E) : \underline{f}^{-1}(E) \to (E,e)$ is an T_χ-fibration.

Proof. Let $(Z,u) \in \chi$, $g : (Z,u) \to (f^{-1}_{ce}(E),c)$ be an S-map and $G : (Z,u) \to P(E,e)$ be an S-homotopy with $G_0 = f_{ce} \circ g$. Let $i : (f^{-1}_{ce}(E),c) \to (S_2,c)$ and $j : (E,e) \to (O,e)$ be inclusion S-maps. Then $[\widehat{j} \circ G]_0 = f_{ce} \circ (i \circ g)$. Since $\underline{f}$ is an T_χ-fibration, then there is an S-homotopy $H : (Z,u) \to P(S_1,a)$ such that $H_0 = \gamma \circ (i \circ g) = \gamma | f^{-1}_{ce}(E) \circ g$ and $f_{ae} \circ H_t = [\widehat{j} \circ G]_t = j \circ G_t = G_t$ for all $t \in I$. By the last part, note that $H(z)(t) \in f^{-1}_{ae}(E)$ for all $z \in Z$, $t \in I$. That is, we can consider H as S-homotopy $:(Z,u) \to P(f^{-1}_{ae}(E),a)$. Hence $\underline{f}|\underline{f}^{-1}(E)$ is an T_χ-fibration. □

Theorem 4.6. Let $\underline{f} : S(ac\gamma) \to (O,e)$ be an t-map. If at least one of the S-maps f_{ae} and f_{ce} is an S_χ-fibration then $\underline{f}$ is an T_χ-fibration.

Proof. Firstly, let $f_{ae} : (S_1,a) \to (O,e)$ be an S_χ-fibration. Let $(Z,u) \in \chi$, $g : (Z,u) \to (S_2,c)$ be an S-map and $G : (Z,u) \to P(O,e)$ is an S-homotopy with $G_0 = f_{ce} \circ g$. Then $G_0 = f_{ce} \circ g = f_{ae} \circ (\gamma \circ g)$. Since $\gamma \circ g$ is an S-map from (Z,u) into (S_1,a) and f_{ae} is an S_χ-fibration, then there is an S-homotopy $H : (Z,u) \to P(S_1,a)$ such that $H_0 = \gamma \circ g$ and $f_{ae} \circ H_t = G_t$ for all $t \in I$. That is $\underline{f}$ is an T_χ-fibration.

The other case, let $f_{ce} : (S_2,a) \to (O,e)$ be an S_χ-fibration. Let $(Z,u) \in \chi$, $g : (Z,u) \to (S_2,c)$ be an S-map and $G : (Z,u) \to P(O,e)$ be an S-homotopy with $G_0 = f_{ce} \circ g$. Then there is an S-homotopy $F : (Z,u) \to P(S_2,c)$ such that $F_0 = g$ and $f_{ce} \circ F_t = G_t$ for all $t \in I$. Define an S-homotopy $H : (Z,u) \to P(S_1,a)$ by $H = \widehat{\gamma} \circ F$. Then $H_0 = \gamma \circ F_0 = \gamma \circ g$ and

$$f_{ae} \circ H_t = f_{ae} \circ (\gamma \circ F_t) = f_{ce} \circ F_t = G_t$$

for all $t \in I$.That is $\underline{f}$ is an T_χ-fibration. □

Let $S(ac\gamma)$ be an st-space. If there exists an S-map $\gamma' : (S_1,a) \to (S_2,c)$ such that $\gamma \circ \gamma' = id$ then $S(ac\gamma)$ will be called an *extendable* by an S-map γ'.

Theorem 4.7. Let $S(ac\gamma)$ be an extendable by an S-map γ'. Then for every T_χ-fibration $\underline{f} : S(ac\gamma) \to (O,e)$, f_{ae} is an S_χ-fibration.

Proof. Let $(Z,u) \in \chi$, $g : (Z,u) \to (S_1,a)$ be an S-map and $G : (Z,u) \to P(O,e)$ be an S-homotopy with $G_0 = f_{ae} \circ g$. Then $G_0 = f_{ae} \circ g = f_{ce} \circ (\gamma' \circ g)$. Since $\gamma' \circ g$ is an S-map from (Z,u) into (S_2,c) and $\underline{f}$ is an T_χ-fibration, then there is an S-homotopy $H : (Z,u) \to P(S_1,a)$ such that $H_0 = \gamma \circ (\gamma' \circ g) = g$ and $f_{ae} \circ H_t = G_t$ for all $t \in I$. That is f_{ae} is an S_χ-fibration. □

5 Pullback t-maps

One notable exception is that the pullback of approximate fibration need not be an approximate fibration. In this section, we show that the pullbacks of T_χ-fibrations are T_χ-fibrations.

Proposition 5.1 Let $\underline{f}: S(ac\gamma) \to (O,e)$ be an t-map and $k:(O',e') \to (O,e)$ be an S-map. Then the triple $S(ac\gamma)_k = \{(S_{k1}, e' \times a), (S_{k2}, e' \times c), \gamma^k\}$ is an st-space such that

$$S_{k1} = \{(x,s) \in O' \times S_1 | k(x) = f_{ae}(s)\},$$
$$S_{k2} = \{(x,s) \in O' \times S_2 | k(x) = f_{ce}(s)\},$$

and $\gamma^k(x,s) = (x, \gamma(s))$ for all $(x,s) \in S_{k2}$.

Proof. Since the maps k and f_{ae} are S-maps, then for all $(x,s),(x',s') \in S_{k1}$,

$$k(xe'x') = k(x)ek(x') = f_{ae}(s)ef_{ae}(s') = f_{ae}(sas')$$

Hence $(x,s)(e' \times a)(x',s') = (xe'x', sas') \in S_{k1}$. That is, $(S_{k1}, e' \times a)$ is an S-subspace of $(O' \times S_1, e' \times a)$. Similarly, $(S_{k2}, e' \times c)$ is an S-subspace of $(O' \times S_2, e' \times c)$.

