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Global Solvability of a Continuous and Restricted Coagulation Process in a Moving Medium

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Abstract: In this paper, existence and uniqueness of a global solution to continuous, non-common and non-linear convection-coagulation equations are investigated by means of various techniques. The method of characteristics (Mizohata, 1973), substochastic methods and Kato-Voigt pertubation (Banasiak et al., 2006) are exploited to show that the linear operator (transport-coagulation) is the infinitesimal generator of a strongly continuous semigroup. Then, uniqueness of the solution to the full nonlinear problem follows by showing that the coagulation term is globally Lipschitz, hereby addressing the problem of existence and uniqueness for the combined coagulation and transport processes.

Keywords: Transport processes; semigroups; non-linear Cauchy problems; coagulation; global solution, uniqueness

1 Motivation and introduction

In many branches of natural sciences like biology, ecology, physics, chemistry, engineering, and numerous domains of applied sciences, it is possible to see clusters undergoing coagulation (coalescence) process (or its inverse, the fragmentation see [10,11,15]). Among concrete examples we count agglutination and splitting of blood cells, formation and splitting of aerosol droplets, evolution of phytoplankton aggregates, depolymerization, rock fractures and breakage of droplets. The coagulation kernel can be size and position dependent and new particles resulting from the coagulation can be spatially distributed across the space. Coagulation equations, combined with transport terms (sometime combined with fragmentation process), have been used to describe a wide range of phenomena. For instance, in ecology or aquaculture, we have phytoplankton population evolving in flowing water. In chemical engineering, the process is often accompanied by growth or decay of aggregates e.g. by surface deposition or dissolution, see, e.g., [7] or by birth or division processes in biological considerations, see, e.g., [2,19]. We have applications describing polymerization and polymer degradation, solid drugs break-up in organisms or in solutions. We also have external processes such as oxidation, melting, or dissolution, cause the exposed surface of particles to recede, resulting in the loss of mass of the system. Simultaneously, they widen the surface pores of the particle, causing the loss of connectivity and thus fragmentation, as the pores join each other, see [14,7,18, 20] and references therein.

Various types of coagulation equations have been comprehensively analyzed in numerous works: The authors in [12,16] only considered growth processes modeled by a first order partial differential operator and showed existence result for fragmentation-coagulation model with coagulation kernel taking into account that not all particles in an aggregate have the same ability to combine with particles of other aggregates which results in a damped coagulation process. In [1], the authors used similar kernels to model the evolution of phytoplankton. The author in [17] exploited the contraction mapping principle to prove existence and uniqueness results for the non-autonomous coagulation and multiple-fragmentation equation. But transport processes combined with coagulation or fragmentation in the same model are still barely touched in the domain of mathematical and abstract analysis. A special and non-common type of transport model is analyzed in [9,**?**] where the authors proved the existence of the smallest substochastic semigroup generated by the linear part, consisting of the

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transport operator combined to fragmentation terms. Kinetic-type Models with diffusion were globally investigated in [4] and later extended in [5], where the author showed that the diffusive part does not affect the breach of the conservation laws and very recently, in [3], the author investigated equations describing fragmentation and coagulation processes with growth or decay and proved an analogous result.

In the present work the model we analyze is presented as follows: In social grouping population, if we define a spatial dynamical system in which locally group-size distribution can be estimated, but in which we also allow immigration and emigration from adjacent areas with different distributions, we obtain the general model consisting of transport, direction changing, fragmentation and coagulation processes describing the dynamics a population of, for example, phytoplankton which is a spatially explicit group-size distribution model as presented in [8]. We analyze the model consisting of transport and coagulation processes with the coagulation part different from the classical one where the kernel $k(m,n)$ is defined as the rate at which particles of mass m coalesce with particles of mass n and is derived by assuming that the average number of coalescences between particles having mass in $(m;m+dm)$ and those having mass in $(n;n+dn)$ is $k(m,n)p(t,m)p(t,n)dmdndt$ during the time interval $(t;t+dt)$, where p is the *concentration* of particles. In our model, we assume that any individual in the populations is viewed as a collection of joined cells.

Working in the space $L_1(\mathbb{R}^3\times\mathbb{R}_+, mdmdx)$, we will make use, as in [9], of and Friedrichs lemma [21] to show that the transport operator generates a stochastic dynamical system, with the assumption that the velocity field is globally Lipschitz continuous and divergence free.

