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Identities Involving Some New Special Polynomials Arising from the Applications of Fractional Calculus

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Abstract: Inspired by a number of recent investigations, we introduce the new analogues of the Apostol-Bernoulli polynomials and the Apostol-Euler polynomials, the Apostol-Genocchi polynomials based on Mittag-Leffler function. Making use of the Caputo-fractional derivative, we derive some new interesting identities of these polynomials. It turns out that some known results are derived as special cases.

Keywords: Mittag-Leffler function, Caputo-Fractional derivative, Apostol-Bernoulli polynomials, Apostol-Genocchi polynomials, Apostol-Euler polynomials

1 Introduction

The concept of fractional is popularly appeared from a question raised in the year 1695 by L'Hôpital (1661-1704) to Leibniz (1646-1716), which searched the way of Leibniz's notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N}^* := \mathbb{N} \cup \{0\}$ when $n = \frac{1}{2}$ (What if $n = \frac{1}{2}$). In his response, dated 30 September 1695, Leibniz wrote to L' Hôpital as follows:"$\cdots$This is an apparent paradox from which, one day, useful consequences will be drawn$\cdots$" (see [1], [2]).

Subsequent mention of fractional derivatives was made, in some context, by Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Grünwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917 (for details, see [1], [2]).

One of the most recent works on the subject of fractional calculus is the book of Podlubny [1] published in 1999, which deals with theory of fractional differential equations.

One of the fundamental functions of the fractional calculus is Euler's gamma function $\Gamma(\xi)$, which generalizes the fractional "$n!$" defined by

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt, \tag{1}$$

which converges in the right half of the complex plane $\mathrm{Re}(\xi) > 0$ (see [1], [2]). This function satisfies the following functional equations:

$$\Gamma(\xi+1) = \xi\Gamma(\xi) \text{ and } B(\xi,\gamma) = \frac{\Gamma(\xi)\Gamma(\gamma)}{\Gamma(\xi+\gamma)} \tag{2}$$

where $B(\xi,\gamma)$ is known as Beta function (see [1], [2], [6], [10]).

The Mittag-Leffler function, which generalizes the exponential function e^z, is given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)} \text{ (see [1], [2]).} \tag{3}$$

Obviously that

$$E_1(z) = e^z.$$

The Mittag-Leffler function plays a vital and important role in the concept of *fractional calculus.*

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The generalization of Mittag-Leffler function is also defined by the following series expansion:

$$E_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ (\alpha>0,\beta>0)\ \text{(see [1], [2])}. \tag{4}$$

By (4), we have

$$\begin{aligned}E_{1,1}(z)&=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+1)}=e^z,\\ E_{1,2}(z)&=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+2)}=\frac{e^z-1}{z},\\ E_{1,3}(z)&=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k+3)}=\frac{e^z-z-1}{z^2}.\end{aligned} \tag{5}$$

Continuing this process gives

$$E_{1,m}(z)=\frac{1}{z^{m-1}}\left\{e^z-\sum_{k=0}^{m-2}\frac{z^k}{k!}\right\}\ \text{(see [1], [2])}.$$

Note that various generalizations of the Mittag-Leffler function which are studied by Humbert and Delerue [1] and by Chak [3], were further extended by Srivastava [2].

In [1],[2], the Riemann-Liouville fractional integral of order α for a function f is defined by

$$\mathscr{I}^{(\alpha)}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,ds, \tag{6}$$
$$(f:(0,\infty)\to\mathbb{R}\ \text{and}\ \alpha>0).$$

The Caputo-fractional derivative of higher order for a continuous function f is given by

$$\mathscr{D}^{(\alpha)}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{f^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \tag{7}$$
$$(f:(0,\infty)\to\mathbb{R}\ \text{and}\ \alpha>0),$$

where n is the smallest integer greater than or equal to α (see [1], [2]).

