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Distribution and survival functions and application in intuitionistic random approximation

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Abstract: In this paper, first, we consider the distribution and survival functions. Next, we define intuitionistic random φ-normed spaces which improve and generalize the definition of an intuitionistic Menger space. As an application, we prove the stability of some functional equations in intuitionistic random φ-normed spaces by the modified method which provides a better estimation.

Keywords: Distribution function; survival function; stability; cubic functional equation; intuitionistic φ-random normed space.

1 Introduction

Distribution and survival functions are important in probability theory. We use these functions to define intuitionistic random φ-normed spaces and find an application about stability of some functional equations. The study of stability problems for functional equations is related to a question of Ulam [26] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [11]. Subsequently, the result of Hyers was generalized by Aoki [2] for additive mappings and by Rassias [22] for linear mappings by considering an unbounded Cauchy difference. The paper [22] of Rassias has provided a lot of influence in the development of what we now call Hyers–Ulam–Rassias stability of functional equations. We refer the interested readers for more information on such problems to the papers [4, 6, 12, 15, 23].

2 Preliminaries

Now, we give some definitions and lemmas for our main results in this paper.

Definition 1. A function $\mu : \mathbb{R} \to [0,1]$ is called a *distribution function* if it is left continuous on $\mathbb{R}$, non-decreasing and

$$\inf_{t\in\mathbb{R}} \mu(t) = 0, \quad \sup_{t\in\mathbb{R}} \mu(t) = 1.$$

We denote by D the family of all measure distribution functions and by H a special element of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Forward, $\mu(x)$ is denoted by μ_x.

Definition 2. A function $\nu : \mathbb{R} \to [0,1]$ is called a *survival function* if it is right continuous on $\mathbb{R}$, non-increasing and

$$\inf_{t\in\mathbb{R}} \nu(t) = 0, \quad \sup_{t\in\mathbb{R}} \nu(t) = 1.$$

We denote by B the family of all survival functions and by G a special element of B defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0, \\ 0, & \text{if } t > 0. \end{cases}$$

Forward, $\nu(x)$ is denoted by ν_x.

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Lemma 1.([8]) *Consider the set L^* and the operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \, and \, x_1 + x_2 \leq 1\},$$

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1, \; x_2 \geq y_2,$$

for all $(x_1, x_2), (y_1, y_2) \in L^*$. Then $(L^*, \leq_{L^*})$ is a complete lattice.

We denote the bottom and the top elements of lattices by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, the *triangular norm* $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying

$$T(1,x) = 1 * x = x$$

for all $x \in [0,1]$. The *triangular conorm* $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 3.([8]) A *triangular norm* (*t–norm*) on L^* is a mapping $\mathscr{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

(a) for all $x \in L^*$, $\mathscr{T}(x, 1_{L^*}) = x$ (: boundary condition);

(b) for all $(x,y) \in (L^*)^2$, $\mathscr{T}(x,y) = \mathscr{T}(y,x)$ (: commutativity);

(c) for all $(x,y,z) \in (L^*)^3$, $\mathscr{T}(x, \mathscr{T}(y,z)) = \mathscr{T}(\mathscr{T}(x,y),z)$ (: associativity);

(d) for all $(x,x',y,y') \in (L^*)^4$, $x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathscr{T}(x,y) \leq_{L^*} \mathscr{T}(x',y')$ (: monotonicity).

In this paper, $(L^*, \leq_{L^*}, \mathscr{T})$ has an Abelian topological monoid with the top element 1_{L^*} and so $\mathscr{T}$ is a *continuous t–norm*.

Definition 4.A continuous t–norm $\mathscr{T}$ on L^* is said to be *continuous representable t-norm* if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathscr{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example,

$$\mathscr{T}(a,b) = (a_1 b_1, \min\{a_2 + b_2, 1\})$$

and

$$\mathbf{M}(a,b) = (\min\{a_1, b_1\}, \max\{a_2, b_2\})$$

for all $a = (a_1, a_2), b = (b_1, b_2) \in L^*$ are the continuous representable t-norm.

