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Solvability of Nonlinear Quadratic Functional Equations

H. H. G. Hashem and A. M. A. El-Sayed*

Faculty of Science, Alexandria University, Alexandria, Egypt

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Abstract: We give an existence theorem for some quadratic functional equations which includes many key integral and functional equations that arise in nonlinear analysis and its applications. In particular, we extend the class of characteristic functions appearing in Chandrasekhar's classical integral equation from astrophysics and retain existence of its solutions. Also, some counter examples are considered.

Keywords: Quadratic functional integral equation; Positive monotonic solutions; Measure of noncompactness.

1 Introduction

The study of integral equations have gained much attention due to extensive applications of these equations in describing numerous events and problems of real world, and the theory of integral equations is rapidly developing with the help of several tools of functional analysis, topology and fixed point theory. For details, we refer to [2]-[11] and [17]-[27].

One of the kinds of integral equations is quadratic integral equations which have received increasing attention during recent years due to its applications in numerous diverse fields of science and engineering for example, the theory of radiative transfer, kinetic theory of gases, the theory of neutron transport and the traffic theory. Many authors have studied different kinds of nonlinear quadratic integral equations in different classes (see[2], [3], [7]-[14] and [17] - [27]). Especially, Chandrasekar's integral equation which has been a subject of much investigation since its appearance around fifty years ago [13].

Let $L_1 = L_1[0,T]$ be the class of Lebesgue integrable functions on $I = [0,T]$ with the standard norm.

Here, we are concerning with the nonlinear quadratic functional equation

$$x(t) = f(t,x(\phi_1(t))) + g(t,x(\phi_2(t))).\psi\left(t, \int_0^{\alpha(t)} u(t,s,x(\phi_3(s)))\, ds\right), \quad t \in I. \tag{1}$$

which includes as special cases numerous function, integral and functional integral equations encountered in nonlinear analysis. For example, the quadratic integral equation of Chandrasekhar type

$$x(t) = 1 + \lambda\, x(t) \int_0^t \frac{t\,\phi(s)}{t+s}\, x(s)\, ds$$

and the quadratic integral equation of fractional order

$$x(t) = f(t) + g(t,x(t)) \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\, h(s,x(s))\, ds, \quad \beta > 0.$$

2 Preliminaries

In this section, we introduce some notations and preliminary facts which are used in the paper.
Now let E be a Banach space with zero element θ and let X be a nonempty bounded subset of E. Moreover denote by $B_r = B(\theta, r)$ the closed ball in E centered at θ and with radius r. In the sequel we shall need some criteria for compactness in measure; the complete description of compactness in measure was given by Fréchet [4], but the following sufficient condition will be more convenient for our purposes (see[4]).

Theorem 1. *Let X be a bounded subset of L_1. Assume that there is a family of subsets $(\Omega_c)_{0\le c\le b-a}$ of the interval (a,b) such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \le x(t_2)$, $(t_1 \in \Omega_c, t_2 \notin \Omega_c)$, then the set X is compact in measure.*

* Corresponding author e-mail: hendhghashem@yahoo.com

The measure of weak noncompactness defined by De Blasi [1] and [15] is given by,

$$\beta(X) = inf(r>0: \text{ there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + K_r)$$

The function $\beta(X)$ possesses several useful properties which may be found in [15].
The convenient formula for the function $\beta(X)$ in L_1 was given by Appel and De Pascale (see [1])

$$\beta(X) = \lim_{\varepsilon\to 0}(sup_{x\in X}(sup[\int_D |x(t)|dt : D \subset [a,b], \text{meas } D \leq \varepsilon])), \quad (2)$$

where the symbol meas D stands for Lebesgue measure of the set D.
Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see[4]) defined by

$$\chi(X) = inf(r>0: \text{ there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + K_r)$$

In the case when the set X is compact in measure, the Hausdorff and De Blasi measures of noncompactness will be identical. Namely we have (see[1] and [15])

Theorem 2.*Let X be an arbitrary nonempty bounded subset of L_1. If X is compact in measure then $\beta(X) = \chi(X)$.*

Finally, we will recall the fixed point theorem due to Darbo [6].

