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Contracting Quadratic Operators of Bisexual population

Nasir N. Ganikhodjaev[1] *and Uygun U. Jamilov*[1,2,*]

[1] Department of Computational and Theoretical Sciences, Faculty of Science,IIUM, 25200, Kuantan, Malaysia
[2] Institute of Mathematics, National University of Uzbekistan, 100125, Tashkent, Uzbekistan

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Abstract: In this paper we find a sufficient condition under which the operator of bisexual population is contraction and show that this condition is not necessary.

Keywords: Quadratic stochastic operator, fixed point,trajectory, contracting operators.

1 Introduction

The action of genes is manifested statistically in sufficiently large communities of matching individuals (belonging to the same species). These communities are called populations [2]. The population exists not only in space but also in time, i.e. it has its own life cycle. The basis for this phenomenon is reproduction by mating. Mating in a population can be free or subject to certain restrictions.

The whole population in space and time comprises discrete generations $F_0, F_1, \dots$. The generation F_{n+1} is the set of individuals whose parents belong to the F_n generation. A state of a population is a distribution of probabilities of the different types of organisms in every generation. Type partition is called differentiation. The simplest example is sex differentiation. In bisexual population any kind of differentiation must agree with the sex differentiation, i.e. all the organisms of one type must belong to the same sex. Thus, it is possible to speak of male and female types.

The evolution (or dynamics) of a population comprises a determined change of state in the next generations as a result of reproductions and selection. This evolution of a population can be studied by a dynamical system (iterations) of a quadratic stochastic operator.

The history of the quadratic stochastic operators can be traced back to the work of S. Bernshtein [1]. For more than 80 years this theory has been developed and many papers were published (see [1]-[7],[10]-[17]). Several problems of physical and biological systems lead to necessity of study the asymptotic behavior of the trajectories of quadratic stochastic operators.

Let $E=\{1,2,\dots,m\}$. By the $(m-1)-$ simplex we mean the set

$$S^{m-1}=\{\mathbf{x}=(x_1,\dots,x_m)\in R^m : x_i\geq 0, \sum_{i=1}^m x_i=1\}.$$

Each element $\mathbf{x}\in S^{m-1}$ is a probability measure on E and so it may be looked upon as the state of a biological (physical and so on) system of m elements.

A quadratic stochastic operator $V: S^{m-1}\mapsto S^{m-1}$ has the form

$$V: x'_k=\sum_{i,j=1}^m p_{ij,k}x_ix_j, \quad (k=1,\dots,m), \tag{1}$$

where $p_{ij,k}-$ coefficient of heredity and

$$p_{ij,k}=p_{ji,k}\geq 0, \quad \sum_{k=1}^m p_{ij,k}=1, \quad (i,j,k=1,\dots,m).$$

For a given $\mathbf{x}^{(0)}\in S^{m-1}$, the trajectory $\{\mathbf{x}^{(n)}\},\ n=0,1,2,\dots$ of $\mathbf{x}^{(0)}$ under the action of QSO (1) is defined by $\mathbf{x}^{(n+1)}=V(\mathbf{x}^{(n)})$, where $n=0,1,2,\dots$

One of the main problems in mathematical biology is to study the asymptotic behavior of the trajectories. There are many papers devoted to study of the evolution of the free population, i.e. to study of dynamical system generated by quadratic stochastic operator (1), see e.g.

* Corresponding author e-mail: jamilovu@yandex.ru

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[3]-[16]. In [15] a survey of theory quadratic stochastic operators is given.

In this paper we find a condition under which the evolutionary operators of bisexual population is contraction.