From (6) and (7), we have

$$\mathscr{D}^{(\alpha)}f(t)=\mathscr{D}^{(k)}\left[\mathscr{I}^{(k-\alpha)}f(t)\right],k\in\mathbb{N}. \tag{8}$$

From (8), it follows that

$$\mathscr{D}^{(\alpha)}t^n=\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}\text{(see [1], [2])}. \tag{9}$$

The fractional derivative of the product fg, which is called Leibniz rule, is given by

$$\mathscr{D}^{(\alpha)}[f(t)g(t)]=\sum_{k=0}^{\infty}\binom{\alpha}{k}f^{(k)}(t)\mathscr{D}^{(\alpha-k)}g(t)\ \text{(see [1], [2])}. \tag{10}$$

In the next section, we consider analogues of Bernoulli, Euler and Genocchi polynomials which are introduced using (3).

2 Analogues of Bernoulli, Euler and Genocchi Polynomials and their Properties

Recently, analogues of Bernoulli, Euler and Genocchi polynomials were studied by many mathematicians [4-24]. We are now ready to give the definition of generating functions, corresponding to Mittag-Leffler function, of Bernoulli, Euler and Genocchi polynomials.

Definition 1. *Let $\alpha>0$ and $\lambda>0$, define*

$$\begin{aligned}\mathscr{K}(x,z:\alpha\mid\lambda)&=\sum_{n=0}^{\infty}\mathscr{B}_n(x:\alpha\mid\lambda)\frac{z^n}{n!}=\frac{z}{\lambda E_\alpha(z)-1}e^{xz},\\ \mathscr{I}(x,z:\alpha\mid\lambda)&=\sum_{n=0}^{\infty}\mathscr{E}_n(x:\alpha\mid\lambda)\frac{z^n}{n!}=\frac{2}{\lambda E_\alpha(z)+1}e^{xz},\\ \mathscr{M}(x,z:\alpha\mid\lambda)&=\sum_{n=0}^{\infty}\mathscr{G}_n(x:\alpha\mid\lambda)\frac{z^n}{n!}=\frac{2z}{\lambda E_\alpha(z)+1}e^{xz},\end{aligned}$$

where $\mathscr{B}_n(x:\alpha\mid\lambda)$, $\mathscr{E}_n(x:\alpha\mid\lambda)$ and $\mathscr{G}_n(x:\alpha\mid\lambda)$ are called, respectively, Bernoulli-type, Euler-type and Genocchi-type polynomials.

Corollary 1. *Taking $\alpha=1$ in Definition 1, we have*

$$\begin{aligned}\sum_{n=0}^{\infty}B_n(x\mid\lambda)\frac{z^n}{n!}&=\frac{z}{\lambda e^z-1}e^{xz},\\ \sum_{n=0}^{\infty}E_n(x\mid\lambda)\frac{z^n}{n!}&=\frac{2}{\lambda e^z+1}e^{xz},\end{aligned}$$

and

$$\sum_{n=0}^{\infty}G_n(x\mid\lambda)\frac{z^n}{n!}=\frac{2z}{\lambda e^z+1}e^{xz},$$

where $B_n(x\mid\lambda)$, $E_n(x\mid\lambda)$ and $G_n(x\mid\lambda)$ are called Apostol-Bernoulli polynomials, Apostol-Euler polynomials and Apostol-Genocchi polynomials, respectively (see [8], [9], [18], [19], [22], [23], [25]).

Corollary 2. *Substituting $\alpha=\lambda=1$ in Definition 1, we have*

$$\begin{aligned}\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!}&=\frac{z}{e^z-1}e^{xz},\\ \sum_{n=0}^{\infty}E_n(x)\frac{z^n}{n!}&=\frac{2}{e^z+1}e^{xz},\end{aligned}$$

and

$$\sum_{n=0}^{\infty}G_n(x)\frac{z^n}{n!}=\frac{2z}{e^z+1}e^{xz},$$

where $B_n(x)$, $E_n(x)$ and $G_n(x)$ are called classical Bernoulli polynomials, classical Euler polynomials and classical Genocchi polynomials, respectively (see [8], [9], [18], [19], [22], [23], [25]).