Theorem 3.*Let Q be a nonempty, bounded, closed and convex subset of E and let $H: Q \to Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ, i.e., there exists a constant $\alpha \in [0,1)$ such that $\chi(HX) \leq \alpha\, \chi(X)$ for any nonempty subset X of Q. Then H has at least one fixed point in the set Q.*

3 Existence Theorem

Let the functional operator H be defined as

$$(Hx)(t) = \psi\left(t, \int_0^{\alpha(t)} u(t,s,x(s))\, ds\right),$$

$$(Fx)(t) = f(t,x(t))$$

Then equation (1) may be written in operator form as:

$$(Ax)(t) = (Fx(\phi_1))(t) + (Gx(\phi_2))(t).(Hx(\phi_3))(t)$$

where $(Gx)(t) = g(t,x(t))$.

Consider the following assumptions:

(i) $f,\ g: I \times R \to R$ are functions such that $f,g: I \times R_+ \to R_+$. Moreover, the functions f, g satisfy Carathèodory condition (i.e. are measurable in t for all $x \in R$ and continuous in x for all $t \in I$) and there exist two functions $m_1,\ m_2 \in L_1$ and constants $b_1 > 0,\ b_2 > 0$ such that

$$|f(t,x)| \leq m_1(t) + b_1\,|x|, \quad |g(t,x)| \leq m_2(t) + b_2\,|x|$$

$\forall\,(t,x) \in I \times R.$
Apart from this the functions f and g are nondecreasing in both variables.

(ii) $u: I \times I \times R \to R$ is such that $u(t,s,x) \geq 0$ for $(t,s,x) \in I \times I \times R_+$ and $u(t,s,x)$ satisfies Carathéodory condition (i.e. it is measurable in (t,s) for all $x \in R$ and continuous in x for almost all $(t,s) \in I \times I$).

(iii)There exist a positive constant b_3, a function $m_3 \in L_1$ and a measurable (in both variables) function $k(t,s) = k: I \times I \to R_+$ such that

$$|u(t,s,x)| \leq k(t,s)(m_3(t) + b_3\,|x|) \quad \forall\, t,s \in I \text{ and for } x \in R$$

and the integral operator K, generated by the function k and defined by

$$(Kx)(t) = \int_0^t k(t,s)\, x(s)\, ds, \quad t \in I. \quad (3)$$

maps continuously L_1 into L_∞ on I;

(iv) $t \to u(t,s,x)$ is a.e. nondecreasing on I for almost all fixed $s \in I$ and for each $x \in R_+$;

(v) $\psi: I \times R \to R$ is a function such that $\psi: I \times R_+ \to R_+$. Moreover, the function ψ satisfies Carathèodory condition (i.e. is measurable in t for all $x \in R$ and continuous in x for all $t \in I$) and there exist bounded and measurable function $m(t)$ and a constant $b > 0$, such that

$$|\psi(t,x)| \leq m(t) + b\,|x| \ \forall\,(t,x) \in I \times R.$$

Apart from this the function ψ is nondecreasing in both variables.

(vi) $\alpha: I \to I$ is continuous.

(vii) $\phi_i: I \to I,\ i = 1,2,3$ are increasing, absolutely continuous on and there exist positive constants $B_i,\ i = 1,2,3$ such that $\phi_i' \geq B_i$ a.e. on I;

(viii)Let
$d > \sqrt{4\,M\,b\,b_2\,b_3\,B_1{}^2\,B_2\,B_3\,(||m_1|| + b\,M.||m_2||\,||m_3|| + N\,||m_2||)}$,
$|m(t)| \leq N, \quad M = ||K||_{L_\infty}$

where we assume that
$d = B_1\,B_2\,B_3 - b_1\,B_2\,B_3 - b\,b_2\,M\,B_1\,B_3\,||m_3|| - M\,b\,b_3\,B_1\,B_2\,||m_2|| + b\,b_2\,N\,B_1\,B_3.$

Moreover, we assume that there exists a positive solution r of the quadratic equation

$$b\,b_2\,b_3\,B_1\,M\,r^2 - d\,r + B_1\,B_2\,B_3(||m_1|| + b\,M.||m_2||\,||m_3|| + N\,||m_2||) = 0.$$

and define the set

$$B_r \;=\; \{x \in L_1 : ||x|| \le r\}.$$

For the existence of at least one L_1-positive solution of the quadratic functional equation (1) we have the following theorem.