2 Definitions

In this section following [2], we describe the evolution operator of a bisexual population. Assuming that the population is bisexual we suppose that the set of females can be partitioned into finitely many different types indexed by $\{1,2,\ldots,n\}$ and, similarly, that the male types are indexed by $\{1,2,\ldots,\nu\}$. The number $n+\nu$ is called the dimension of the population. The population is described by its state vector $(\mathbf{x},\mathbf{y})$ in $S^{n-1}\times S^{\nu-1}$, the product of two unit simplexes in $\mathbb{R}^n$ and $\mathbb{R}^\nu$ respectively. Vectors $\mathbf{x}$ and $\mathbf{y}$ are the probability distributions of the females and males over the possible types:

$$x_i\geq 0,\ \ \sum_{i=1}^{n}x_i=1;\ \ y_j\geq 0,\ \ \sum_{j=1}^{\nu}y_j=1.$$

Denote $S=S^{n-1}\times S^{\nu-1}$. We call the partition into types hereditary if for each possible state $\mathbf{z}=(\mathbf{x},\mathbf{y})\in S$ describing the current generation, the state $\mathbf{z}'=(\mathbf{x}',\mathbf{y}')\in S$ is uniquely defined describing the next generation. This means that the association $\mathbf{z}\mapsto\mathbf{z}'$ defined a map $V:S\mapsto S$ called the evolution operator. For any point $\mathbf{z}^{(0)}\in S$ the sequence $\mathbf{z}^{(t)}=V(\mathbf{z}^{(t-1)}),\ \ t=1,2,\ldots$ is called the trajectory of $\mathbf{z}^{(0)}$. Let $p^{(f)}_{ij,k}$ and $p^{(m)}_{ij,l}$ be inheritance coefficients defined as the probability that a female offspring is type k and, respectively, that a male offspring is type l, when the parental pair is $i,j(i,k=1,2,\ldots,n;$ and $j,l=1,2,\ldots,\nu)$. We have

$$p^{(f)}_{ij,k}\geq 0,\sum_{k=1}^{n}p^{(f)}_{ij,k}=1,\ \ p^{(m)}_{ij,l}\geq 0,\sum_{l=1}^{\nu}p^{(m)}_{ij,l}=1.$$

Let $\mathbf{z}'=(\mathbf{x}',\mathbf{y}')$ be the state of the offspring population at the birth stage. This is obtained from inheritance coefficients as

$$W:\begin{cases}x'_k=\sum\limits_{i,j=1}^{n,\nu}p^{(f)}_{ij,k}x_iy_j,\ \ (1\leq k\leq n)\\ y'_l=\sum\limits_{i,j=1}^{n,\nu}p^{(m)}_{ij,l}x_iy_j,\ \ (1\leq l\leq \nu).\end{cases}\tag{2}$$

We see from (2) that for a bisexual population the evolution operator is a quadratic mapping of S into itself. But for free population the operator is quadratic mapping of the simplex into itself given by (1).

In [8,9] an algebra of the bisexual population is defined as the following:
Consider $\{\mathbf{e}_1,\ldots,\mathbf{e}_{n+\nu}\}$ the canonical basis on $\mathbb{R}^{n+\nu}$ and divide the basis as $\mathbf{e}^{(f)}_i=\mathbf{e}_i, i=1,\ldots,n$ and $\mathbf{e}^{(m)}_i=\mathbf{e}_{n+i}, i=1,\ldots,\nu$. Introduce on $\mathbb{R}^{n+\nu}$ a multiplication defined by

$$\begin{aligned}&\mathbf{e}^{(f)}_i\mathbf{e}^{(m)}_j=\mathbf{e}^{(m)}_j\mathbf{e}^{(f)}_i=\tfrac{1}{2}\left(\sum_{k=1}^{n}p^{(f)}_{ij,k}\mathbf{e}^{(f)}_k+\sum_{l=1}^{\nu}p^{(m)}_{ij,l}\mathbf{e}^{(m)}_l\right);\\&\mathbf{e}^{(f)}_i\mathbf{e}^{(f)}_k=0,\ \ i,k=1,\ldots,n;\\&\mathbf{e}^{(m)}_j\mathbf{e}^{(m)}_l=0,\ \ j,l=1,\ldots,\nu;\end{aligned}\tag{3}$$