2 Conservativeness in the coagulation process

The model of coagulation dynamics occurring in a moving process [8,9,12] is given by

$$\begin{cases} \frac{\partial}{\partial t}p(t,x,m) = -div(\omega(x,m)p(t,x,m)) - d(x,m)p(t,x,m) \\ \qquad +\chi_{U_{\mathbb{R}}}(m,x)\dfrac{\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(y,n)p(t,y,n)(m-n)d(y,m-n)p(t,y,m-n)dndy}{m\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\eta d(y,\eta)p(t,y,\eta)d\eta dy} \\ p(0,x,m) = \overset{o}{p}(x,m), \quad a.e.\ (x,m)\in\mathbb{R}^3\times\mathbb{R}_+ \end{cases} \tag{1}$$

where in terms of the mass size m and the position x, the state of the system is characterized at any moment t by the particle-mass-position distribution $p = p(t,x,m)$, (p is also called the *density* or *concentration* of particles), with $p:\mathbb{R}_+\times\mathbb{R}^3\times(m_0,\infty)\to\mathbb{R}_+$. the velocity $\omega = \omega(x,m)$ of the transport is supposed to be a known quantity, depending on the size m of aggregates and their position x, but independent of t. The space variable x is supposed to vary in the whole of $\mathbb{R}^3$. The function $\overset{o}{p}(x,m)$ represents the density of groups of size m at position x at the beginning ($t=0$). In the model (1), we assume that the quasi nonlocal coagulation process at a position x occurs in the following way: Two clusters in a neighborhood of x coalesce to form a third group which becomes located at x. The coefficient d characterizes the competence of aggregates to joint (also called coagulation propensity). We define the other terms and elements in the following subsection.

Because the space variable x varies in the whole of $\mathbb{R}^3$ (unbounded) and since the total number of individuals in a population is not modified by interactions among groups, the following conservation law is supposed to be satisfied:

$$\frac{d}{dt}\mathscr{I}(t) = 0, \tag{2}$$

where $\mathscr{I}(t) = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} p(t,x,m)mdmdx$, is the total number of individuals in the space (or total mass of the ensemble) with the assumption that $m_0 > 0$ is the smallest mass/size a monomer can have in the system. Henceforth we assume that for each $t>0$, the density of groups of size m at the position x and time t is the function $(x,m)\longrightarrow p(t,x,m)$ taken from the Banach space

$$\mathscr{X}_1 := L_1(\mathbb{R}^3\times\mathbb{R}_+, mdmdx)$$

and $\overset{o}{p}\in\mathscr{X}_1$. When any subspace $S\subseteq\mathscr{X}_1$, we will denote by S_+ the subset of S defined as $S_+ = \{g\in S; g(x,m)\geq 0, m\in\mathbb{R}_+, x\in\mathbb{R}^3\}$.

In $\mathscr{X}_1$, we define from the right-hand side of (1), the coagulation expression $\mathscr{N}$ given by

$$[\mathscr{N}p](x,m) := [\mathscr{C}p - \mathscr{D}p](x,m) \tag{3}$$

where

$$[\mathscr{C}p](x,m) = \chi_{U_{\mathbb{R}}}(m,x)\frac{\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(y,n)p(t,y,n)(m-n)d(y,m-n)p(t,y,m-n)dndy}{m\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\eta d(y,\eta)p(t,y,\eta)d\eta dy}, \tag{4}$$

for any $p\in\mathscr{X}_{1+}\setminus\{0\}$,

$$\mathscr{C}(0) = 0,$$

and

$$[\mathscr{D}p](x,m) = d(x,m)p(x,m). \tag{5}$$

We assume that no particle of mass $m < 2m_0$ can emerge as a result of coagulation, then $\chi_{U_{\mathbb{R}}}$ is the characteristic function of the set $U_{\mathbb{R}} = \mathbb{R}^3\times U = \mathbb{R}^3\times[2m_0,\infty)$. Following [1], we assume that only a part of the aggregates has the competence to join. This could for example be due to the fact that only cells of some species have the necessary devices to glue or to attach to others. The coefficient of competence is a function $d(x,m)$ depending also on the position of the cluster. We assume that d is a positive and bounded function in the sense that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1 \alpha_m \le d(x,m) \le \theta_2 \alpha_m \quad \text{and} \quad \text{ess} \sup_{\mathbb{R}^3 \times (m_0,\infty)} d(x,n) < \infty, \tag{6}$$

with $\alpha_m \in \mathbb{R}_+$, independent of x and uniform in m.