Definition 5.A *negator* on L^* is any decreasing mapping $\mathscr{N} : L^* \longrightarrow L^*$ satisfying $\mathscr{N}(0_{L^*}) = 1_{L^*}$ and $\mathscr{N}(1_{L^*}) = 0_{L^*}$.

If $\mathscr{N}(\mathscr{N}(x)) = x$ for all $x \in L^*$, then $\mathscr{N}$ is called an *involutive negator*.

A *negator* on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the *standard negator* on $[0,1]$ defined by

$$N_s(x) = 1 - x$$

for all $x \in [0,1]$.

Let φ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(a) $\varphi(-t) = \varphi(t)$ for all $t \in \mathbb{R}$;

(b) $\varphi(1) = 1$;

(c) φ is strictly increasing and continuous on $[0,\infty)$, $\varphi(0) = 0$ and $\lim_{\alpha \to \infty} \varphi(\alpha) = \infty$.

Some examples of such functions are: $\varphi(t) = |t|$; $\varphi(t) = |t|^p$, $p \in (0,\infty)$; $\varphi(t) = \frac{2t^{2n}}{|t|+1}$ for all $n \in \mathbb{N}$.

3 Intuitionistic random space

The notation of intuitionistic Menger space was introduced in [16] and the notation of random φ-normed spaces introduced in [9,18].

In the sequel, we adopt the usual terminology, notations and conventions of the theory of intuitionistic random φ-normed spaces as in [5,10,16,17,24,25].

Definition 6. Let μ and ν be a distribution function and a survival function from $X \times (0,+\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \leq 1$ for all $x \in X$ and $t > 0$. The 3-tuple $(X, \mathscr{P}_{\mu,\nu}, \mathscr{T})$ is said to be an *intuitionistic random φ-normed space* (briefly φ-IRN-space) if X is a vector space, $\mathscr{T}$ is a continuous representable t-norm and $\mathscr{P}_{\mu,\nu}$ is a mapping $X \times (0,+\infty) \to L^*$ satisfying the following conditions: for all $x,y \in X$ and $t,s > 0$,

(a) $\mathscr{P}_{\mu,\nu}(x,0) = 0_{L^*}$;

(b) $\mathscr{P}_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x = 0$;

(c) $\mathscr{P}_{\mu,\nu}(\alpha x,t) = \mathscr{P}_{\mu,\nu}(x, \frac{t}{\varphi(\alpha)})$ for all $\alpha \neq 0$;

(d) $\mathscr{P}_{\mu,\nu}(x+y,t+s) \geq_{L^*} \mathscr{T}(\mathscr{P}_{\mu,\nu}(x,t), \mathscr{P}_{\mu,\nu}(y,s))$.

In this case, $\mathscr{P}_{\mu,\nu}$ is called an *intuitionistic random norm*. Here,

$$\mathscr{P}_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)).$$

Note that, if $(X, \mathscr{P}_{\mu,\nu}, \mathscr{T})$ is a φ-IRN-space and define $\mathscr{P}_{\mu,\nu}(x-y,t) = \mathscr{M}_{\mu,\nu}(x,y,t)$, then $(X, \mathscr{M}_{\mu,\nu}, \mathscr{T})$ is an intuitionistic Menger spaces.

*Example 1.*Let $(X, \|\cdot\|)$ be a normed space. Let $\mathscr{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all

$a=(a_1,a_2)$, $b=(b_1,b_2)\in L^*$ and μ,ν be a distribution function and a survival function defined by

$$\mathscr{P}_{\mu,\nu}(x,t)=(\mu_x(t),\nu_x(t))=\left(\frac{t}{t+\|x\|^p},\frac{\|x\|^p}{t+\|x\|^p}\right)$$

for all $t\in\mathbb{R}^+$ and $0<p\leq 1$. Then $(X,\mathscr{P}_{\mu,\nu},\mathscr{T})$ is a φ-IRN-space.