Thus the coefficients of bisexual inheritance is the structure constants of an algebra, i.e. a bilinear mapping of $\mathbb{R}^{n+\nu}\times\mathbb{R}^{n+\nu}$ to $\mathbb{R}^{n+\nu}$. The general formula for the multiplication is the extension of (3) by bilinearity, i.e. for $\mathbf{z},\mathbf{t}\in\mathbb{R}^{n+\nu}$,

$$\mathbf{z}=(\mathbf{x},\mathbf{y})=\sum_{i=1}^{n}x_i\mathbf{e}^{(f)}_i+\sum_{j=1}^{\nu}y_j\mathbf{e}^{(m)}_j,$$

$$\mathbf{t}=(\mathbf{u},\mathbf{v})=\sum_{i=1}^{n}u_i\mathbf{e}^{(f)}_i+\sum_{j=1}^{\nu}v_j\mathbf{e}^{(m)}_j$$

using (3), we obtain

$$\begin{aligned}\mathbf{z}\mathbf{t}=&\tfrac{1}{2}\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{\nu}p^{(f)}_{ij,k}(x_iv_j+u_iy_j)\right)\mathbf{e}^{(f)}_k+\\&+\tfrac{1}{2}\sum_{l=1}^{\nu}\left(\sum_{i=1}^{n}\sum_{j=1}^{\nu}p^{(m)}_{ij,l}(x_iv_j+u_iy_j)\right)\mathbf{e}^{(m)}_l.\end{aligned}\tag{4}$$

From (4) and using (2), in the particular case that $\mathbf{z}=\mathbf{t}$, i.e. $\mathbf{x}=\mathbf{u}$ and $\mathbf{y}=\mathbf{v}$, we obtain

$$\mathbf{z}\mathbf{z}=\mathbf{z}^2=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{\nu}p^{(f)}_{ij,k}x_iy_j\right)\mathbf{e}^{(f)}_k+$$

$$+\sum_{l=1}^{\nu}\left(\sum_{i=1}^{n}\sum_{j=1}^{\nu}p^{(m)}_{ij,l}x_iy_j\right)\mathbf{e}^{(m)}_l=W(\mathbf{z}).$$

for any $\mathbf{z}\in S$. This algebraic interpretation is very useful. For example, a bisexual population state $\mathbf{z}=(\mathbf{x},\mathbf{y})$ is an equilibrium (fixed point) precisely when $\mathbf{z}$ is an idempotent element of the set S, i.e. $\mathbf{z}=\mathbf{z}^2$.

The algebra $\mathscr{B}=\mathscr{B}_W$ generated by the evolution operator W (see (2)) is called the *evolution algebra of the bisexual population*.

In [8] it was shown that if $\mathbf{z}$ is a fixed point then $\mathbf{z}\in R^{n+\nu}_0\bigcup R^{n+\nu}_1$, where

$$R^{n+\nu}_\eta=\{\mathbf{z}=(\mathbf{x},\mathbf{y}):\sum_{i=1}^{n}x_i=\sum_{j=1}^{\nu}y_j=\eta\},\ \ \eta=0,1.$$

For simplex $S=S^{n-1}\times S^{\nu-1}$ by tangent space we get

$$R^{n+\nu}_0=\{\mathbf{z}=(\mathbf{x},\mathbf{y}):\sum_{i=1}^{n}x_i=\sum_{j=1}^{\nu}y_j=0\}.$$

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3 Contracting operators

In operator theory, a bounded operator $W : X \to Y$ between normed vector spaces X and Y is said to be a *contraction* if its operator norm $\|W\| \leq 1$.

An extremal example of a quadratic contraction is the constant operator. In this case the coefficients $p_{ij,k}^{(f)}, p_{ij,l}^{(m)}$ do not depend on i and j. This suggests that for a sufficiently small scattering of coefficient for every fixed k,l the quadratic operator will be a contraction. This remark can be expressed as a precise theorem.