Proposition 1. *The coagulation model described by (3) is formally conservative.*

Proof. We aim to show that (2) is satisfies, that is $\frac{d}{dt}\mathscr{I}(t) = \frac{d}{dt}\int_{\mathbb{R}^3}\int_{m_0}^{\infty} p(t,x,m) m dm dx = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} m\frac{\partial}{\partial t}p(t,x,m) dm dx = 0.$ By assumption (6), we just need to prove that

$$\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\left(\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right)dmdx = \int_{\mathbb{R}^3}\int_{m_0}^{\infty} md(x,m)p(t,x,m)dmdx \cdot \int_{\mathbb{R}^3}\int_{m_0}^{\infty} \eta d(x,\eta)p(t,x,\eta)d\eta dx. \tag{7}$$

Making use of the Fubini integration theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{m_0}^{\infty}\left[\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right]dmdx \\
&= \int_{\mathbb{R}^3}\int_{2m_0}^{\infty}\left[\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} nd(x,n)p(t,x,n)(m-n)d(x,m-n)p(t,x,m-n)dndx\right]dmdx \\
&= \int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)\left[\int_{\mathbb{R}^3}\int_{n+m_0}^{\infty}(m-n)d(x,m-n)p(t,x,m-n)dmdx\right]dndx \\
&= \int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)\left[\int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\eta)d(x,\eta)p(t,x,\eta)d\eta dx\right]dndx \\
&= \int_{\mathbb{R}^3}\int_{m_0}^{\infty} nd(x,n)p(t,x,n)dndx \times \int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\eta)d(x,\eta)p(t,x,\eta)d\eta dx,
\end{aligned}$$

which ends the proof.

The total number of cells in all aggregates that, at time t, are implicated in the coagulation process is given by:

$$M(t) := \int_{\mathbb{R}^3}\int_{m_0}^{\infty} \eta d(x,\eta)p(t,x,\eta)\, d\eta dx,$$

and

$$f(t,x,m) := \frac{md(x,m)p(t,x,m)}{M(t)}$$

is the fraction of cells in size-m aggregates and position x competent for the coagulation process with respect to the total population of cells in aggregates prone to join. In terms of the quantities introduced so far, we can express the time rate of cells forming aggregates of size m and position x:

$$M(t,x)\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} f(t,y,m-n)f(t,y,n)\, dndy,$$

If coagulation were the only process, the equation would read:

$$\frac{\partial}{\partial t}mp(t,x,m) = M(t)\chi_{U_{\mathbb{R}}}(m,x)\int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} f(t,y,m-n)f(t,y,n)dndx - md(y,m)p(t,y,m)dndy,$$

which, after basic algebra, leads to:

$$\frac{\partial}{\partial t}p(t,x,m) = [\mathscr{C}p - \mathscr{D}p](t,x,m) \tag{8}$$

with $\mathscr{C}$ and $\mathscr{D}$ given by (4) and (5) respectively.

3 Cauchy problem for the transport operator in $\Lambda = \mathbb{R}^3 \times \mathbb{R}_+$

Λ is endowed with the Lebesgue measure $d\mu = d\mu_{m,x} = dmdx$. Our primary objective in this section is to analyze the solvability of the transport problem

$$\frac{\partial}{\partial t}p(t,x,m) = -div(\omega(x,m)\, p(t,x,m)), \tag{9}$$

$$p(0,x,m) = \overset{o}{p}(x,m), \quad m \in \mathbb{R}_+,\ x \in \mathbb{R}^3$$

in the space $\mathscr{X}_1$.

Furthermore, to simplify the notation we put $\mathbf{v} = (x,m) \in \Lambda$. With the assumption that ω is divergence free and globally Lipschitz continuous, then $div\,\omega(\mathbf{v}) := \nabla \cdot \omega(\mathbf{v}) = 0$. To properly study the transport operator, we consider the function $\omega : \Lambda \longrightarrow \mathbb{R}^3$ and $\tilde{\mathscr{T}}$ the expression appearing on the right-hand side of the equation (9). Then

$$\begin{aligned}\tilde{\mathscr{T}}[p(t,\mathbf{v})] &:= -div\,(\omega(\mathbf{v})\, p(t,\mathbf{v})) \\ &= (\nabla \cdot \omega(\mathbf{v}))p(t,\mathbf{v}) + \omega(\mathbf{v}) \cdot (\nabla p(t,\mathbf{v})),\end{aligned} \tag{10}$$

which becomes

$$\tilde{\mathscr{T}}[p(t,\mathbf{v})] := \omega(\mathbf{v}) \cdot (\nabla p(t,\mathbf{v})). \tag{11}$$