Taking $x=0$ in the above definition, we have

$$\mathscr{B}_n(0:\alpha\mid\lambda) := \underbrace{\mathscr{B}_n(\alpha\mid\lambda)}_{\text{Bernoulli-type number}}$$
$$\mathscr{E}_n(0:\alpha\mid\lambda) := \underbrace{\mathscr{E}_n(\alpha\mid\lambda)}_{\text{Euler-type number}}$$
$$\mathscr{G}_n(0:\alpha\mid\lambda) := \underbrace{\mathscr{G}_n(\alpha\mid\lambda)}_{\text{Genocchi-type number}}$$

and from the above, we write

$$\begin{aligned}\mathscr{K}(0,z:\alpha\mid\lambda) &:= \mathscr{K}(z:\alpha\mid\lambda),\\ \mathscr{I}(0,z:\alpha\mid\lambda) &:= \mathscr{I}(z:\alpha\mid\lambda),\\ \mathscr{M}(0,z:\alpha\mid\lambda) &:= \mathscr{M}(z:\alpha\mid\lambda).\end{aligned} \tag{11}$$

Matching Definition 1 and (11), we get the following corollary.

Corollary 3. *The following functional equations hold true:*

$$\begin{aligned}\mathscr{K}(x,z:\alpha\mid\lambda) &= \mathscr{K}(z:\alpha\mid\lambda)e^{xz},\\ \mathscr{I}(x,z:\alpha\mid\lambda) &= \mathscr{I}(z:\alpha\mid\lambda)e^{xz},\\ \mathscr{M}(x,z:\alpha\mid\lambda) &= \mathscr{M}(z:\alpha\mid\lambda)e^{xz}.\end{aligned}$$

By using (4) and Corollary 3, becomes

$$\sum_{n=0}^{\infty}\mathscr{B}_n(x:\alpha\mid\lambda)\frac{z^n}{n!} = \left(\sum_{n=0}^{\infty}\mathscr{B}_n(\alpha\mid\lambda)\frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{z^n}{n!}\right).$$

From the rule of Cauchy product, we get

$$\sum_{n=0}^{\infty}\mathscr{B}_n(x:\alpha\mid\lambda)\frac{z^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k(\alpha\mid\lambda)x^{n-k}\right)\frac{z^n}{n!}. \tag{12}$$

Comparing the coefficients of $\frac{z^n}{n!}$ in (12), we have

$$\mathscr{B}_n(x:\alpha\mid\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k(\alpha\mid\lambda)x^{n-k}. \tag{13}$$

Similarly, we can get identities for Euler-type and Genocchi-type polynomials. Therefore, we discover the following theorem.

Theorem 1. *The following identities hold true:*

$$\begin{aligned}\mathscr{B}_n(x:\alpha\mid\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathscr{B}_k(\alpha\mid\lambda)x^{n-k},\\ \mathscr{E}_n(x:\alpha\mid\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathscr{E}_k(\alpha\mid\lambda)x^{n-k},\\ \mathscr{G}_n(x:\alpha\mid\lambda) &= \sum_{k=0}^{n}\binom{n}{k}\mathscr{G}_k(\alpha\mid\lambda)x^{n-k}.\end{aligned}$$

Let us now apply the familiar derivative $\frac{d}{dx}$ in the both sides of Definition 1,

$$\begin{aligned}\frac{d}{dx}\left(\sum_{n=0}^{\infty}\mathscr{B}_n(x:\alpha\mid\lambda)\frac{z^n}{n!}\right) &= \sum_{n=0}^{\infty}\left(\frac{d}{dx}\mathscr{B}_n(x:\alpha\mid\lambda)\right)\frac{z^n}{n!}\\ &= \frac{z}{\lambda E_\alpha(z)-1}\left[\frac{d}{dx}e^{xz}\right]\\ &= \frac{z^2}{\lambda E_\alpha(z)-1}e^{xz}\\ &= \sum_{n=0}^{\infty}\mathscr{B}_n(\alpha\mid\lambda)\frac{z^{n+1}}{n!}.\end{aligned}$$

Similarly, we can procure the derivatives of Euler-type and Genocchi-type polynomials. Thus, we state the following theorem.