Note that for all $(x,s) \in S_{k2}$, $f_{ae}(\gamma(s)) = f_{ce}(s) = k(x)$, that is, $(x, \gamma(s)) \in S_{k1}$. Hence γ^k is a function takes S_{k2} into S_{k1}. Since $\gamma^k = id \times \gamma | S_{k2}$, then γ^k is an S-map. Hence the triple $S(ac\gamma)_k$ is an st-space.□

In the last proposition, the st-space $S(ac\gamma)_k$ is called a *pullback st-space* of $S(ac\gamma)$ induced from $\underline{f}$ by k.

Let $\underline{f}: S(ac\gamma) \to (O,e)$ be an t-map and $k:(O',e') \to (O,e)$ be an S-map. The t-map $\underline{f}^k: S(ac\gamma)_k \to (O',e')$ which is given by $\underline{f}^k = \{f_a^k, f_c^k\}$ is called a *pullback t-map* of $\underline{f}$ induced by k, where $f_a^k(x,s) = x$ and $f_c^k(x,s') = x$ for all $(x,s) \in S_{k1}$, $(x,s') \in S_{k2}$.

Theorem 5.2. Let $\underline{f}: S(ac\gamma) \to (O,e)$ be an T_χ-fibration and $k:(O',e') \to (O,e)$ be an S-map. Then the pullback $\underline{f}^k$ of $\underline{f}$ induced by k is an T_χ-fibration.

Proof. Let $(Z,u) \in \chi$, $g':(Z,u) \to (S_{k2}, e' \times c)$ be an S-map and $G':(Z,u) \to P(O',e')$ be an S-homotopy with $G'_0 = f_c^k \circ g'$. Define an S-map $g:(Z,u) \to (S_2,c)$ by $g(z) = \mathscr{P}_2(g'(z))$ and an S-homotopy $G:(Z,u) \to P(O,e)$ by $G(z) = k \circ G'(z)$ for all $z \in Z$. Note that

$$\begin{aligned} G(z)(0) &= (k \circ G'(z))(0) = k(G'(z)(0)) = k[f_c^k(g'(z))] \\ &= k(\mathscr{P}_1(g'(z))) = f_{ce}(\mathscr{P}_2(g'(z))) = f_{ce}(g(z)) \end{aligned}$$

for all $z \in Z$. That is, $G_0 = f_{ce} \circ g$. Since $\underline{f}$ is an T_χ-fibration, then there is an S-homotopy $H:(Z,u) \to P(S_1,a)$ such that $H_0 = \gamma \circ g$ and $f_{ae} \circ H_t = G_t$ for all $t \in I$.

Define an S-homotopy $H':(Z,u) \to P(S_{k1}, e' \times a)$ by $H'(z)(t) = [G'(z)(t), H(z)(t)]$ for all $z \in Z, t \in I$. Note that $f_a^k \circ H' = G'$ and

$$\begin{aligned} H'(z)(0) &= [G'(z)(0), H(z)(0)] = [f_c^k(g'(z)), \gamma(g(z))] \\ &= [\mathscr{P}_1(g'(z)), \gamma(\mathscr{P}_2(g'(z)))] \\ &= \gamma^k[\mathscr{P}_1(g'(z)), \mathscr{P}_2(g'(z))] \\ &= \gamma^k(g'(z)) = (\gamma^k \circ g')(z) \end{aligned}$$

for all $z \in Z$. That is, $H'_0 = \gamma^k \circ g'$. Hence $\underline{f}^k$ is an T_χ-fibration.□

6 Covering homotopy theorem

The main result of this section is a covering homotopy theorem for st-maps into T_χ-fibrations. We first have need of the following two results which are the corresponding results for a covering homotopy theorem in Hurewicz fibrations [7].

Theorem 6.1. Let $\underline{f}: S(ac\gamma) \to (O,e)$ be an T_χ-fibration and let $k,k':(Z,u) \to P(S_2,c)$ be two S-maps. Let $k_0 \simeq_s k'_0$ and $\widehat{f_{ce}} \circ k \simeq_s \widehat{f_{ce}} \circ k'$ by S-homotopies $G:(Z,u) \to P(S_2,c)$ and $R:(Z,u) \to P[P(O),\underline{e}]$, respectively. If $R_{0t} = f_{ce} \circ G_t$ for all $t \in I$, then there exists an S-homotopy $F:(Z,u) \to P[P(S_1),\underline{a}]$ between $\widehat{\gamma} \circ k$ and $\widehat{\gamma} \circ k'$ such that $F_{0t} = \gamma \circ G_t$ and $f_{ae} \circ F_{rt} = R_{rt}$ for all $r,t \in I$.

Proof. Let

$$A = (I \times \{0\}) \cup (\{0\} \times I) \cup (I \times \{1\}) \subset I \times I.$$

For every $(r,t) \in A$, define an S-map $\ll (r,t) \gg: (Z,u) \to (S_2,c)$ by

$$\ll (r,t) \gg (z) = \begin{cases} k(z)(r), & t = 0; \\ G(z)(t), & r = 0; \\ k'(z)(z), & t = 1 \end{cases}$$

for all $z \in Z$. Recall ([4], P. 100) that there is a homeomorphism $m: I \times I \to I \times I$ taking A onto $I \times \{0\}$. By hypothesis, note that for every $(r,t) \in A$,

$$(f_{ce} \circ \ll (r,t) \gg)(z) = R_{rt}(z) = (R(z)(r))(t)$$

for all $z \in Z$. For every $r \in I$, define an S-map $g^r:(Z,u) \to (S_2,c)$ and an S-homotopy $R^r:(Z,u) \to P(O,e)$ by $g^r(z) = \ll m^{-1}(r,0) \gg (z)$ and

$$R^r(z)(t) = (R(z)(\mathscr{P}_1[m^{-1}(r,t)]))(\mathscr{P}_2[m^{-1}(r,t)])$$

for all $z \in Z, t \in I$. Note that for every $r \in I$,

$$\begin{aligned} R^r(z)(0) &= (R(z)(\mathscr{P}_1[m^{-1}(r,0)]))(\mathscr{P}_2[m^{-1}(r,0)]) \\ &= (f_{ce} \circ \ll (\mathscr{P}_1[m^{-1}(r,0)], \\ &\quad \mathscr{P}_2[m^{-1}(r,0)]) \gg)(z) \\ &= (f_{ce} \circ \ll m^{-1}(r,0) \gg)(z) = (f_{ce} \circ g^r)(z). \end{aligned}$$