For $\mathbf{v} \in \Lambda$ and $t \in \mathbb{R}$, the Cauchy problem

$$\begin{aligned}\frac{d\mathfrak{r}}{ds} &= \omega(\mathfrak{r}), \qquad s \in \mathbb{R} \\ \mathfrak{r}(t) &= \mathbf{v},\end{aligned} \tag{12}$$

has an unique solution $\mathfrak{r}(s)$ with values in Λ. Let the function $\phi : \Lambda \times \mathbb{R}^2 \longrightarrow \Lambda$ be defined by the condition that for $(\mathbf{v},t) \in \Lambda \times \mathbb{R}$,

$$s \longrightarrow \phi(\mathbf{v},t,s), \qquad s \in \mathbb{R}$$

is the unique solution of the Cauchy problem (12). We obtain the characteristics of $\tilde{\mathscr{T}}$ given by the integral curves ϕ-parameterized family $(\mathfrak{r})_\phi$ (with $\mathfrak{r}(s) = \phi(\mathbf{v},t,s)$, $s \in \mathbb{R}$, the only solution of (12)). The function ϕ possesses many desirable properties [13,24,25] that will be relevant for studying the transport operator in $\mathscr{X}_1$. We can take

$$\begin{aligned}&\mathscr{T}p = \tilde{\mathscr{T}}p, \ \text{ with } \tilde{\mathscr{T}}p \ \text{ represented by (11)} \\ &D(\mathscr{T}) := \{p \in \mathscr{X}_1,\ \mathscr{T}p \in \mathscr{X}_1\},\end{aligned} \tag{13}$$

Note that $\mathscr{T}p$ is understood in the sense of distribution. Precisely speaking, if we take $C_0^1(\Lambda)$ as the set of the test functions, each $p \in D(\mathscr{T})$ if and only if $p \in L_1(\Lambda)$ and there exists $g \in \mathscr{X}_1$ such that

$$\int_{\Lambda} \xi g d\mu = \int_{\Lambda} p\partial\cdot(\xi\omega)d\mu = \int_{\Lambda} p\omega\cdot\partial\xi d\mu, \qquad (14)$$

for all $\xi \in C_0^1(\Lambda)$, where

$$\omega\cdot\partial\xi(\mathbf{v}) := \sum_{j=1}^{3} \omega_j\partial_j\xi(\mathbf{v}) \qquad (15)$$

with $\omega_j = \omega_j(\mathbf{v})$, the j^{th} component of the velocity $\omega(\mathbf{v})$. The middle term in (14) exists as ω is globally Lipschitz continuous, and the last equality follows as ω is divergence-free. If this is the case, we define $\mathscr{T}p = g$.
Now we prove that the operator $\mathscr{T}$ is the generator of a stochastic semigroup on $\mathscr{X}_1$

Theorem 1. *If the function ω is globally Lipschitz continuous and divergence-free, then the operator $(D(\mathscr{T}),\mathscr{T})$ defined by (13) is the generator of a strongly continuous stochastic semigroup $(G_{\mathscr{T}}(t))_{t\geq 0}$, given by*

$$[G_{\mathscr{T}}(t)p](\mathbf{v}) = p(\phi(\mathbf{v},t,0)) \qquad (16)$$

for any $p \in \mathscr{X}_1$ and $t > 0$.

*Proof.*The proof of this theorem is fully detailed in [9, Theorem 2].

*Remark.*The previous theorem allows us show that the model (9) is conservative in the space $\mathscr{X}_1$, that is, the law (2) is satisfied. In fact the semigroup generated by the operator $\mathscr{T}$ is stochastic, then we have

$$0 = \int_{\Lambda} \mathscr{T}p\, d\mu, \quad \text{for all } p \in D(\mathscr{T}), \text{ then} \qquad (17)$$

$$0 = \int_{\mathbb{R}^3}\int_0^{\infty} m\mathscr{T}p(t,x,m)dmdx, \quad \text{for all } t \geq 0,$$

which leads to

$$\begin{aligned} \frac{d}{dt}\mathscr{I}(t) &= \frac{d}{dt}\left(\int_{\mathbb{R}^3}\int_0^{\infty} mp(t,x,m)dmdx\right) \\ &= \int_{\mathbb{R}^3}\int_0^{\infty} m\partial_t p(t,x,m)dmdx \\ &= \int_{\mathbb{R}^3}\int_0^{\infty} m\mathscr{T}p(t,x,m)dmdx \\ &= 0. \end{aligned}$$

This is an expected result since the flow process alone does not modify the total number of individuals in the system.

4 Coagulation propensity in a moving process

We consider the coalescence competence operator $(D,D(\mathscr{D}))$ defined in (5), as a perturbation of the transport system (9). We obtain the abstract Cauchy problem

$$\begin{aligned} \partial_t p(t,\mathbf{v}) &= \mathscr{T}p(t,\mathbf{v}) - \mathscr{D}p(t,\mathbf{v}) = Fp(t,\mathbf{v}) \\ p(0,\mathbf{v}) &= \overset{o}{p}(\mathbf{v}), \quad \mathbf{v} \in \Lambda, \end{aligned} \qquad (18)$$

where

$$F = \mathscr{T} - D. \qquad (19)$$

Remark.$(F,D(F)) = (\mathscr{T} - \mathscr{D}, D(\mathscr{T}))$ is a well defined operator.