Theorem 4. *Let the assumptions (i)-(viii) be satisfied. If* $b_1\ B_2\ B_3 + M\ b\ b_2\ B_1\ B_3\ ||m_3|| + \ r\ b\ b_2\ b_3\ M + b_2\ B_1\ B_3\ N < B_1\ B_2\ B_3$, *then the quadratic functional equation (1) has at least one solution* $x \in L_1$ *which is positive and a.e. nondecreasing on* I.

Proof. Take an arbitrary $x \in L_1$, then, we get

$$\begin{aligned} |\,(Ax)(t)\,| \;\le\; & |m_1(t)| + b_1\,|\,x(\phi_1(t))| \\ & + \;(\,m_2(t) + b_2\,|\,x(\phi_2(t))\,|\,) \\ & \left[m(t) + b\int_0^{\alpha(t)} k(t,s)(m_3(s) + b_3\,|\,x(\phi_3(s))\,|\,)\,ds\right] \end{aligned}$$

and

$$||\,(Ax)(t)\,|| \;=\; \int_0^T |\,(Ax)(t)\,|\,dt$$

$$\le \int_0^T |m_1(t)|\,dt \;+\; b_1 \int_0^T |\,x(\phi_1(t))|\,dt$$

$$+ \int_0^T (\,m_2(t)\,dt + b_2\,|\,x(\phi_2(t))\,|\,)\left[m(t) + b\int_0^{\alpha(t)} k(t,s)(m_3(t) + b_3\,|\,x(\phi_3(s))\,|\,)\,ds\right] dt$$

$$\le ||m_1|| + \frac{b_1}{B_1}\int_0^T |\,x(\phi_1(t))|.\phi_1'(t)\,dt + \int_0^T m_2(t)\,m(t)dt + b\int_0^T m_2(t)\int_0^{\alpha(t)} k(t,s)\,m_3(s)\,ds\,dt$$

$$+ b\,b_2 \int_0^T |x(\phi_2(t))| \int_0^{\alpha(t)} k(t,s)\,m_3(s)\,ds\,dt + b\,b_3 \int_0^T m_2(t)\int_0^{\alpha(t)} k(t,s)\,|x(\phi_3(s))|\,ds\,dt$$

$$+ b\,b_2\,b_3 \int_0^T |x(\phi_2(t))| \int_0^{\alpha(t)} k(t,s)\,|x(\phi_3(s))|\,ds\,dt + b_2 \int_0^T m(t)|x(\phi_2(t))|\,dt$$

$$\le ||m_1|| + \frac{b_1}{B_1}\int_0^T |\,x(\phi_1(t))|.\phi_1'(t)\,dt + b\int_0^T m_2(t)\int_0^T k(t,s)\,m_3(s)\,ds\,dt$$

$$+ b\,b_2 \int_0^T |x(\phi_2(t))| \int_0^T k(t,s)\,m_3(s)\,ds\,dt + b\,b_3 \int_0^T m_2(t)\int_0^T k(t,s)\,|x(\phi_3(s))|\,ds\,dt + N\int_0^T m_2(t)\,dt$$

$$+ b\,b_2\,b_3 \int_0^T |x(\phi_2(t))| \int_0^T k(t,s)\,|x(\phi_3(s))|\,ds\,dt + b_2\,N \int_0^T |x(\phi_2(t))|\,dt$$

$$\le ||m_1|| + \frac{b_1}{B_1}\int_{\phi_1(0)}^{\phi_1(T)} |\,x(\phi_1(t))|.\phi_1'(t)\,dt + b\int_0^T m_2(t)\int_0^T k(t,s)\,m_3(s)\,ds\,dt$$

$$+ \frac{b\,b_2}{B_2}\int_{\phi_2(0)}^{\phi_2(T)} |x(\phi_2(t))|.\phi_2'(t)\int_0^T k(t,s)\,m_3(s)\,ds\,dt + \frac{b_2\,N}{B_2}\int_0^T |x(\phi_2(t))|.\phi_2'(t)\,dt + N\,||m_2||$$