That is, $R_0^r = f_{ce} \circ g^r$. Then for every $r \in I$, since $\underline{f}$ is an T_χ-fibration, there exists an S-homotopy $F^r:(Z,u) \to P(S_1,a)$ such that $F_0^r = \gamma \circ g^r$ and $f_{ae} \circ F_t^r = R_t^r$ for all $t \in I$. Define an S-homotopy $F:(Z,u)) \to P[P(S_1),\underline{a}]$ by

$$(F(z)(r))(t) = F^{\mathscr{P}_1[m(r,t)]}(z)(\mathscr{P}_2[m(r,t)])$$

for all $z \in Z, r, t \in I$. Note that

$$\begin{aligned}(F(z)(r))(0) &= F^{\mathscr{P}_1[m(r,0)]}(z)(\mathscr{P}_2[m(r,0)]) \\ &= F^{\mathscr{P}_1[m(r,0)]}(z)(0) \\ &= (\gamma \circ g^{\mathscr{P}_1[m(r,0)]})(z) \\ &= (\gamma \circ \ll m^{-1}(\mathscr{P}_1[m(r,0)], \\ &\quad 0) \gg)(z) \\ &= (\gamma \circ \ll m^{-1}(\mathscr{P}_1[m(r,0)], \\ &\quad \mathscr{P}_2[m(r,0)]) \gg)(z) \\ &= (\gamma \circ \ll m^{-1}(m(r,0)) \gg)(z) \\ &= (\gamma \circ \ll (r,0) \gg)(z) \\ &= (\gamma \circ k(z))(r) \\ &= ((\widehat{\gamma} \circ k)(z))(r)\end{aligned}$$

and similarly, $(F(z)(r))(1) = ((\widehat{\gamma} \circ k')(z))(r)$ for all $r \in I, z \in Z$. That is, F is an S-homotopy between $\widehat{\gamma} \circ k$ and $\widehat{\gamma} \circ k'$. Also note that

$$\begin{aligned}F_{ot}(z) &= (F(z)(0))(t) = F^{\mathscr{P}_1[m(0,t)]}(z)(\mathscr{P}_2[m(0,t)]) \\ &= F^{\mathscr{P}_1[m(r,0)]}(z)(0) = (\gamma \circ g^{\mathscr{P}_1[m(0,t)]})(z) \\ &= (\gamma \circ \ll m^{-1}(\mathscr{P}_1[m(0,t)], 0) \gg)(z) \\ &= (\gamma \circ \ll m^{-1}(\mathscr{P}_1[m(0,t)], \mathscr{P}_2[m(0,t)]) \gg)(z) \\ &= (\gamma \circ \ll m^{-1}(m(0,t)) \gg)(z) \\ &= (\gamma \circ \ll (0,t) \gg)(z) = (\gamma \circ G_t)(z)\end{aligned}$$

and

$$\begin{aligned}(f_{ae} \circ F_{rt})(z) &= (f_{ae} \circ F_r(z))(t) = (f_{ae} \circ F^{\mathscr{P}_1[m(r,t)]}(z)) \\ &\quad (\mathscr{P}_2[m(r,t)]) \\ &= R^{\mathscr{P}_1[m(r,t)]}(z)(\mathscr{P}_2[m(r,t)]) \\ &= \{R(z)(\mathscr{P}_1[m^{-1}\{\mathscr{P}_1[m(r,t)], \\ &\quad \mathscr{P}_2[m(r,t)]\}])\} \\ &\quad (\mathscr{P}_2[m^{-1}\{\mathscr{P}_1[m(r,t)], \mathscr{P}_2[m(r,t)]\}]) \\ &= \{R(z)(\mathscr{P}_1[m^{-1}\{m(r,t)\}])\} \\ &\quad (\mathscr{P}_2[m^{-1}\{m(r,t)\}]) \\ &= \{R(z)(\mathscr{P}_1[r,t])\}(\mathscr{P}_2[r,t]) \\ &= (R(z)(r))(t) = R_{rt}(z)\end{aligned}$$

for all $r, t \in I$, $z \in Z$. That is, $F_{0t} = \gamma \circ G_t$ and $f_{ae} \circ F_{rt} = R_{rt}$ for all $r, t \in I$. □

Corollary 6.2. Let $\underline{f} : S(ac\gamma) \to (O, e)$ be an T_χ-fibration. Let $k, k' : (Z, u) \to P(S_2, c)$ be S-maps such that $k_0 = k'_0$ and $\widehat{f_{ce}} \circ k = \widehat{f_{ce}} \circ k'$. Then there exists S-homotopy $F : (Z, u) \to P[P(S_1), \underline{a}]$ between $\widehat{\gamma} \circ k$ and $\widehat{\gamma} \circ k'$ such that $F_{0t} = \gamma \circ k_0 = \gamma \circ k'_0$ and $f_{ae} \circ F_{rt} = f_{ce} \circ k_r$ for all $r, t \in I$.

Proof. Define an S-homotopy $G : (Z, u) \to P(S_2, c)$ by $G(z)(t) = k_0(z)$ and define an S-homotopy $R : (Z, u)) \to P[P(O), \underline{e}]$ by $((R(z)(r))(t) = (f_{ce} \circ k_r)(z)$ for all $r, t \in I$, $z \in Z$. Then by using the above theorem, one can get the desired S-homotopy. □

Definition 6.3. Let $\underline{f} : S(ac\gamma) \to (O, e)$ and $\underline{f}' : Q(uv\mu) \to (O, e)$ be two t-maps. An st-map $\underline{d} : Q(uv\mu) \to S(ac\gamma)$ is called an $(\underline{f}, \underline{f}')$*−preserving* if $f_{ae} \circ d_{ua} = f'_{ue}$ and the S-homotopy in the definition of $\underline{d}$ between $d_{ua} \circ \mu$ and $\gamma \circ d_{vc}$, say M, can be chosen such that $f_{ae} \circ M_t = f'_{ve}$ for all $t \in I$.