Theorem 2. *The following identities hold true:*

$$\frac{d}{dx}\mathscr{B}_n(x:\alpha\mid\lambda) = n\mathscr{B}_{n-1}(x:\alpha\mid\lambda),$$
$$\frac{d}{dx}\mathscr{E}_n(x:\alpha\mid\lambda) = n\mathscr{E}_{n-1}(x:\alpha\mid\lambda)$$

and

$$\frac{d}{dx}\mathscr{G}_n(x:\alpha\mid\lambda) = n\mathscr{G}_{n-1}(x:\alpha\mid\lambda).$$

Polynomials $\mathscr{A}_n(x)$ are called Appell polynomials when they have the following identity:

$$\frac{d}{dx}\mathscr{A}_n(x:\alpha) = n\mathscr{A}_{n-1}(x:\alpha). \tag{14}$$

So, by Theorem 2 and (14), we have the following corollary.

Corollary 4. *Our polynomials, which are $\mathscr{B}_n(x:\alpha\mid\lambda)$, $\mathscr{E}_n(x:\alpha\mid\lambda)$ and $\mathscr{G}_n(x:\alpha\mid\lambda)$, are Appell polynomials.*

From Theorem 2, we have

$$\int_0^1\mathscr{B}_n(x:\alpha\mid\lambda)\,dx = \frac{\mathscr{B}_{n+1}(1:\alpha\mid\lambda)-\mathscr{B}_n(x:\alpha\mid\lambda)}{n+1}.$$

More generally,

$$\int_x^{x+1}\mathscr{B}_n(y:\alpha\mid\lambda)\,dy = \frac{\mathscr{B}_{n+1}(x+1:\alpha\mid\lambda)-\mathscr{B}_n(x:\alpha\mid\lambda)}{n+1}.$$

Thus, we get the following theorem.

Theorem 3. *The following identities hold true:*

$$\int_x^{x+1}\mathscr{B}_n(y:\alpha\mid\lambda)\,dy = \frac{\mathscr{B}_{n+1}(x+1:\alpha\mid\lambda)-\mathscr{B}_n(x:\alpha\mid\lambda)}{n+1},$$
$$\int_x^{x+1}\mathscr{E}_n(y:\alpha\mid\lambda)\,dy = \frac{\mathscr{E}_{n+1}(x+1:\alpha\mid\lambda)-\mathscr{E}_n(x:\alpha\mid\lambda)}{n+1}$$

and

$$\int_x^{x+1}\mathscr{G}_n(y:\alpha\mid\lambda)\,dy = \frac{\mathscr{G}_{n+1}(x+1:\alpha\mid\lambda)-\mathscr{G}_n(x:\alpha\mid\lambda)}{n+1}.$$

3 Identities Including Special Polynomials Arising from Fractional Calculus

Recent works involving the integral of the product of several type Bernstein polynomials [15], p-adic integral representation for q-Bernoulli numbers and q-integral representation of q-Bernstein polynomials [7], fermionic p-adic integral representation for Frobenius-Euler numbers and polynomials [14], derivative representations of Bernstein polynomials [17] have been investigated.

In this final part, we derive some new interesting identities related to special polynomials by utilizing from Caputo-fractional derivative and Riemann-Liouville integral.