Recall that $\mathscr{T}$ is an unbounded operator then, to show the remark, we need to characterize the domain of $F = \mathscr{T} - D$, so, let us prove that $D(\mathscr{T}) \cap D(\mathscr{D}) = D(F)$. Firstly we note that $D(\mathscr{T}) \cap D(\mathscr{D}) = D(\mathscr{T})$ since $D(\mathscr{D}) = \mathscr{X}_1$.
(i): To prove $D(\mathscr{D}) \supseteq D(F)$, we are going to show that the domain of $\mathscr{D}$ is at least that of F.
Because $\mathscr{T}$ is conservative, integration of (18) over Λ gives $\frac{d}{dt}\|p\|_1 = \frac{d}{dt}\int_{\mathbb{R}^3}\int_0^{\infty} mp(t,x,m)dmdx = -\int_{\mathbb{R}^3}\int_0^{\infty} d(x,m)mp(x,m)dmdx$. Hence (6) leads to $-\int_{\mathbb{R}^3}\int_0^{\infty}\theta_2\alpha_m mp(x,m)dmdx \leq -\int_{\mathbb{R}^3}\int_0^{\infty} d(x,m)mp(x,m)dmdx \leq -\int_{\mathbb{R}^3}\int_0^{\infty}\theta_1\alpha_m mp(x,m)dmdx$ for all $p \in (\mathscr{X}_1)_+$ and using Gronwall's inequality, we obtain

$$-\theta_2\alpha_m\|p\|_1 \leq \frac{d}{dt}\|p\|_1 \leq -\theta_1\alpha_m\|p\|_1,$$

then

$$e^{-\theta_2\alpha_m t}\|\overset{o}{p}\|_1 \leq \|p\|_1 \leq e^{-\theta_1\alpha_m t}\|\overset{o}{p}\|_1.$$

This inequality for $p = G_F(t)\overset{o}{p}$ yields

$$e^{-\theta_2\alpha_m t}\|\overset{o}{p}\|_1 \leq \|G_F(t)\overset{o}{p}\|_1 \leq e^{-\theta_1\alpha_m t}\|\overset{o}{p}\|_1 \qquad (20)$$

where $\overset{o}{p} \in (C_0^{\infty}(\Lambda))_+ \subseteq D(F)_+$. If we take $0 \leq \overset{o}{p} \in \mathscr{X}_1$ then we can use approximations to the identity (mollifiers) $\varpi_\varepsilon(\mathbf{v}) = C_\varepsilon\varpi(\mathbf{v}/\varepsilon)$ where ϖ is a $C_0^{\infty}(\Lambda)$ function defined by $\varpi(\mathbf{v}) = \begin{cases} \exp\left(\frac{1}{|\mathbf{v}|^2-1}\right) & \text{for } |\mathbf{v}| < 1 \\ 0 & \text{for } |\mathbf{v}| \geq 1 \end{cases}$ and C_ε are constants chosen so that $\int_{\Lambda}\varpi_\varepsilon(\mathbf{v})dx = 1$.
Using the mollification of $\overset{o}{p}$ by taking the convolution

$$\overset{o}{p}_\varepsilon := \int_{\Lambda}\overset{o}{p}(\mathbf{v}-\mathbf{y})\varpi_\varepsilon(\mathbf{y})d\mu_y = \int_{\Lambda}\overset{o}{p}(\mathbf{y})\varpi_\varepsilon(\mathbf{v}-\mathbf{y})d\mu_y, \quad (21)$$

we obtain $\overset{o}{p}_\varepsilon$ in $\mathscr{X}_1$ (since $\overset{o}{p} \in \mathscr{X}_1$) and $\overset{o}{p} = \lim_{\varepsilon\longrightarrow 0^+}\overset{o}{p}_\varepsilon$ in $\mathscr{X}_1$. Moreover, $\overset{o}{p}_\varepsilon$ are also nonnegative by (21) since $0 \leq \overset{o}{p}$, and the family $(\overset{o}{p}_\varepsilon)_\varepsilon \subseteq C_0^{\infty}(\Lambda)$. This shows that any non-negative $\overset{o}{p}$ taken in $\mathscr{X}_1$ can be approximated by a sequence of non-negative functions of $C_0^\infty(\Lambda)$. The inequality (20) is therefore valid for any nonnegative $\overset{o}{p} \in \mathscr{X}_1$. Using the fact that any arbitrary element $\overset{o}{g}$ of $\mathscr{X}_1$ (equipped with the pointwise order almost everywhere) can be written in the form $\overset{o}{g} = \overset{o}{g}_+ - \overset{o}{g}_-$, where $\overset{o}{g}_+,\ \overset{o}{g}_- \in (\mathscr{X}_1)_+$, the positive element approach, [6,26] allows us to extend the right inequality of (20) to all $\mathscr{X}_1$ so as to have

$$\|G_F(t)p\|_1 \le e^{-\theta_1 \alpha_m t}\|p\|_1. \tag{22}$$

Using the semigroup representation of the resolvent, we obtain for $\lambda > 0$

$$\begin{aligned} \|R(\lambda,F)p\|_1 &\le \int_0^\infty e^{-\lambda t}\|G_F(t)p\|_1 dt \\ &\le \int_0^\infty e^{-\lambda t} e^{-\theta_1 \alpha_m t}\|p\|_1 dt \\ &\le \frac{1}{\lambda + \theta_1 \alpha_m}\|p\|_1. \end{aligned}$$