Theorem 6.4. Let $\underline{f} : S(ac\gamma) \to (O, e)$ be an T_χ-fibration and $S(ac\gamma)$ be an extendable by an S-map γ'. Let $\underline{d} : Q(uv\mu) \to S(ac\gamma)$ be an st-map and $\underline{D} : Q(uv\mu) \to P(O, e)$ be an t-map such that $\underline{d}$ is an $(\underline{f}, [\underline{D}]_0)$-preserving. Then there exists an st-map $\underline{H} : Q(uv\mu) \to PS(ac\widehat{\gamma})$ such that $[\underline{H}]_0 \equiv \underline{d}$, $\underline{f} \circ [\underline{H}]_r = [\underline{D}]_r$, for all $r \in I$, and $\underline{H}$ is an $(\underline{f}, \underline{D})$-preserving.

Proof. Let $M : (Q_2, v) \to P(S_1, a)$ be an S-homotopy between S-maps $M_0 = \gamma \circ d_{vc}$ and $M_1 = d_{ua} \circ \mu$. Since $\underline{d}$ is an $(\underline{f}, [\underline{D}]_0)$-preserving, then $f_{ae} \circ d_{ua} = [D_{u\underline{e}}]_0$ and $f_{ae} \circ M_t = [D_{v\underline{e}}]_0$ for all $t \in I$. Then

$$f_{ce} \circ d_{vc} = f_{ae} \circ (\gamma \circ d_{vc}) = f_{ae} \circ M_0 = [D_{v\underline{e}}]_0$$

and

$$f_{ce} \circ (\gamma' \circ d_{ua}) = f_{ae} \circ d_{ua} = [D_{u\underline{e}}]_0.$$

Since $\underline{f}$ is an T_χ-fibration, then, for the part $[D_{u\underline{e}}]_0 = f_{ce} \circ (\gamma' \circ d_{ua})$, there exists an S-homotopy $H' : (Q_1, u) \to P(S_1, a)$ such that $H'_0 = \gamma \circ (\gamma' \circ d_{ua}) = d_{ua}$ and $f_{ae} \circ H'_r = [D_{u\underline{e}}]_r$ for all $r \in I$. For the part $[D_{v\underline{e}}]_0 = f_{ce} \circ d_{vc}$, similarly, there exists an S-homotopy $H'' : (Q_2, v) \to P(S_1, a)$ such that $H''_0 = \gamma \circ d_{vc}$ and $f_{ae} \circ H''_r = [D_{v\underline{e}}]_r$ for all $r \in I$.

First we show that the pair

$$\underline{H} = \{H_{u\underline{a}} = H', H_{v\underline{c}} = \widehat{\gamma'} \circ H''\} : Q(uv\mu) \to PS(ac\widehat{\gamma})$$

is an st-map. Consider the two S-homotopies $\widehat{\gamma'} \circ (H_{u\underline{a}} \circ \mu)$, $H_{v\underline{c}} : (Q_2, v) \to P(S_2, c)$. We get that

$$\begin{aligned}[\widehat{\gamma'} \circ (H_{u\underline{a}} \circ \mu)]_0 &= [\widehat{\gamma'} \circ (H' \circ \mu)]_0 = \gamma' \circ (H'_0 \circ \mu) \\ &= \gamma' \circ (d_{ua} \circ \mu) \\ &\simeq_s \gamma' \circ (\gamma \circ d_{vc}) \\ &= \gamma' \circ H''_0 = [H_{v\underline{c}}]_0\end{aligned}$$

$$\begin{aligned}f_{ce} \circ [\widehat{\gamma'} \circ (H_{u\underline{a}} \circ \mu)]_r &= f_{ce} \circ (\gamma' \circ (H'_r \circ \mu)) \\ &= (f_{ae} \circ H'_r) \circ \mu \\ &= [D_{u\underline{e}}]_r \circ \mu = [D_{v\underline{e}}]_r \\ &= f_{ae} \circ H''_r \\ &= (f_{ae} \circ \gamma) \circ (\gamma' \circ H''_r) \\ &= f_{ce} \circ [\widehat{\gamma'} \circ H'']_r \\ &= f_{ce} \circ [H_{v\underline{c}}]_r\end{aligned}$$

for all $r \in I$. Then we can apply Theorem (6.1), take $Z = Q_2$, $k = \widehat{\gamma'} \circ (H_{u\underline{a}} \circ \mu)$ and $k' = H_{v\underline{c}}$. Note that $k_0 \simeq_s k'_0$ by an S-homotopy $G = \widehat{\gamma'} \circ M$ and $\widehat{f_{ce}} \circ k = \widehat{f_{ce}} \circ k'$, here we can define an S-homotopy $R:(Q_2,v)\to P[P(S_1),\underline{a}]$ by $(R(q)(r))(t)=(f_{ce}\circ k'_r)(q)$ for all $q\in Q_2, r,t\in I$. Since

$$\begin{aligned}R_{0t}(q) &= (R(q)(0))(t) = (f_{ce}\circ k'_0)(q) = (f_{ce}\circ [H_{u\underline{a}}]_0)(q)\\ &= (f_{ce}\circ(\gamma'\circ H''_0))(q) = (f_{ae}\circ H''_0)(q)\\ &= [D_{v\underline{e}}]_0(q) = (f_{ae}\circ M_t)(g)\\ &= (f_{ce}\circ(\gamma'\circ M_t))(q) = (f_{ce}\circ G_t)(q)\end{aligned}$$

for all $q\in Q_2, t\in I$, then there is an S-homotopy $F:(Q_2,v)\to P[P(S_1),\underline{a}]$ between $\widehat{\gamma}\circ k$ and $\widehat{\gamma}\circ k'$ such that $F_{0t}=\gamma\circ G_t$ and $f_{ae}\circ F_{rt}=R_{rt}$ for all $r\in I$. Then

$$H_{u\underline{a}}\circ\mu = \widehat{\gamma}\circ(\widehat{\gamma}'\circ(H_{u\underline{a}}\circ\mu)) \simeq_s \widehat{\gamma}\circ H_{v\underline{c}}.$$