The *Lipschitz constant* of an operator $W : \mathbb{R}^{n+\nu} \to \mathbb{R}^{n+\nu}$ is

$$L(W) = \sup_{\mathbf{z}\neq\mathbf{t}} \frac{\|W(\mathbf{z}) - W(\mathbf{t})\|}{\|\mathbf{z}-\mathbf{t}\|},$$

where $\|\cdot\|$ is some norm in $\mathbb{R}^{n+\nu}$. If this norm can be chosen so that $L(W) < 1$ then W will be a strict contraction in this norm with the consequences: unique fixed point, convergence of all trajectories to this point, exponential rate of convergence. Unless otherwise specified, we will use the l_1- norm in the basis $\mathbf{e}_i^{(f)} = \mathbf{e}_i, i = 1,\dots,n$ and $\mathbf{e}_i^{(m)} = \mathbf{e}_{n+i}, i = 1,\dots,\nu$ defined as $\|\mathbf{z}\| = \sum_{i=1}^{n} x_i + \sum_{j=1}^{\nu} y_j$ for $\mathbf{z} = (\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} x_i \mathbf{e}_i^{(f)} + \sum_{j=1}^{\nu} y_j \mathbf{e}_j^{(m)}$.

Lemma 3.1.[2] Let Δ be a convex $n-$ dimensional compact in $\mathbb{R}^n$, $F : \Delta \to \Delta$ be a smooth map. Then (for any norm) $L(F) \equiv \max_{\mathbf{z}\in\Delta} \|d_{\mathbf{z}}F\|$.

Lemma 3.2.[2]. Let a matrix $A = (a_{ij})_{i,j=1}^{n}$ satisfies

$$\sum_{i=1}^{n} a_{i1} = \sum_{i=1}^{n} a_{i2} = \dots = \sum_{i=1}^{n} a_{in}.$$

Then

$$\|A|\mathbb{R}_0^n\| = \frac{1}{2} \max_{j_1\neq j_2} \sum_{i=1}^{n} |a_{ij_1} - a_{ij_2}|,$$

where $A|\mathbb{R}_0^n$ is restriction operator A on $\mathbb{R}_0^n$.

For each $\mathbf{z} \in \mathscr{B}$ we have linear operator $M_{\mathbf{z}} : \mathscr{B} \to \mathscr{B}$ defined by $M_{\mathbf{z}}(\mathbf{t}) = \mathbf{zt}$.

Theorem 3.3. The following inequality holds for the Lipschitz's constant

$$L(W) \leq \max_{i_1,i_2,j} \left(\sum_{k=1}^{n} |p_{i_1j,k}^{(f)} - p_{i_2j,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{i_1j,l}^{(m)} - p_{i_2j,l}^{(m)}| \right) +$$

$$\max_{j_1,j_2,i} \left(\sum_{k=1}^{n} |p_{ij_1,k}^{(f)} - p_{ij_2,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{ij_1,l}^{(m)} - p_{ij_2,l}^{(m)}| \right).$$

Proof. For the operator W in S the derivative is

$$d_{\mathbf{z}}W = \frac{1}{2} \begin{pmatrix} \sum_{j=1}^{\nu} p_{1j,1}^{(f)} y_j & \cdots & \sum_{j=1}^{\nu} p_{nj,1}^{(f)} y_j & \sum_{i=1}^{n} p_{i1,1}^{(f)} x_i & \cdots & \sum_{i=1}^{n} p_{i\nu,1}^{(f)} x_i \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{\nu} p_{1j,n}^{(f)} y_j & \cdots & \sum_{j=1}^{\nu} p_{nj,n}^{(f)} y_j & \sum_{i=1}^{n} p_{i1,n}^{(f)} x_i & \cdots & \sum_{i=1}^{n} p_{i\nu,n}^{(f)} x_i \\ \sum_{j=1}^{\nu} p_{1j,1}^{(m)} y_j & \cdots & \sum_{j=1}^{\nu} p_{nj,1}^{(m)} y_j & \sum_{i=1}^{n} p_{i1,1}^{(m)} x_i & \cdots & \sum_{i=1}^{n} p_{i\nu,1}^{(m)} x_i \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{\nu} p_{1j,\nu}^{(m)} y_j & \cdots & \sum_{j=1}^{\nu} p_{nj,\nu}^{(m)} y_j & \sum_{i=1}^{n} p_{i1,\nu}^{(m)} x_i & \cdots & \sum_{i=1}^{n} p_{i\nu,\nu}^{(m)} x_i \end{pmatrix}$$