By the right inequality of (6), we obtain that

$$\|\mathscr{D}R(\lambda,F)p\|_1 \le \frac{\theta_2 \alpha_m}{\lambda + \theta_1 \alpha_m}\|p\|_1 \le \frac{\theta_2}{\theta_1}\|p\|_1,$$

since α_m is uniform in m. This last relation means that the domain of $\mathscr{D}$ is at least that of F and then, $D(\mathscr{D}) \supseteq D(F)$.

(ii): Next we prove $D(\mathscr{T}) \supseteq D(F)$.

Because $F = \mathscr{T} - \mathscr{D}$ and $\mathscr{D}$ is bounded, we exploit the following relation for resolvent in $\mathscr{X}_1$:

$$\begin{aligned} \lambda I - F &= \lambda I - \mathscr{T} + \mathscr{D}R(\lambda,F)(\lambda I - F) \\ I &= (\lambda I - \mathscr{T})R(\lambda,F) + \mathscr{D}R(\lambda,F) \\ R(\lambda,\mathscr{T}) &= R(\lambda,F) + R(\lambda,\mathscr{T})\mathscr{D}R(\lambda,F) \\ R(\lambda,F) &= R(\lambda,\mathscr{T})(I - \mathscr{D}R(\lambda,F)) \end{aligned}$$

for every $m \in \mathbb{R}_+$. This leads to $D(\mathscr{T}) \supseteq D(F)$ and therefore $D(F) \subseteq D(\mathscr{T}) \cap D(\mathscr{D})$.

(iii): Finally, we show $D(\mathscr{T}) \cap D(\mathscr{D}) \subseteq D(F)$.

If $p \in D(\mathscr{T}) \cap D(\mathscr{D})$ then $\|\mathscr{T}p\|_1 < \infty$ and $\|\mathscr{D}p\|_1 < \infty$. Therefore

$$\|\mathscr{T}p - \mathscr{D}p\|_1 \le \|\mathscr{T}p\|_1 + \|\mathscr{D}p\|_1 < \infty,$$

meaning that $p \in D(F)$ and thus $D(\mathscr{T}) \cap D(\mathscr{D}) \subseteq D(F)$. Hence, $D(\mathscr{T}) \cap D(\mathscr{D}) = D(F)$ and the remark is completed.

The assumption (6) implies that the operator $\mathscr{D}$ generates a C_0-semigroup of contractions $(G_{\mathscr{D}}(t))_{t\ge 0}$, which yields the following theorem [9, Theorem 5].

Theorem 2. *The operator $(F, D(F))$ is the infinitesimal generator of a substochastic semigroup $(G_F(t))_{t\ge 0}$ defined by*

$$[G_F(t)p](\nu) = \left[\lim_{\nu \longrightarrow \infty}\left[G_{\mathscr{T}}\left(\frac{t}{\nu}\right)G_{\mathscr{D}}\left(\frac{t}{\nu}\right)\right]^\nu p\right](\nu) \tag{23}$$

for $p \in \mathscr{X}_1$ and $t > 0$, where $(G_{\mathscr{T}}(t))_{t\ge 0}$ is defined by (16).

In the next section, we use the non-linear perturbation to analyze the full model (1).

5 Global solution for the full model

The coagulation process appearing in a moving medium mathematically reads as:

$$\begin{aligned} \partial_t p(t,x,m) &= \mathscr{T}p(t,x,m) - \mathscr{D}p(t,x,m) + \mathscr{C}p(t,x,m) \\ p(0,x,m) &= \overset{o}{p}(x,m), \quad a.e.\ (x,m) \in \mathbb{R}^3 \times \mathbb{R}_+ \end{aligned}$$

where $\mathscr{C}$, given by (4), is defined on the set $\mathscr{X}_{1+} = \{g \in \mathscr{X}_1 : g \ge 0\}$. We recall that $\mathscr{C}(0) = 0$. We need the following lemma:

Lemma 1. *The operator $\mathscr{C}$ satisfies a global Lipschitz condition on the set $\mathscr{X}_{1+}$.*