That is, $\underline{H}=\{H_{u\underline{a}},H_{v\underline{c}}\}:Q(uv\mu)\to PS(ac\widehat{\gamma})$ is an st-map. Note that $[H_{u\underline{a}}]_0\circ\mu = d_{ua}\circ\mu$,

$$\gamma\circ[H_{v\underline{c}}]_0 = \gamma\circ d_{vc},\ f_{ae}\circ[H_{u\underline{a}}]_r = [D_{u\underline{e}}]_r$$

and $f_{ce}\circ[H_{v\underline{c}}]_r = [D_{v\underline{e}}]_r$. That is, $[\underline{H}]_0\equiv\underline{d}$ and $\underline{f}\circ[\underline{H}]_r = [\underline{D}]_r$ for all $r\in I$. For a preserving property, we get that $f_{ae}\circ[H_{ua}]_r = [D_{u\underline{e}}]_r$ and

$$\begin{aligned}(f_{ae}\circ F_{rt})(q) &= (f_{ae}\circ F_{rt})(q) = (f_{ce}\circ k'_r)(q)\\ &= (f_{ce}\circ[H_{v\underline{c}}]_r)(q)\\ &= (f_{ce}\circ(\gamma'\circ H''_r)(q)\\ &= (f_{ae}\circ H''_r)(q) = [D_{v\underline{e}}]_r(q)\end{aligned}$$

for all $q\in Q_2, r,t\in I$. That is, $\underline{H}$ is an $(\underline{f},\underline{D})$-preserving. □

Theorem 6.5. Let $\underline{f}:S(ac\gamma)\to(O,e)$ be an T_χ-fibration and $S(ac\gamma)$ be an extendable by an S-map γ'. Let $\underline{d},\underline{d}':Q(uv\mu)\to PS(ac\widehat{\gamma})$ be two st-maps such that there exist an st-map $\underline{g}:Q(uv\mu)\to PS(ac\widehat{\gamma})$ and an t-map $\underline{R}:Q(uv\mu)\to P[P(O),\underline{e}]$ with

$$[\underline{g}]_0\equiv[\underline{d}]_0,\ [\underline{g}]_1\equiv[\underline{d}']_0,\ [\underline{R}]_{r0}=\underline{f}\circ[\underline{d}]_r,\ [\underline{R}]_{r1}=\underline{f}\circ[\underline{d}']_r,$$

and $[\underline{g}]_t$ is an $(\underline{f},[\underline{R}]_{0t})$-preserving for all $r,t\in I$. Then there exists an st-map $\underline{H}:Q(uv\mu)\to PPS(\widehat{ac\widehat{\gamma}})$ such that

$$[\underline{H}]_{0t}\equiv[\underline{g}]_t,\ [\underline{H}]_{r0}\equiv[\underline{d}]_r,\ [\underline{H}]_{r1}\equiv[\underline{d}']_r,$$

for all $r,t\in I$, and $\underline{H}$ is an $(\underline{f},\underline{R})$-preserving.

Proof. Since for every $t\in I$, $[\underline{g}]_t$ is an $(\underline{f},[\underline{R}]_{0t})$-preserving, then there exists an S-homotopy $E^t:(Q_2,v)\to P(S_1),a)$ between two S-maps $E^t_0=[g_{u\underline{a}}]_t\circ\mu$ and $E^t_1=\gamma\circ[g_{v\underline{c}}]_t$ such that $f_{ae}\circ E^t_s=[R_{v\underline{e}}]_{0t}$ and $f_{ae}\circ[g_{u\underline{a}}]_t=[R_{u\underline{e}}]_{0t}$ for all $s,t\in I$.

First we show that for every $r\in I$, $[\underline{d}]_r$ is an $(\underline{f},[\underline{R}]_{r0})$-preserving and $[\underline{d}']_r$ is an $(\underline{f},[\underline{R}]_{r1})$-preserving. For an st-map $[\underline{d}]$, in Theorem (6.1), consider $k=\widehat{\gamma}'\circ([d_{u\underline{a}}]\circ\mu)$,

$$k'=(\widehat{\gamma}'\circ\widehat{\gamma})\circ[d_{v\underline{c}}],\ G(q)(s)=(\gamma'\circ E^0_s)(q),$$

and $(R(q)(r))(s)=([R_{u\underline{e}}]_{r0}\circ\mu)(q)$ for all $s,r\in I, q\in Q_2$. Note that

$$G_0=\gamma'\circ E^0_0=\gamma'\circ([g_{u\underline{a}}]_0\circ\mu)=\gamma'\circ([d_{u\underline{a}}]_0\circ\mu)=k_0,$$

$$G_1=\gamma'\circ E^0_1=\gamma'\circ(\gamma\circ[g_{v\underline{c}}]_0)=\gamma'\circ(\gamma\circ[d_{v\underline{c}}]_0)=k'_0,$$

$$R_{r0}=[R_{u\underline{e}}]_{r0}\circ\mu=(f_{ce}\circ\gamma')\circ([d_{ua}]_r\circ\mu)=f_{ce}\circ k_r,$$

$$R_{r1}=[R_{u\underline{e}}]_{r0}\circ\mu=(f_{ce}\circ\gamma')\circ(\gamma\circ[d_{v\underline{c}}]_r)=f_{ce}\circ k'_r,$$

and

$$R_{0s}=[R_{u\underline{e}}]_{00}\circ\mu=f_{ae}\circ E^0_s=(f_{ce}\circ\gamma')\circ E^0_s=f_{ce}\circ G_s$$

for all $s,r\in I$. Then there exists an S-homotopy $F:(Q_2,v)\to P[P(S_1),\underline{a}]$ between $\widehat{\gamma}\circ k=[d_{u\underline{a}}]\circ\mu$ and $\widehat{\gamma}\circ k'=\widehat{\gamma}\circ[d_{v\underline{c}}]$ such that $F_{0s}=\gamma\circ G_s=E^0_s$ and

$$f_{ae}\circ F_{rs}=R_{rs}=[R_{u\underline{e}}]_{r0}\circ\mu=[R_{v\underline{e}}]_{r0}$$

for all $r,s\in I$. For every $r\in I$, define $K^r:(Q_2,v)\to P(S_1),a)$ by $K^r(q)(s)=F_{rt}(q)$ for all $s,r\in I$, $q\in Q_2$; note that K^r is homotopy between two S-maps $K^r_0=[d_{u\underline{a}}]_r\circ\mu$ and $K^r_1=\gamma\circ[d_{v\underline{c}}]_r$ such that $K^0_s=E^0_s$, $f_{ae}\circ K^r_s=[R_{v\underline{e}}]_{r0}$, and $f_{ae}\circ[d_{u\underline{a}}]_r=[R_{u\underline{e}}]_{r0}$ for all $s\in I$.