$$d_{\mathbf{z}}W = 2M_{\mathbf{z}} = 2\sum_{k=1}^{n} x_k M_k^{(f)} + 2\sum_{l=1}^{\nu} y_l M_l^{(m)},$$

where $M_k^{(f)} = M_{\mathbf{e}_k^{(f)}}$ and $M_l^{(m)} = M_{\mathbf{e}_l^{(m)}}$ are the multiplication maps with matrixes $(p_{ij,k}^{(f)})_{i,k=1}^{n}$ and respectively $(p_{ij,l}^{(m)})_{j,l=1}^{\nu}$.

By Lemma 3.1 we have

$$L(W) = 2\max_{\mathbf{z}\in S} \|M_{\mathbf{z}}\| \leq 2\max_{k} \|M_k^{(f)}\| + 2\max_{l} \|M_l^{(m)}\|.$$

By Lemma 3.2,

$$\|M_k^{(f)}\| = \frac{1}{2} \max_{i_1,i_2,j} \left(\sum_{k=1}^{n} |p_{i_1j,k}^{(f)} - p_{i_2j,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{i_1j,l}^{(m)} - p_{i_2j,l}^{(m)}| \right),$$

$$\|M_l^{(m)}\| = \frac{1}{2} \max_{i,j_1,j_2} \left(\sum_{k=1}^{n} |p_{ij_1,k}^{(f)} - p_{ij_2,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{ij_1,l}^{(m)} - p_{ij_2,l}^{(m)}| \right).$$

Corollary 3.4. An evolutionary operator (2) is a strict contraction if

$$\max_{i_1,i_2,j} \left(\sum_{k=1}^{n} |p_{i_1j,k}^{(f)} - p_{i_2j,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{i_1j,l}^{(m)} - p_{i_2j,l}^{(m)}| \right) +$$

$$\max_{j_1,j_2,i} \left(\sum_{k=1}^{n} |p_{ij_1,k}^{(f)} - p_{ij_2,k}^{(f)}| + \sum_{l=1}^{\nu} |p_{ij_1,l}^{(m)} - p_{ij_2,l}^{(m)}| \right) < 1. \quad (5)$$

For evolutionary operators with positive coefficients there is a multiplicative estimate of the distance from the evolutionary operator to the constant one. Let

$$\mu^f \equiv \mu^f(W) = \max_{i_1,i_2,j,k} \frac{p_{i_1j,k}^{(f)}}{p_{i_2j,k}^{(f)}}, \ \mu^m \equiv \mu^m(W) = \max_{i,j_1,j_2,l} \frac{p_{ij_1,l}^{(m)}}{p_{ij_2,l}^{(m)}},$$

and let $\zeta(W)$ equal to LHS of (5).