$$+ \frac{b\,b_3}{B_3}\int_0^T m_2(t)\int_{\phi_3(0)}^{\phi_3(T)} k(t,s)\,|x(\phi_3(s))|.\phi_3'(s)\,ds\,dt$$

$$+ \frac{b\,b_2\,b_3}{B_2 B_3}\int_{\phi_2(0)}^{\phi_2(T)} |x(\phi_2(t))|.\phi_2'(t)\int_{\phi_3(0)}^{\phi_3(T)} k(t,s)\,|x(\phi_3(s))|.\phi_3'(s)\,ds\,dt$$

$$\le ||m_1|| + \frac{b_1}{B_1}\int_0^T |\,x(\theta)|\,d\theta + b\,M\,||m_2||\,||m_3|| + \frac{M\,b\,b_2\,||m_3||}{B_2}\int_0^T |x(\theta)|\,d\theta + N\,||m_2||,$$

$$+ \frac{M\,b\,b_3\,||m_2||}{B_3}\int_0^T |x(\theta)|\,d\theta + \frac{M\,b\,b_2\,b_3}{B_2\,B_3}\int_0^T |x(\theta)|\,d\theta.\int_0^T |x(\theta)|\,d\theta + \frac{b_2\,N}{B_2}\int_{\phi_2(0)}^{\phi_2(T)} |x(u)|\,du$$

$$\le ||m_1|| + \frac{b_1}{B_1}\,||\,x|| + b\,M\,||m_2||\,||m_3|| + \frac{M\,b\,b_2\,||m_3||}{B_2}\,||x|| + N\,||m_2||$$

$$+ \frac{M\,b\,b_3\,||m_2||}{B_3}\,||x|| + \frac{M\,b\,b_2\,b_3}{B_2\,B_3}\,||x||^2 + \frac{b\,b_2\,N}{B_2}||x||.$$

which shows that the operator A maps the ball B_r into itself with

$$r \;=\; \frac{d - \sqrt{d^2 - 4\,b\,M\,b_2\,b_3\,{B_1}^2\,B_2\,B_3\,(||m_1|| + b\,M.||m_2||\,||m_3|| + N\,||m_2||)}}{2\,b\,M\,b_2\,b_3\,B_1}. \tag{4}$$

Assumption (vii) implies

$$0 \;<\; d^2 - 4\,b\,M\,b_2\,b_3\,{B_1}^2\,B_2\,B_3\,(||m_1|| + b\,M.||m_2||\,||m_3|| + N\,||m_2||) \;<\; d^2,$$

which implies that

$$0 \;<\; \sqrt{d^2 - 4\,b\,M\,b_2\,b_3\,{B_1}^2\,B_2\,B_3\,(||m_1|| + b\,M.||m_2||\,||m_3|| + N\,||m_2||)} \;<\; d.$$

Then d is positive and hence r is a positive constant.
Let Q_r be a subset of $B_r \in L_1$ consisting of all functions which are a.e. nondecreasing on I.
Clearly, the set Q_r is nonempty, bounded, convex and closed [4]. Moreover this set is compact in measure [5].
Then we deduce that the operator A maps Q_r into itself.
Since the operator $(Ux)(t) \;=\; \int_0^{\alpha(t)} u(t,s,x(s))\,ds$ is continuous, then the operator H is continuous and hence the product $G.H$ is continuous. Also, F is continuous. Thus the operator A is continuous on Q_r.
Let X be a nonempty subset of Q_r. Fix $\varepsilon > 0$ and take a measurable subset $D \subset I$ such that meas $D \le \varepsilon$. Then, for any $x \in X$, using the same reasoning as in [4] and [5], we get

$$||\,Ax\,||_{L_1(D)} = \int_D |\,(Ax)(t)\,|\,dt$$

$$\le \int_D |m_1(t)|\,dt \;+\; b_1 \int_D |\,x(\phi_1(t))|\,dt$$

$$+ \int_D (\,m_2(t)\,dt + b_2\,|\,x(\phi_2(t))\,|\,)\left[m(t) + b\int_0^{\alpha(t)} k(t,s)(m_3(s) + b_3\,|\,x(\phi_3(s))\,|\,)\,ds\right] dt$$