For an st-map $[\underline{d}']$, similarly, for every $r\in I$, there exists an S-homotopy $K'^r:(Q_2,v)\to P(S_1),a)$ between two S-maps $K'^r_0=[d'_{u\underline{a}}]_r\circ\mu$ and $K'^r_1=\gamma\circ[d'_{v\underline{c}}]_r$ such that $K'^0_s=E^0_s$, $f_{ae}\circ K'^r_s=[R_{v\underline{e}}]_{r1}$, and $f_{ae}\circ[d'_{u\underline{a}}]_r=[R_{u\underline{e}}]_{r1}$ for all $s\in I$.

Let $A=(I\times\{0\})\cup(\{0\}\times I)\cup(I\times\{1\})\subset I\times I$. For every $(r,t)\in A$, define an st-map $[\underline{h}]_{(r,t)}:Q(uv\mu)\to S(ac\gamma)$ and an S-homotopy $M^{(r,t)}:(Q_2,v)\to P(S_1,a)$ by

$$[\underline{h}]_{(r,t)}=\begin{cases}[\underline{d}]_r & t=0;\\ [\underline{g}]_t & r=0;\\ [\underline{d}']_r & t=1\end{cases}\quad\text{and}\quad M^{(r,t)}=\begin{cases}K^r & t=0;\\ E^t & r=0;\\ K'^r & t=1,\end{cases}$$

respectively. Note that for every $(r,t)\in A$,

$$f_{ae}\circ[h_{u\underline{a}}]_{(r,t)}=[R_{u\underline{e}}]_{rt},\ f_{ae}\circ M^{(r,t)}_s=[R_{v\underline{e}}]_{rt},$$

for all $s\in I$, and $M^{(r,t)}_0$ is S-homotopy between $M^{(r,t)}_0=[h_{u\underline{a}}]_{(r,t)}\circ\mu$ and $M^{(r,t)}_1=\gamma\circ[h_{v\underline{c}}]_{(r,t)}$.

Recall ([4], P. 100) that there is a homeomorphism $m:I\times I\to I\times I$ taking A onto $I\times\{0\}$. For every $r\in I$, define an st-map $\underline{D}^r:Q(uv\mu)\to P(O,e)$ by $[\underline{D}^r]_t=\{[D^r_{u\underline{e}}],[D^r_{v\underline{e}}]\}$ where

$$[D^r_{u\underline{e}}]_t=[R_{u\underline{e}}]_{\mathscr{P}_1[m^{-1}(r,t)]\mathscr{P}_2[m^{-1}(r,t)]}$$

and

$$[D^r_{v\underline{e}}]_t=[R_{v\underline{e}}]_{\mathscr{P}_1[m^{-1}(r,t)]\mathscr{P}_2[m^{-1}(r,t)]}$$

for all $t\in I$. Consider an st-map $\underline{h}^r=[\underline{h}]_{m^{-1}(r,0)}$ and an S-homotopy $N^r=M^{m^{-1}(r,0)}$, we get that $f_{ae}\circ N^r_s=[D^r_{v\underline{e}}]_{r0}$ for all $s\in I$, $f_{ae}\circ[h_{u\underline{a}}]_{m^{-1}(r,0)}=[D^r_{u\underline{e}}]_0$, and N^r is an S-homotopy between $N^r_0=[h_{u\underline{a}}]_{m^{-1}(r,0)}\circ\mu$ and $N^r_1=\gamma\circ[h_{v\underline{c}}]_{m^{-1}(r,0)}$. That is, for every $r\in I$, an st-map $\underline{h}^r$ is an $(\underline{f},[\underline{D}^r]_0)$-preserving. Then by the Theorem (6.4), there exist an st-map $\underline{H}^r:Q(uv\mu)\to PS(ac\widehat{\gamma})$ such that

$[\underline{H}^r]_0 \equiv \underline{h}^r$, $\underline{f} \circ [\underline{H}^r]_t = [\underline{D}^r]_t$ for all $t \in I$, and $\underline{H}^r$ is an $(\underline{f}, \underline{D}^r)$-preserving.

Hence the desired an st-map $\underline{H} : Q(uv\mu) \to PPS(\underline{ac\widehat{\widehat{\gamma}}})$ is given by

$$[\underline{H}]_{rt} = [\underline{H}^{\mathscr{P}_1[m(r,t)]}]_{\mathscr{P}_2[m(r,t)]}$$

for all $r,t \in I$. □

Corollary 6.6. Let $\underline{f} : S(ac\gamma) \to (O,e)$ be an T_χ-fibration and $S(ac\gamma)$ be an extendable by an S-map γ'. Let $\underline{d}, \underline{d'} : Q(uv\mu) \to PS(\underline{ac\widehat{\gamma}})$ be two st-maps such that there exists an st-map $\underline{g} : Q(uv\mu) \to PS(\underline{ac\widehat{\gamma}})$ with $[\underline{g}]_0 \equiv [\underline{d}]_0$, $[\underline{g}]_1 \equiv [\underline{d'}]_0$, $\underline{f} \circ [\underline{d}]_r = \underline{f} \circ [\underline{d'}]_r$ for all $r \in I$, and $\underline{g}$ is an $(\underline{f}, \underline{f} \circ \underline{d})$-preserving. Then there exists an st-map $\underline{H} : Q(uv\mu) \to PPS(\underline{ac\widehat{\widehat{\gamma}}})$ such that

$$[\underline{H}]_{0t} \equiv [\underline{g}]_t,\ [\underline{H}]_{r0} \equiv [\underline{d}]_r,\ [\underline{H}]_{r1} \equiv [\underline{d'}]_r$$

for all $r,t \in I$, and $\underline{H}$ is an $(\underline{f}, \underline{f} \circ \underline{d})$-preserving.

Proof. Define an t-map $\underline{R} : Q(uv\mu) \to P[P(O), \underline{e}]$ by $[\underline{H}]_{rt} = \underline{f} \circ [\underline{d}]_r$ for all $r,t \in I$. Then by using the above theorem, one can get the desired st-map H. □

7 S_χ-approximate fibrations

In this section, we first give the notion of an approximate fibration in homotopy theory for topological semigroups. Next, we give the relation between the $T_{\mathscr{N}_\pi}$-fibration and $S_{\mathscr{N}_\pi}$-approximate fibration.