Lemma 3.5.

$$\zeta(W) \leq 4\frac{\mu^f - 1}{\mu^f + 1} + 4\frac{\mu^m - 1}{\mu^m + 1}.$$

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Proof. If $\alpha,\beta > 0$ and $\mu = \max(\frac{\alpha}{\beta},\frac{\beta}{\alpha})$ then obviously

$$|\alpha-\beta| = \frac{\mu-1}{\mu+1}(\alpha+\beta).$$

Hence

$$|p^{(f)}_{i_1j,k} - p^{(f)}_{i_2j,k}| \le \frac{\mu^f-1}{\mu^f+1}(p^{(f)}_{i_1j,k} + p^{(f)}_{i_2j,k}),$$

$$|p^{(m)}_{i_1j,l} - p^{(m)}_{i_2j,l}| \le \frac{\mu^m-1}{\mu^m+1}(p^{(m)}_{i_1j,l} + p^{(m)}_{i_2j,l}),$$

and respectively

$$|p^{(f)}_{ij_1,k} - p^{(f)}_{ij_2,k}| \le \frac{\mu^f-1}{\mu^f+1}(p^{(f)}_{ij_1,k} + p^{(f)}_{ij_2,k}),$$

$$|p^{(m)}_{ij_1,l} - p^{(m)}_{ij_2,l}| \le \frac{\mu^m-1}{\mu^m+1}(p^{(m)}_{ij_1,l} + p^{(m)}_{ij_2,l}).$$

It remains to sum these inequalities over k and respectively over l, keeping in mind that

$$\sum_{k=1}^{n} p^{(f)}_{i_1j,k} = \sum_{k=1}^{n} p^{(f)}_{i_2j,k} = \sum_{l=1}^{\nu} p^{(m)}_{ij_1,l} = \sum_{l=1}^{\nu} p^{(m)}_{ij_2,l} = 1.$$

Corollary 3.6.

$$L(W) \le 4\frac{\mu^f-1}{\mu^f+1} + 4\frac{\mu^m-1}{\mu^m+1}.$$

Corollary 3.7. If $7\mu^f\mu^m - (\mu^f+\mu^m) < 9$ then the evolutionary operator (2) is a strict contraction.

Corollary 3.8. Let $\mu = \max(\mu^f,\mu^m)$. Then

$$L(W) \le 8\frac{\mu-1}{\mu+1}$$

and if $\mu < \frac{9}{7}$ then the evolutionary operator (2) is a strict contraction.

Remark 3.9. For free and bisexual populations the conditions of contractility are essentially different.

Let us give several examples and check the condition of Corollary 3.4.

Example 3.10. Consider the operator

$$W: \begin{cases} x_1' = \frac{3}{7}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + \frac{4}{7}x_2y_2, \\ x_2' = \frac{4}{7}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + \frac{3}{7}x_2y_2, \\ y_1' = \frac{4}{7}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + \frac{3}{7}x_2y_2, \\ y_2' = \frac{3}{7}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_1 + \frac{4}{7}x_2y_2. \end{cases} \tag{6}$$

The coefficients of the operator (6) are the following

$$\begin{array}{llll} p^{(f)}_{11,1} = \frac{3}{7} & p^{(f)}_{12,1} = \frac{1}{2} & p^{(f)}_{21,1} = \frac{1}{2} & p^{(f)}_{22,1} = \frac{4}{7} \\ p^{(f)}_{11,2} = \frac{4}{7} & p^{(f)}_{12,2} = \frac{1}{2} & p^{(f)}_{21,2} = \frac{1}{2} & p^{(f)}_{22,2} = \frac{3}{7} \\ p^{(m)}_{11,1} = \frac{4}{7} & p^{(m)}_{12,1} = \frac{1}{2} & p^{(m)}_{21,1} = \frac{1}{2} & p^{(m)}_{22,1} = \frac{3}{7} \\ p^{(m)}_{11,2} = \frac{3}{7} & p^{(m)}_{12,2} = \frac{1}{2} & p^{(m)}_{21,2} = \frac{1}{2} & p^{(m)}_{22,2} = \frac{4}{7} \end{array}$$