$$\le ||m_1||_{L_1(D)} + \frac{b_1}{B_1}\int_D |\,x(\phi_1(t))|.\phi_1'(t)\,dt + b\int_D m_2(t)\int_0^T k(t,s)m_3(s)\,ds\,dt + N\int_D m_2(t)\,dt$$

$$+ b\,b_2 \int_D |x(\phi_2(t))| \int_0^T k(t,s)\,m_3(s)\,ds\,dt + M\,b\,b_3 \int_D m_2(t)\int_0^T |x(\phi_3(s))|\,ds\,dt$$

$$+ M\,b\,b_2\,b_3 \int_D |x(\phi_2(t))| \int_0^T |x(\phi_3(s))|\,ds\,dt + \frac{b_2\,N}{B_2}\int_D |x(\phi_2(t))|\,\phi_2'(t)\,dt$$

$$\le ||m_1||_{L_1(D)} + \frac{b_1}{B_1}\int_D |\,x(\theta)|\,d\theta + b\,M\,||m_2||_{L_1(D)}\,||m_3||_{L_1} + \frac{M\,b\,b_2\,||m_3||_{L_1}}{B_2}\int_D |x(\theta)|\,d\theta$$

$$+ \frac{M\,b\,b_3\,||m_2||_{L_1(D)}}{B_3}\int_0^T |x(\theta)|\,d\theta + \frac{M\,b\,b_2\,b_3}{B_2\,B_3}\int_D |x(\theta)|\,d\theta.\int_0^T |x(\theta)|\,d\theta$$

$$+ \frac{b_2\,N}{B_2}\int_D |x(\theta)|\,d\theta + N\,||m_2||_{L_1(D)}$$

$$\le ||m_1||_{L_1(D)} + \frac{b_1}{B_1}\,||\,x||_{L_1(D)} + M\,b\,||m_2||_{L_1(D)}||m_3||_{L_1} + \frac{M\,b\,b_2\,||m_3||_{L_1}}{B_2}\,||x||_{L_1(D)}$$

$$+ \frac{M\,b\,b_3\,||m_2||_{L_1(D)}}{B_3}\,||x|| + \frac{M\,b\,b_2\,b_3}{B_2\,B_3}\,||x||_{L_1(D)}.||x||$$

$$+ \frac{b_2\,N}{B_2}\,||x||_{L_1(D)} + N\,||m_2||_{L_1(D)}$$

$$\leq ||m_1||_{L_1(D)} + \frac{b_1}{B_1} ||x||_{L_1(D)} + M\, b\, ||m_2||_{L_1(D)}\, ||m_3||_{L_1} + \frac{M\, b\, b_2\, ||m_3||_{L_1}}{B_2} ||x||_{L_1(D)}$$
$$+ \frac{r\, b\, M\, b_3\, ||m_2||_{L_1(D)}}{B_3} + \frac{r\, b\, M\, b_2\, b_3}{B_2\, B_3} ||x||_{L_1(D)}$$
$$+ \frac{b_2\, N}{B_2} ||x||_{L_1(D)} + N\, ||m_2||_{L_1()}$$

Since

$$\lim_{\varepsilon \to 0} \{ sup\{ \int_D |m_i(t)|\, dt : D \subset I, \quad \text{meas}\, D < \varepsilon \} \} = 0, \quad i = 1,2,3.$$

We obtain

$$\beta(Ax(t)) \leq [\frac{b_1}{B_1} + \frac{M\, b\, b_2\, ||m_3||_{L_1}}{B_2} + \frac{M\, b\, b_2\, b_3\, r}{B_2\, B_3} + \frac{b_2\, N}{B_2}]\, \beta(x(t)).$$

This implies

$$\beta(AX) \leq [\frac{b_1}{B_1} + \frac{M\, b\, b_2\, ||m_3||_{L_1}}{B_2} + \frac{M\, b\, b_2\, b_3\, r}{B_2\, B_3} + \frac{b_2\, N}{B_2}]\, \beta(X), \tag{5}$$

where β is the De Blasi measure of weak noncompactness.
Keeping in mind Theorem 2 we can write (5) in the form

$$\chi(AX) \leq [\frac{b_1}{B_1} + \frac{M\, b\, b_2\, ||m_3||_{L_1}}{B_2} + \frac{M\, b\, b_2\, b_3\, r}{B_2\, B_3} + \frac{b_2\, N}{B_2}]\, \chi(X),$$

where χ is the Hausedorff measure of noncompactness.