As well known, Apostol-Bernoulli polynomials are given to be:

$$\mathscr{F}(t,z\mid\lambda)=\frac{z}{\lambda e^{z}-1}e^{tz}=\sum_{n=0}^{\infty}B_{n}(t\mid\lambda)\frac{z^{n}}{n!}. \tag{15}$$

Note that Apostol-Bernoulli polynomials are analytic on the region $\mathfrak{D}=\{z\in\mathbb{C}\mid|z+\log\lambda|<2\pi\}$ (see [19], [22], [23]). Differentiating in the both sides of (15), we have

$$\frac{d}{dt}B_{n}(t\mid\lambda)=nB_{n-1}(t\mid\lambda)\ \text{(see [19], [22], [23])}. \tag{16}$$

When $t=0$ in (15), we have $B_{n}(0\mid\lambda):=B_{n}(\lambda)$ that are called Bernoulli numbers, and can be generated by

$$\mathscr{F}(z\mid\lambda)=\frac{z}{\lambda e^{z}-1}=\sum_{n=0}^{\infty}B_{n}(\lambda)\frac{z^{n}}{n!}. \tag{17}$$

By (15) and (17), we have the following functional equation:

$$\mathscr{F}(t,z\mid\lambda)=e^{tz}\mathscr{F}(z\mid\lambda).$$

By using Taylor's formula in the last identity, we have

$$B_{m}(t\mid\lambda)=\sum_{k=0}^{m}\binom{m}{k}t^{m-k}B_{k}(\lambda)=\sum_{k=0}^{m}\binom{m}{k}t^{k}B_{m-k}(\lambda)$$

(see [19], [22], [23]).

Taking $f(t)=B_{m}(t\mid\lambda)$ in (7) gives

$$\begin{aligned}\mathscr{D}^{\alpha}B_{m}(t\mid\lambda)&=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{\frac{d^{n}}{dt^{n}}B_{m}(t\mid\lambda)\mid_{t=s}}{(t-s)^{\alpha-n+1}}ds\\&=m(m-1)\cdots(m-n+1)\\&\quad\times\sum_{k=0}^{m-n}\binom{m-n}{k}B_{m-n-k}(\lambda)\\&\quad\times\left[\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{s^{k}}{(t-s)^{\alpha-n+1}}ds\right]\\&=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\\&\quad\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}B_{m-n-k}(\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}.\end{aligned}$$

Therefore, we procure the following theorem.

Theorem 4. *The following identity*

$$\mathscr{D}^{\alpha}B_{m}(t\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}B_{m-n-k}(\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}$$

is true.

In [19], Apostol-Bernoulli polynomials of higher order are defined by

$$\underbrace{\frac{z}{\lambda e^{z}-1}\frac{z}{\lambda e^{z}-1}\cdots\frac{z}{\lambda e^{z}-1}}_{h-times}e^{tz}=\sum_{m=0}^{\infty}B_{m}^{(h)}(t\mid\lambda)\frac{z^{n}}{n!}. \tag{18}$$

Taking into account that $B_{m}^{(h)}(t)$ is analytic on $\mathfrak{D}$. It follows from (18), we have

$$\frac{d}{dt}B_{m}^{(h)}(t\mid\lambda)=mB_{m-1}^{(h)}(t\mid\lambda) \tag{19}$$

and

$$\frac{d^{n}}{dt^{n}}B_{m}^{(h)}(t\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}B_{m-n}^{(h)}(t\mid\lambda)\ \text{(see [19])}. \tag{20}$$

Substituting $t=0$ into (18), we have $B_{m}^{(h)}(0\mid\lambda):=B_{m}^{(h)}(\lambda)$ that are called Bernoulli polynomials of higher order.