It is easy to check that condition (5) fulfilled for (6). Indeed,

$$\max_{i_1,i_2,j}\left(\sum_{k=1}^{n}|p^{(f)}_{i_1j,k} - p^{(f)}_{i_2j,k}| + \sum_{l=1}^{\nu}|p^{(m)}_{i_1j,l} - p^{(m)}_{i_2j,l}|\right) +$$
$$\max_{j_1,j_2,i}\left(\sum_{k=1}^{n}|p^{(f)}_{ij_1,k} - p^{(f)}_{ij_2,k}| + \sum_{l=1}^{\nu}|p^{(m)}_{ij_1,l} - p^{(m)}_{ij_2,l}|\right) = \frac{4}{7}.$$

Consequently, this operator is a strict contraction and it has unique fixed point $(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2})$. Moreover any trajectory of (6) converges to the fixed point.

The following example shows that the condition of Corollary 3.4 is not fulfilled and evolutionary operator has periodic trajectory.

Example 3.11. Consider the operator

$$W: \begin{cases} x_1' = x_1y_1 \\ x_2' = x_1y_2 + x_2 \\ y_1' = x_2y_2 \\ y_2' = x_1 + x_2y_1 \end{cases} \tag{7}$$

It easy to check that operator (7) does not satisfy the condition of Corollary 3.4.

We rewrite the operator (7) in the form

$$W: \begin{cases} x_1' = x_1y_1 \\ y_1' = (1-x_1)(1-y_1) \end{cases} \tag{8}$$

Denote $x_n = x_1^{(n)}$, $y_n = y_1^{(n)}$ then from (8) we have

$$\begin{cases} x_{n+1} = x_ny_n \\ y_{n+1} = (1-x_n)(1-y_n) \end{cases} \tag{9}$$

Since $0 \le x_ny_n \le x_n$ from the first equation of (9) it follows that $\lim_{n\to\infty} x_n = x^* = 0$. Indeed, for

$$\mathbf{z}^0 \in int(S^1\times S^1) = \{\mathbf{z} \in S^1\times S^1 : x_i > 0, y_i > 0, i = 1,2\}.$$

we get from (9)

$$\frac{x_{n+2}}{x_{n+1}} = (1-x_n)\left(1-\frac{x_{n+1}}{x_n}\right),$$

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$$x_{n+2}x_n = x_{n+1}(1-x_n)(x_n - x_{n+1}),$$
$$\lim_{n\to\infty} x_{n+2}x_n = \lim_{n\to\infty} x_{n+1}(1-x_n)(x_n - x_{n+1}),$$
$$(x^*)^2 = 0, \quad x^* = 0.$$

Now consider the operator

$$W^2 : \begin{cases} x' = xy - x^2y - xy^2 + x^2y^2 \\ y' = x + y - xy - x^2y - xy^2 + x^2y^2 \end{cases}$$

Clearly, the operator W^2 has fixed points $(0,y)$, $0 \le y \le 1$. The point $(0,y)$ is a saddle point.

It is easy to check that the set $\{(x,y) \in S^1 \times S^1 : x_1 = 0\}$ is an invariant subset for (7). Any point of the invariant subset is periodic point with period two for operator (7). So trajectory of the operator with an initial point from invariant subset does not converge. Thus operator (7) has a trajectory which does non-converge to the fixed point $(0,1,\frac{1}{2},\frac{1}{2})$.

The following example shows that condition of Corollary 3.4 is sufficient but is not necessary.

Example 3.12. Consider the operator with coefficients of inheritance

$$\begin{array}{llll} p^{(f)}_{11,1} = 0 & p^{(f)}_{12,1} = 0 & p^{(f)}_{21,1} = \frac{1}{2} & p^{(f)}_{22,1} = \frac{1}{2} \\ p^{(f)}_{11,2} = 1 & p^{(f)}_{12,2} = 1 & p^{(f)}_{21,2} = \frac{1}{2} & p^{(f)}_{22,2} = \frac{1}{2} \\ p^{(m)}_{11,1} = 0 & p^{(m)}_{12,1} = \frac{1}{2} & p^{(m)}_{21,1} = 0 & p^{(m)}_{22,1} = \frac{1}{2} \\ p^{(m)}_{11,2} = 1 & p^{(m)}_{12,2} = \frac{1}{2} & p^{(m)}_{21,2} = 1 & p^{(m)}_{22,2} = \frac{1}{2} \end{array}$$