Since $\frac{b_1}{B_1} + \frac{M\, b\, b_2\, ||m_3||_{L_1}}{B_2} + \frac{M\, b\, b_2\, b_3\, r}{B_2\, B_3} + \frac{b_2\, N}{B_2} < 1$, from Theorem 3 follows that A is contraction with respect to the measure of noncompactness χ. Thus A has at least one fixed point in Q_r which is a solution of the quadratic functional equation.■

4 Special cases and applications

As particular cases of Theorem 4 we can obtain theorems on the existence of positive and a.e. nondecreasing solutions belonging to the space $L_1(I)$ of the following quadratic functional equations:

1. If $\psi(t,x) = \int_0^{\alpha(t)} u(t,s,x(\phi_3(s)))\, ds$, then we obtain the quadratic functional equation [26]

$$x(t) = f(t,x(\phi_1(t))) + g(t,x(\phi_2(t))) \int_0^{\alpha(t)} u(t,s,x(\phi_3(s)))\, ds, \ t \in I.$$

2. If $\psi(t,x) = \int_0^t u(t,s,x(\phi_3(s)))\, ds$, then we obtain the quadratic functional equation

$$x(t) = f(t,x(\phi_1(t))) + g(t,x(\phi_2(t))) \int_0^t u(t,s,x(\phi_3(s)))\, ds, \ t \in I.$$

that was studied in [14].

3. If $\psi(t,x) = \int_0^t u(t,s,x(\phi_3(s)))\, ds$, $g(t,x) = 1$ and $f(t,x) = a(t)$ then we obtain the functional integral equation of Urysohn type

$$x(t) = a(t) + \int_0^t u(t,s,x(\phi_3(s)))\, ds, \ t \in I.$$

4. If $f(t,x) = a(t)$, $u(t,s,x) = h(t,x)$, then we obtain the quadratic integral equation

$$x(t) = a(t) + g(t,x(\phi_1(t))) \int_0^t h(s,x(\phi_2(s)))\, ds, \ t \in I$$

that was proved in [22].

5. If $g(t,x) = 0$, then we obtain the functional equation

$$x(t) = f(t,x(\phi_1(t))), \ t \in I$$

which is the same results proved by Banas [4].

6. If $f(t,x) = a(t)$, $u(t,s,x) = k(t,s)\, h(t,x)$, then we obtain the quadratic integral equation

$$x(t) = a(t) + g(t,x(t)) \int_0^t k(t,s)\, h(s,x(\phi(s)))\, ds, \ t \in I$$

which is the same results proved in [21].

7. If $f(t,x) = a(t)$, $u(t,s,x) = k(t,s)\, h(t,x)$ and $g(t,x) = 1$ then we obtain the functional integral equation

$$x(t) = a(t) + \int_0^t k(t,s)\, h(s,x(\phi(s)))\, ds, \ t \in I$$

which is the same results proved in [5].

8. If $f(t,x) = 0$ for any $t \in I$ and $x \in R_+$ we have

$$x(t) = g(t,x(\phi_2(t))) \int_0^t u(t,s,x(\phi_3(s)))\, ds, \ t \in I.$$

9. If $\phi_i(t) = t$, $i = 1,2,3$, $\psi(t,x) = \int_0^t u(t,s,x(s))\, ds$ for any $t \in I$ and $x \in R_+$ we have

$$x(t) = f(t,x(t)) + g(t,x(t)) \int_0^t u(t,s,x(s))\, ds, \ t \in I.$$

4.1 Chanrasekhar's integral equation

Example 1:
Let us consider the quadratic integral equation of Volterra type having the form