Proof. We set: $\Psi h(x,m) = md(x,m)h(x,m)$ and $\alpha(h) = \int_{\mathbb{R}^3}\int_{m_0}^\infty \Psi h(x,m)\,dmdx.$

Thanks to (6) we also set $\vartheta = \operatorname{ess\,sup}_{\mathbb{R}^3 \times (m_0,\infty)} d(x,n) < \infty.$

Remark. We note that for every $h \in \mathscr{X}_{1+} \setminus \{0\} \subset \mathscr{X}_{1+} = D(\mathscr{D})$, the operator α satisfies $\alpha(h) = \int_{\mathbb{R}^3}\int_{m_0}^\infty \Psi h(x,m)dmdx = \int_{\mathbb{R}^3}\int_{m_0}^\infty md(x,m)h(x,m)\,dmdx = \|\mathscr{D}h\|_1 \le \vartheta\|h\|_1 < \infty.$

In terms Ψ and α the operator $\mathscr{C}$ takes the expression

$$\mathscr{C}h(x,m) = \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi h * \Psi h)(m)}{m\alpha(h)},$$

where $h \in \mathscr{X}_{1+} \setminus \{0\}$ and

$$(\Psi h * \Psi h)(m) := \int_{\mathbb{R}^3}\int_{m_0}^{m-m_0} \Psi h(x,n)\Psi h(x,m-n)\,dndx.$$

Let g_0 be a function fixed in $\mathscr{X}_{1+} \setminus \{0\}$. We set $\kappa := \alpha(g_0)\vartheta^{-1}$. Let g be any function from $\mathscr{X}_{1+} \setminus \{0\}$ such that $\|g - g_0\|_1 \le \kappa$. Then

$$\alpha(g) = \alpha(g_0) + \alpha(g - g_0) \le 2\alpha(g_0). \tag{24}$$

Making use of the linearity of α and properties of the convolution $*$ we have the following:

$$\begin{aligned} &\mathscr{C}g(x,m) - \mathscr{C}g_0(x,m) \\ &= \chi_{U_{\mathbb{R}}}(m,x)\frac{[(\Psi g * \Psi g)(m)]\alpha(g_0 - g)}{m\alpha(g_0)\alpha(g)} + \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi g * \Psi g)(m)}{m\alpha(g_0)} + \chi_{U_{\mathbb{R}}}(m,x)\frac{(\Psi g_0 * \Psi g_0)(m)}{m\alpha(g_0)} \\ &= \chi_{U_{\mathbb{R}}}(m,x)\frac{[(\Psi g * \Psi g)(m)]\alpha(g_0 - g)}{m\alpha(g_0)\alpha(g)} + \chi_{U_{\mathbb{R}}}(m,x)\frac{[\Psi(g + g_0) * \Psi(g - g_0)](m)}{m\alpha(g_0)}. \end{aligned}$$

It follows that

$$\|\mathscr{C}g - \mathscr{C}g_0\|_1 \le \frac{\alpha(|g_0 - g|)\int_{\mathbb{R}^3}\int_{m_0}^\infty (\Psi g * \Psi g)(m)\,dmdx}{\alpha(g_0)\alpha(g)} + \frac{\int_{\mathbb{R}^3}\int_{m_0}^\infty [\Psi(g + g_0) * |\Psi(g - g_0)|](m)\,dmdx}{\alpha(g_0)} \tag{25}$$

By the Remark 5 we have

$$\int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\Psi g*\Psi g)(m)\,dmdx=\left[\int_{\mathbb{R}^3}\int_{m_0}^{\infty}(\Psi g)(m)\,dmdx\right]^2=(\alpha(g))^2\le\vartheta\|h\|_1<\infty,$$

and

$$\int_{\mathbb{R}^3}\int_{m_0}^{\infty}[\Psi(g+g_0)*|\Psi(g-g_0)|](m)dmdx=\alpha(|g-g_0|)\alpha(g_0+g).$$

Therefore using again the linearity of α and applying (24) yield

$$\begin{aligned}\|\mathscr{C}g-\mathscr{C}g_0\|_1&\le\frac{\alpha(g)\alpha(|g_0-g|)}{\alpha(g_0)}+\frac{\alpha(g_0+g)\alpha(|g-g_0|)}{\alpha(g_0)}\\&\le 5\alpha(|g_0-g|)\\&\le 5\vartheta\|g-g_0\|_1.\end{aligned}\tag{26}$$