Definition 7.1. Let (S,a) and (O,e) be S-spaces with compact metrizable spaces S and O. An S-map $f : (S,a) \to (O,e)$ is called an S_χ*-approximate fibration* if for every S-space $(Z,u) \in \chi$ and given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $g : (Z,u) \to (S,a)$ and $G : (Z,u) \to P(O,e)$ are S-maps with $d[G(z)(0), (f \circ g)(z)] < \delta$, then there is an S-homotopy $H : (Z,u) \to P(S,a)$ such that $H_0 = g$ and $d[G(z)(t), (f \circ H(z))(t)] < \varepsilon$ for all $z \in Z, t \in I$.

One easily check that the map $f : S \to O$ is an approximate fibration if and only if the S-map $f : (S,\pi) \to (O,\pi)$ is an $S_{\mathscr{N}_\pi}$-approximate fibration.

Theorem 7.2. The composition of S_χ-approximate fibrations is an S_χ-approximate fibration.

Proof. Let $f : (S,a) \to (O,e)$ and $f' : (O,e) \to (O',e')$ be S_χ-approximate fibrations. Let d and d' denote the metrics on O and O', respectively. Let $(Z,u) \in \chi$ and let $\varepsilon > 0$ be given. Let $g : (Z,u) \to (S,a)$ and $G : (Z,u) \to P(O',e')$ be S-maps. Since $f \circ g : (Z,u) \to (O,e)$ is an S-map and f' is an S_χ-approximate fibration, then there exists $\delta > 0$ such that whenever

$$d'[G(z)(0), [f' \circ (f \circ g)](z)] < \delta$$

for all $z \in Z$, then there exists an S-homotopy $F : (Z,u) \to P(O,e)$ such that $F_0 = f \circ g$ and

$$d'[G(z)(t), (f' \circ F(z))(t)] < \varepsilon/2 \quad (1)$$

for all $z \in Z,\ t \in I$. Since f' is continuous and $\varepsilon/2 > 0$, then there exists $\delta' > 0$ such that

$$d(x,y) < \delta' \Longrightarrow d'(f'(x), f'(y)) < \varepsilon/2 \quad (2)$$

for all $x,y \in O$. For $\delta' > 0$, since $F_0 = f \circ g$, then $d[F(z)(0), (f \circ g)(z)] = 0 < \delta$ for all $z \in Z$. And since f is an S_χ-approximate fibration, then there is an S-homotopy $H : (Z,u) \to P(S,a)$ such that $H_0 = g$ and $d[F(z)(t), (f \circ H(z))(t)] < \delta'$ for all $z \in Z, t \in I$. From (2), we get

$$d'[(f' \circ F)(z)(t), [(f' \circ f) \circ H(z)](t)] < \varepsilon/2 \quad (3)$$

for all $z \in Z, t \in I$. From (1) and (3), then

$$\begin{aligned} &d'[G(z)(t), [(f' \circ f) \circ H(z)](t)] \\ &\le d'[G(z)(t), (f' \circ F)(z)(t)] \\ &+ d'[(f' \circ F(z))(t), [(f' \circ f) \circ H](z)(t)] \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $z \in Z, t \in I$. Hence $f' \circ f : (S,a) \to (O',e')$ is an S_χ-approximate fibration. □

Let $f : (S,\pi) \to (O,\pi)$ be an S-map with compact metrizable spaces S and O. Let d_s and d_o be metric functions on S and O, respectively. Let $(S \times O, \pi)$ be the product S-space of (S,π) and (O,π). Define a metric function $d((x,y),(x',y')) = \max\{d_s(x,x'), d_o(y,y')\}$ on $S \times O$. It is clear that $(\mathscr{G}(f),\pi)$ is an S-subspace of $(S \times O, \pi)$, where $\mathscr{G}(f) = \{(s, f(s)) : s \in S\}$ is the graph of f. For a positive integer $n > 0$, let $(\mathscr{G}(f)_n, \pi)$ be an S-subspace of $(S \times O, \pi)$, where $\mathscr{G}(f)_n$ denotes the $(1/n)-$neighborhood of $\mathscr{G}(f)$ in $S \times O$. For every positive integers $m \ge n > 0$, define an st-spaces $\mathscr{G}_f(\gamma_{nm})$ and an t-map $\underline{\mathscr{G}_{fnm}} : \mathscr{G}_f(\gamma_{nm}) \to (O,\pi)$ by

$$\mathscr{G}_f(\gamma_{nm}) = \{(\mathscr{G}(f)_n, \pi), (\mathscr{G}(f)_m, \pi), \gamma_{nm}\}$$

and $\underline{\mathscr{G}_f} =$

$$\{f_n : (\mathscr{G}(f)_n, \pi) \to (O,\pi),\ f_m : (\mathscr{G}(f)_m, \pi) \to (O,\pi)\},$$

where $\gamma_{nm} : \mathscr{G}(f)_m \to \mathscr{G}(f)_n$ is an inclusion S-map and f_n and f_m are S-maps given by $f_n(s,x) = x$ and $f_m(s',x') = x'$ for all $(s,x) \in \mathscr{G}(f)_n, (s',x') \in \mathscr{G}(f)_m$.

Theorem 7.3. An S-map $f : (S,\pi) \to (O,\pi)$ is an $S_{\mathscr{N}_\pi}$-approximate fibration if and only if for every a positive integer $n > 0$, there exists a positive integer $m \ge n$ such that the t-map $\underline{\mathscr{G}_{fnm}} : \mathscr{G}_f(\gamma_{nm}) \to (O,\pi)$ is an $T_{\mathscr{N}_\pi}$-fibration.

Proof. Suppose for every a positive integer $n > 0$, there exists a positive integer $m \ge n$ such that the t-map $\underline{\mathscr{G}_{fnm}} : \mathscr{G}_f(\gamma_{nm}) \to (O,\pi)$ is an $T_{\mathscr{N}_\pi}$-fibration. Let $\varepsilon > 0$ be given. Since f is a continuous function, then let δ' be chosen such that if $s,s' \in S$ and $d_s(s,s') < \delta'$, then $d_o(f(s), f(s')) < \varepsilon/2$. Choose a positive integer $n > 0$ such that $1/n \le \delta', \varepsilon/2$. By hypothesis, there exists a positive integer $m \ge n$ such that $\underline{\mathscr{G}_{fnm}}$ is an $T_{\mathscr{N}_\pi}$-fibration.