$$x(t) = a(t) + x(t) \int_0^t \frac{t}{t+s}\, u(t,s,x(s))\, ds. \tag{6}$$

This equation represents the Volterra counterpart of the famous Chandrasekhar quadratic integral equation which has numerous application (cf. [2], [3], [7] and [13]). It arose originally in connection with scattering through a homogeneous semi- infinite plane atmosphere [13]. In astrophysical applications of the Chandraskhar's equation the only restriction, that $\int_0^1 \phi(s) ds \leq 1/2$ is treated a necessary condition in [12].
In case $a(t) = 1$ and $u(t,s,x(s)) = \lambda\, \phi(s)\, x(s)$, λ is a positive constant and on I. Then Eqn.(6) has the form

$$x(t) = 1 + \lambda\, x(t) \int_0^t \frac{t\, \phi(s)}{t+s}\, x(s)\, ds.$$

In order to apply our results we have to impose an additional condition that the so-called "characteristic" function ϕ need not be continuous on I it sufficient only to be bounded and measurable on I.
In this case $r = \frac{1-\sqrt{1-4\lambda\, k}}{2\lambda\, k}$ and the assumption (vii) may be reduced to $4\lambda k \leq 1$ where $\sup_{s\in I}\phi(s) = k$.

Example 2:
Consider the following Chandrasekar's integral equation

$$x(t) = 1 + \frac{1}{10}\, x(t) \int_0^1 \frac{s\, t}{t+s}\, x(s)\, ds, \quad t \in [0,1] \tag{7}$$

where $\lambda = \frac{1}{2}$, $\phi(s) = \frac{s}{5}$ and $k = \frac{1}{10}$, then the condition $4\,\lambda\, k \leq 1$, is satisfied and $r = 4$.

Example 3:
Consider the following Chandrasekar's integral equation

$$x(t) = 1 + \frac{1}{15}\, x(t) \int_0^1 \frac{exp(-s)\, t}{t+s}\, x(s)\, ds, \quad t \in \tag{8}$$

where $\lambda = \frac{1}{5}$, $\phi(s) = \frac{1}{3}exp(-s)$ and $k \leq 1$, then the condition $4\,\lambda\, k \leq 1$ is satisfied.

4.2 Fractional integral equation

Example 1:
Now, consider the quadratic integral equation of fractional order

$$x(t) = f(t) + g(t,x(t)) \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, h(s,x(s))\, ds, \quad \alpha > 0$$

which was studied in [11] (an existence theorem for continuous solutions was proved) and in [21] (an existence for integrable solutions was proved). In our case, $u(t,s,x(s)) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, h(s,x(s))$ and h satisfies $|h(s,x(s))| \leq m(s) + \frac{b'}{(t-s)^{\alpha-1}}\, |x|$, $m \in L_1$. Then $|u(t,s,x(s))| \leq \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, m(s) + b|x|$, b is positive constant and $k(t,s) = \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, m(s)$. The assumption $\int_0^t k(t,s)ds \leq M$ is reduced by $I^{\beta} m(t) \leq M'$, $\beta \leq \alpha$.
For,

$$\int_0^t k(t,s)ds = I^{\alpha} m(t) = I^{\alpha-\beta} I^{\beta} m(t) \leq M' \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds = \frac{M'}{\Gamma(\alpha-\beta+1)} = M$$

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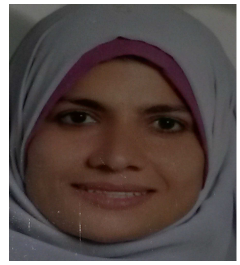

Hind Hashem received the PhD degree in Mathematics at Alexandria University, Egypt. Here research interests are in the areas of pure mathematics (quadratic integral equations of different kinds, coupled systems of differential and integral equations, functional differential and integral equations of fractional orders. She is referee and member of the editorial board of several international journals in the frame of pure and applied mathematics.

Ahmed El-Sayed is Professor of pure mathematics at Alexandria University, faculty of Science, Mathematics and computer science department , Egypt. He is the Managing Editor of Journal of Fractional Calculus andApplications. He is referee and Editor of several international journals in the frame of pure and applied mathematics. His main research interests are: dynamical systems, fractional calculus, modeling, computational methods, the fractional-order logistic equation, Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models, fractional and integral equations.