i.e. the evolution operator has the form

$$W : \begin{cases} x_1' = \frac{1}{2}x_2 \\ x_2' = x_1 + \frac{1}{2}x_2 \\ y_1' = \frac{1}{2}y_2 \\ y_2' = y_1 + \frac{1}{2}y_2 \end{cases} \tag{10}$$

It easy to check that operator (10) does not satisfy the condition of Corollary 3.4.

$$\max_{i_1,i_2,j}\left(\sum_{k=1}^{n}|p^{(f)}_{i_1j,k} - p^{(f)}_{i_2j,k}| + \sum_{l=1}^{\nu}|p^{(m)}_{i_1j,l} - p^{(m)}_{i_2j,l}|\right) +$$
$$\max_{j_1,j_2,i}\left(\sum_{k=1}^{n}|p^{(f)}_{ij_1,k} - p^{(f)}_{ij_2,k}| + \sum_{l=1}^{\nu}|p^{(m)}_{ij_1,l} - p^{(m)}_{ij_2,l}|\right) = 2 > 1.$$

But any trajectory of (10) converges to $(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3})$.

Indeed, from (10) we have

$$x_1^{(n+1)} = \frac{1}{2}(1 - x_1^{(n)})$$

We consider following one dimensional dynamical system.

$$f(x) = \frac{1}{2}(1-x)$$

It has unique fixed point $x = \frac{1}{3}$ and decreasing on $[0,1]$. Easy to check that $f'(x) = -\frac{1}{2}$ and $|f'(\frac{1}{3})| = \frac{1}{2} < 1$ therefore the fixed point $x = \frac{1}{3}$ is attracting.

We claim that any trajectory of $f(x)$ converges to the fixed point $x = \frac{1}{3}$. Indeed, we have

$$f^n(x) = \sum_{k=1}^{n}\frac{(-1)^k}{2^k} + (-1)^n \cdot \frac{x}{2^n}$$

and

$$\lim_{n\to\infty} f^{2n}(x) = \lim_{n\to\infty}\left(\frac{1}{3}\cdot\frac{2^{2n}-1}{2^{2n}} + \frac{x}{2^{2n}}\right) = \frac{1}{3},$$

$$\lim_{n\to\infty} f^{2n+1}(x) = \lim_{n\to\infty}\left(\frac{1}{3}\cdot\frac{2^{2n}-1}{2^{2n}} + \frac{1}{2^{2n+1}} - \frac{x}{2^{2n+1}}\right) = \frac{1}{3}.$$

So for any initial point trajectory of (10) converges to $(\frac{1}{3},\frac{2}{3},\frac{1}{3},\frac{2}{3})$.

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Nasir Ganikhodjaev is Professor at International Islamic University Malaysia. He has graduated the Tashkent State University (1971). He got PhD (1975) degree from the Institute of Mathematics, Tashkent and Rehabilitation Degree Doctor of Sciences in Physics and Mathematics (1991) from the Institute for Low Temperature Physics and Engineering Academy of Sciences of Ukraine, Kharkov. He is known for his publications in lattice models of statistical mechanics and nonlinear dynamical systems.

Uygun Jamilov was born in 1976 in Bukhara, Uzbekistan. He graduated from Tashkent State University in 1998. He got Ph.D. degree in Physics and Mathematics in National University of Uzbekistan in 2012. His research interests include dynamical systems and ergodic theory, functional analysis. He has important results on non-Volterra quadratic stochastic operators. He has about 25 papers published in the leading journals. Presently he is a senior scientific fellow in the Institute of Mathematics at National University of Uzbekistan,Tashkent, Uzbekistan.

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