Next we prove that the later inequality is valid for all $h,g\in\mathscr{X}_{1+}\setminus\{0\}$. Let us fix h,g in $\mathscr{X}_{1+}\setminus\{0\}$ and let $h_t=(1-t)h+tg$ for $t\in[0,1]$. Since the function $t\mapsto\alpha(h_t)$ is continuous and $\alpha(h_t)>0$ for each $t\in[0,1]$ we have $\inf_t\alpha(h_t)>0$. Let $\overline{\kappa}=\vartheta^{-1}\inf_t\alpha(h_t)$. Then (26) yields
$\|\mathscr{C}h_s-\mathscr{C}h_t\|_1\le 5\vartheta\|h_s-h_t\|_1$ provided that $\|h_s-h_t\|_1\le\overline{\kappa}$.
Let $\mathbf{n}$ be an integer such that $\mathbf{n}\ge\|h-g\|_1/\overline{\kappa}$ and let $t_i=i/\mathbf{n}$ for $i=0,1,\dots,\mathbf{n}$. Then $\|h_{t_i}-h_{t_{i-1}}\|_1\le\overline{\kappa}$ and then:

$$\begin{aligned}\|\mathscr{C}h-\mathscr{C}g\|_1&\le\sum_{i=1}^{\mathbf{n}}\|\mathscr{C}h_{t_i}-\mathscr{C}h_{t_{i-1}}\|_1\\&\le 5\vartheta\sum_{i=1}^{\mathbf{n}}\|h_{t_i}-h_{t_{i-1}}\|_1\\&=5\vartheta\|h-g\|_1,\end{aligned}\tag{27}$$

where we used the fact that $h_{t_i}-h_{t_{i-1}}=\dfrac{g-h}{\mathbf{n}}$ for any $i=0,1,\dots,\mathbf{n}$. Furthermore by (7) and Remark 5, $\|\mathscr{C}g-\mathscr{C}0\|_1=\|\mathscr{C}g\|_1\le\int_{\mathbb{R}^3}\int_{m_0}^{\infty}md(x,m)g(x,m)\,dmdx\le\vartheta\|g\|_1$ for any $g\in\mathscr{X}_{1+}$. This concludes that the operator $\mathscr{C}$ is continuous at 0. Therefore inequality (27) passes to the limit at $h=0$ or $g=0$, which concludes the proof.

Theorem 3. *Let $\overset{o}{p}\in D(F)\cap\mathscr{X}_{1+}$, the Cauchy problem*

$$\begin{aligned}\frac{du}{dt}(t)&=F[p(t)]+\mathscr{C}[p(t)]\\p_{|t=0}&=\overset{o}{p},\end{aligned}\tag{28}$$

has a global unique solution.

Proof. First we recall that the solution p of (28) is the unique solution of the integral equation

$$p(t)=G_F(t)p_0+\int_0^t G_F(t-s)\mathscr{C}[p(s)]\,ds,\quad t\ge 0,\tag{29}$$

where $(G_F(t))_{t\ge0}$ is the semigroup generated by the operator F given in (23). We consider

$$\mathscr{Y}:=C([0,t_1],\mathscr{X}_{1+})$$

and its norm

$$\|g\|_{\mathscr{Y}}:=\max\{\|g(t)\|_1:0\le t\le t_1\}.$$

furthermore, we let

$$\Xi:=\{g\in\mathscr{Y}:g(t)\in\overline{B}(p_0,r_1)\cap\mathscr{X}_{1+}\ \forall t\in[0,t_1]\},$$

with $r_1\in\mathbb{R}_+$. Now we define $\mathscr{M}$ on Ξ as the mapping

$$(\mathscr{M}g)(t):=G_F(t)f+\int_0^t G_F(t-s)\mathscr{C}[g(s)]\,ds,\ 0\le t\le t_1.$$

Then $\mathscr{M}(\Xi)\subset\mathscr{Y}$ and $(\mathscr{M}g)(t)\in\mathscr{X}_{1+}$ for all $t\in[0,t_1]$. The proof of the existence of a unique solution $p\in\Xi$ to the equation $p=\mathscr{M}p$ follows in the standard way [22, Theorem 6.1.2] since $\mathscr{X}_{1+}$ is a complete metric space as a closed subspace of a Banach space. Consequently the integral equation (29) has a unique solution $p\in C([0,t_1],\mathscr{X}_{1+})$. The existence of a global strong solution to problem (28) immediately follows from the fact that $\mathscr{C}$ is globally Lipschitz, as shown in the Lemma 1.

6 Concluding Remarks

In this article, we used the theory of strongly continuous semigroups of operators [22] to analyze the well-posedness and show existence result of an integro-differential equation modelling convection-coagulation processes. This work generalizes the preceding ones with the inclusion of the spatial transportation kernel which was not considered before. We proved that the full model with combined coagulation-transport operator has global unique solution, thereby addressing the problem of existence of solutions for this model. This may help us analyze in the same way a model with combined coagulation-fragmentation-transport-direction changing whose the full identification of the generator and characterization of its domain remain an open problem.

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