Let $\delta = 1/m$. Let $(Z,\pi) \in \overline{\mathscr{N}_\pi}$ be a natural S-space, $g : (Z,\pi) \to (S,\pi)$ be an S-map, and $G : (Z,\pi) \to P(O,\pi)$ be an S-homotopy with

$$d_o[G(z)(0), (f \circ g)(z)] < \delta$$

for all $z \in Z$. Define an S-map $g' : (Z,\pi) \to (\mathscr{G}(f)_m,\pi)$ by $g'(z) = (g(z), G(z)(0))$ for all $z \in Z$. Since $G_0 = f_m \circ g'$ and $\underline{\mathscr{G}_{fnm}}$ is an T_χ-fibration, there exists an S-homotopy $F : (Z,\pi) \to P(\mathscr{G}(f)_n,\pi)$ such that $F_0 = \gamma_{nm} \circ g' = g'$ and $f_n \circ F_t = G_t$ for all $t \in I$. By the last part, we can define an S-homotopy $H : (Z,\pi) \to P(S,\pi))$ by $H(z)(t) = \mathscr{P}_1[F(z)(t)]$ for all $z \in Z, t \in I$. We get that $F(z)(t) = (H(z)(t), G(z)(t))$. Since $F(z)(t) \in \mathscr{G}(f)_n$, then there exists $s \in S$ such that $d[(s, f(s)), F(z)(t)] < 1/n$. Then

$$d_s(s,H(z)(t)) < 1/n \leqslant \delta',\ d_o(f(s),G(z)(t)) < 1/n \leqslant \varepsilon/2,$$

and $d_o(f(s), f(H(z)(t))) < 1/n \leqslant \varepsilon/2$; thus

$$\begin{aligned} d_o(G(z)(t), f(H(z)(t))) &\leqslant d_o(f(H(z)(t)), f(s)) \\ &\quad + d_o(f(s),G(z)(t)) < \varepsilon \end{aligned}$$

for all $z \in Z, t \in I$. Hence f is an $S_{\mathscr{N}_\pi}$-approximate fibration.

Conversely, suppose that f is an $S_{\mathscr{N}_\pi}$-approximate fibration. Let n be a positive integer. For $\varepsilon = 1/n > 0$, let δ be given in the definition of $S_{\mathscr{N}_\pi}$-approximate fibration. Since $\delta/2 > 0$ and f is a continuous function, then let δ' be chosen such that if $s,s' \in S$ and $d_s(s,s') < \delta'$, then $d_o(f(s), f(s')) < \varepsilon/2$. Choose a positive integer $m \geqslant n$, such that $1/m \leq \delta', \delta/2$.

Now let $(Z,\pi) \in \mathscr{N}_\pi$ be a natural S-space, $g : (Z,\pi) \to (\mathscr{G}(f)_m,\pi)$ be an S-map, and $G : (Z,\pi) \to P(O,\pi)$ be an S-homotopy with $G_0 = f_m \circ g$. Define an S-map $g' : (Z,\pi) \to (S,\pi)$ by $g'(z) = \mathscr{P}_1[g(z)]$ for all $z \in Z$. We get that $g(z) = (g'(z), G(z)(0))$ for all $z \in Z$. Since $g(z) \in \mathscr{G}(f)_m$, then there exists $s \in S$ such that $d[(s, f(s)), g(z)] < 1/m$. Then

$$d_s(s,g'(z)) < 1/m \leqslant \delta',\ d_o(f(s),G(z)(0)) < 1/m \leqslant \delta/2,$$

and $d_o(f(s), f(g'(z))) < 1/m \leqslant \delta/2$; thus

$$\begin{aligned} d_o(f(g'(z)),G(z)(0)) &\leqslant d_o(f(g'(z)), f(s)) \\ &\quad + d_o(f(s),G(z)(0)) < \varepsilon \end{aligned}$$

Hence, since f is an $S_{\mathscr{N}_\pi}$-approximate fibration, there exists an S-homotopy $H' : (Z,\pi) \to P(S,\pi)$ such that $H'_0 = g'$ and $d_s(G(z)(t), (f \circ H'(z))(t)) < \varepsilon$ for all $z \in Z$, $t \in I$. Define an S-homotopy $H : (Z,\pi) \to (\mathscr{G}(f)_n,\pi)$ by $H(z)(t) = (H'(z)(t), G(z)(t))$ for all $z \in Z$, $t \in I$. Then we get that for $z \in Z$, $t \in I$,

$$\begin{aligned} H(z)(0) &= (H'(z)(0), G(z)(0)) = (g'(z), G(z)(0)) \\ &= g(z) = (\gamma_{nm} \circ g)(z) \end{aligned}$$

and $f_n \circ H_t = G_t$. Hence $\underline{\mathscr{G}_{fnm}}$ is an T_χ-fibration. □

References

[1] A. Hatcher, Algebraic Topology, Cambridge University Press, Cambridge, (2002).

[2] D. Coram and F. Duvall, Approximate fibrations, Rocky Mountain J. Math. **7**, 275-288(1977).

[3] D. Coram and F. Duvall, Approximate fibrations and amovability condition for maps, Pacific J. Math. **72**, 41-56 (1977).

[4] E.H. Spanier, Algebraic Topology, McGraw-Hill, New York, (1966).

[5] M. Lawson, J. Matthews and T. Porter, The homotopy theory of inverse semigroups, International Journal of Algebra and Computation **12**,755-790 (2002).

[6] R. C. Kirby and L. C. Siebenmann, Foundational Essays On Topological Manifolds, Smoothings, And Triangulations. Annals of Math. Studies 88, Princeton University Press, Princeton, NJ, (1977).

[7] W. Hurewicz, On the concept of fiber space, Proc. Nat. Acad. Sci. USA **41**, 956-961 (1955).

[8] Z. Cerin, Homotopy theory of topological semigroup, Topology and its Applications **123** 57-68 (2002).

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