Definition 7.(1) A sequence $\{x_n\}$ in a φ-IRN-space $(X,\mathscr{P}_{\mu,\nu},\mathscr{T})$ is called a *Cauchy sequence* if, for any $\varepsilon>0$ and $t>0$, there exists $n_0\in\mathbb{N}$ such that

$$\mathscr{P}_{\mu,\nu}(x_n-x_m,t)>_{L^*}(N_s(\varepsilon),\varepsilon)$$

for all $n,m\geq n_0$, where N_s is the standard negator.

(2) A sequence $\{x_n\}$ is said to be *convergent* to a point $x\in X$ (denoted by $x_n\overset{\mathscr{P}_{\mu,\nu}}{\longrightarrow}x$) if $\mathscr{P}_{\mu,\nu}(x_n-x,t)\longrightarrow 1_{L^*}$ as $n\longrightarrow\infty$ for all $t>0$.

(3) A φ-IRN-space $(X,\mathscr{P}_{\mu,\nu},\mathscr{T})$ is said to be *complete* if every Cauchy sequence in X is convergent to a point $x\in X$.

4 Applications

The functional equation

$$\begin{aligned}&f(2x+y)+f(2x-y)\\&=2f(x+y)+2f(x-y)+12f(x)\end{aligned}\tag{1}$$

is said to be the *cubic functional equation* since the function $f(x)=cx^3$ is its solution. Every solution of the cubic functional equation is said to be a *cubic mapping*. The stability problem for the cubic functional equation was proved by Jun and Kim [13] for mappings $f:X\longrightarrow Y$, where X is a real normed space and Y is a Banach space. Later, a number of mathematicians have worked on the stability of some types of the cubic equation [14,22]. In addition, Mirmostafaee, Mirzavaziri and Moslehian [19,20], Alsina [1], Miheţ and Radu [17] investigated the stability in the settings of fuzzy, probabilistic and random normed spaces.

We start our work with the main result in a φ-IRN-space with an additional condition for a φ i.e.,

$$\varphi(st)=\varphi(s)\varphi(t)$$

for all $t,s>0$.

Theorem 1.*Let X be a linear space, $(Z,\mathscr{P}'_{\mu,\nu},\mathbf{M})$ be a φ-IRN-space and $\phi:X\times X\longrightarrow Z$ be a function such that, for some $0<\alpha<\varphi(8)$,*

$$\mathscr{P}'_{\mu,\nu}(\phi(2x,0),t)\geq_{L^*}\mathscr{P}'_{\mu,\nu}(\alpha\phi(x,y),t)\tag{2}$$

and

$$\lim_{n\to\infty}\mathscr{P}'_{\mu,\nu}(\phi(2^nx,2^ny),\varphi(8^n)t)=1_{L^*}$$

for all $x,y\in X$ and $t>0$. Let $(Y,\mathscr{P}_{\mu,\nu},\mathbf{M})$ be a complete φ-IRN-space. If $f:X\to Y$ is a mapping such that

$$\begin{aligned}&\mathscr{P}_{\mu,\nu}(f(2x+y)+f(2x-y)-2f(x+y)\\&\quad-2f(x-y)-12f(x),t)\\&\quad\geq_{L^*}\mathscr{P}'_{\mu,\nu}(\phi(x,y),t)\end{aligned}\tag{3}$$

for all $x,y\in X$ and $t>0$, then there exists a unique cubic mapping $C:X\to Y$ such that

$$\begin{aligned}&\mathscr{P}_{\mu,\nu}(f(x)-C(x),t)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}(\phi(x,0),\varphi(2)(\varphi(8)-\alpha)t))\end{aligned}\tag{4}$$

for all $x,y\in X$ and $t>0$

*Proof.*Putting $y=0$ in (3), we get

$$\mathscr{P}_{\mu,\nu}\left(\frac{f(2x)}{8}-f(x),t\right)\geq_{L^*}\mathscr{P}'_{\mu,\nu}(\phi(x,0),\varphi(16)t)\tag{5}$$