Owing to (18) and (20), we see that

$$\begin{aligned}\mathscr{D}^{\alpha}B_{m}^{(h)}(t\mid\lambda)&=\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{\frac{d^{n}}{dt^{n}}B_{m}^{(h)}(t\mid\lambda)\mid_{t=s}}{(t-s)^{\alpha-n+1}}ds\\&=m(m-1)\cdots(m-n+1)\\&\quad\times\sum_{k=0}^{m-n}\binom{m-n}{k}B_{m-n-k}^{(h)}(\lambda)\\&\quad\times\left[\frac{1}{\Gamma(n-\alpha)}\int_{0}^{t}\frac{s^{k}}{(t-s)^{\alpha-n+1}}ds\right]\\&=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}B_{m-n-k}^{(h)}(\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}\\&=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}\\&\quad\times\left(\begin{array}{c}\sum_{\substack{s_{1}+s_{2}+\cdots+s_{h}=m-n-k\\ s_{h}\geq0}}\binom{m-n-k}{s_{1},s_{2},\cdots,s_{h}}\\ \times\left(\prod_{j=1}^{h}B_{s_{j}}\right)\end{array}\right).\end{aligned}$$

Therefore, we can state the following theorem.

Theorem 5. *The following equality*

$$\mathscr{D}^{\alpha}B_m^{(h)}(t\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}\times\left(\begin{array}{c}\sum_{\substack{s_1+s_2+\cdots+s_h=m-n-k\\ s_h\geq 0}}\binom{m-n-k}{s_1,s_2,\cdots,s_h}\\ \times\left(\prod_{j=1}^{h}B_{s_j}\right)\end{array}\right)$$

is true.

We recall the definition of generating function for Bernoulli-type polynomials as follows:

$$\sum_{n=0}^{\infty}\mathscr{B}_n(x:\alpha\mid\lambda)\frac{z^n}{n!}=\frac{z}{\lambda E_{\alpha}(z)-1}e^{xz}$$

and also

$$\frac{d}{dx}\mathscr{B}_n(x:\alpha\mid\lambda)=n\mathscr{B}_{n-1}(x:\alpha\mid\lambda).$$

Taking $f(t)=\mathscr{B}_n(x:\alpha\mid\lambda)$ in (7), we compute

$$\begin{aligned}D^{\alpha}\mathscr{B}_n(x:\alpha\mid\lambda)&=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\frac{d^n}{dt^n}\mathscr{B}_n(x:\alpha\mid\lambda)\mid_{t=s}}{(t-s)^{\alpha-n+1}}ds\\&=m(m-1)\cdots(m-n+1)\\&\quad\times\sum_{k=0}^{m-n}\binom{m-n}{k}\mathscr{B}_n(\alpha\mid\lambda)\\&\quad\times\left[\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{s^k}{(t-s)^{\alpha-n+1}}ds\right]\\&=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\\&\quad\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}\mathscr{B}_n(\alpha\mid\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}.\end{aligned}$$

We can acquire similar identities for Euler-type polynomials and Genocchi-type polynomials. So, we state the following interesting theorem.

Theorem 6. *The following equalities hold true:*

$$D^{\alpha}\mathscr{B}_n(x:\alpha\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}\mathscr{B}_n(\alpha\mid\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n},$$

$$D^{\alpha}\mathscr{E}_n(x:\alpha\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}\mathscr{E}_n(\alpha\mid\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n},$$

and

$$D^{\alpha}\mathscr{G}_n(x:\alpha\mid\lambda)=\frac{\Gamma(m+1)}{\Gamma(m-n+1)}\times\sum_{k=0}^{m-n}\frac{k!\binom{m-n}{k}\mathscr{G}_n(\alpha\mid\lambda)}{\Gamma(n+k-\alpha+1)}t^{k-\alpha+n}.$$

4 Conclusion

In this article, we have introduced the new analogues of the Apostol-Bernoulli polynomias, the Apostol-Euler polynomials, the Apostol-Genocchi polynomials based on the Mittag-Leffler function. As applications of these definitions, we get some new recurrence relations for aforementioned polynomials. Based upon these relations, we also derive some new interesting identities of these polynomials by using the Caputo-fractional derivative.

It would be interesting to apply the results of this paper to other known special polynomials, e.g., to some other Hermite polynomials, Legendre polynomials and related polynomials. We plan to deal with such related problems in our subsequent works.

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