for all $x\in X$ and $t>0$. Replacing x by 2^nx in (5), we obtain

$$\begin{aligned}&\mathscr{P}_{\mu,\nu}\left(\frac{f(2^{n+1}x)}{8^{n+1}}-\frac{f(2^nx)}{8^n},t\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}(\phi(2^nx,0),t\varphi(16)\varphi(8^n))\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t\varphi(16)\varphi(8^n)}{\alpha^n}\right)\end{aligned}\tag{6}$$

for all $x\in X$ and $t>0$. It follows from $\frac{f(2^nx)}{8^n}-f(x)=\sum_{k=0}^{n-1}(\frac{f(2^{k+1}x)}{8^{k+1}}-\frac{f(2^kx)}{8^k})$ and (6) that

$$\begin{aligned}&\mathscr{P}_{\mu,\nu}\left(\frac{f(2^nx)}{8^n}-f(x),t\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{\varphi(16)\varphi(8^k)}}\right)\end{aligned}\tag{7}$$

for all $x\in X$ and $t>0$. By replacing x with 2^mx in (7), we observe that

$$\begin{aligned}&\mathscr{P}_{\mu,\nu}\left(\frac{f(2^{n+m}x)}{8^{n+m}}-\frac{f(2^mx)}{8^m},t\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(2^mx,0),\frac{t\varphi(8^m)}{\sum_{k=0}^{n-1}\frac{\alpha^k}{\varphi(16)\varphi(8^k)}}\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+m}}{\varphi(16)\varphi(8^k)\varphi(8^m)}}\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^{k+m}}{\varphi(16)\varphi(8^{k+m})}}\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{\varphi(16)\varphi(8^k)}}\right)\\&\geq_{L^*}\mathscr{P}'_{\mu,\nu}\left(\phi(x,0),\frac{t}{\sum_{k=m}^{n+m-1}\frac{\alpha^k}{\varphi(16)\varphi(8)^k}}\right)\end{aligned}\tag{8}$$

for all $x\in X$ and $t>0$. Then $\{\frac{f(2^nx)}{8^n}\}$ is a Cauchy sequence in $(Y,\mathscr{P}_{\mu,\nu},\mathbf{M})$. Since $(Y,\mathscr{P}_{\mu,\nu},\mathbf{M})$ is a complete φ-IRN-space, this sequence convergent to a point $C(x)\in Y$. Fix

$x \in X$ and put $m = 0$ in (8). Then we obtain

$$\mathscr{P}_{\mu,\nu}\Big(\frac{f(2^n x)}{8^n} - f(x), t\Big) \geq_{L^*} \mathscr{P}'_{\mu,\nu}\Big(\phi(x,0), \frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{\varphi(16)\varphi(8)^k}}\Big) \tag{9}$$

for all $x \in X$ and $t > 0$ and so, for all $\delta > 0$,

$$\begin{aligned}
&\mathscr{P}_{\mu,\nu}(C(x) - f(x), t+\delta) \\
\geq_{L^*}\ & \mathbf{M}\Big\{\mathscr{P}_{\mu,\nu}\Big(C(x) - \frac{f(2^n x)}{8^n}, \delta\Big), \mathscr{P}_{\mu,\nu}\Big(\frac{f(2^n x)}{8^n} - f(x), t\Big)\Big\} \\
\geq_{L^*}\ & \mathbf{M}\Big\{\mathscr{P}_{\mu,\nu}\Big(C(x) - \frac{f(2^n x)}{8^n}, \delta\Big), \mathscr{P}'_{\mu,\nu}\Big(\phi(x,0), \frac{t}{\sum_{k=0}^{n-1}\frac{\alpha^k}{\varphi(16)\varphi(8)^k}}\Big)\Big\}
\end{aligned} \tag{10}$$

for all $x \in X$ and $t > 0$. Taking the limit as $n \longrightarrow \infty$ and using (10), we get

$$\mathscr{P}_{\mu,\nu}(C(x) - f(x), t+\delta) \geq_{L^*} \mathscr{P}'_{\mu,\nu}(\phi(x,0), \varphi(2)(\varphi(8)-\alpha)) \tag{11}$$

for all $x \in X$ and $t > 0$. Since δ was arbitrary, by taking $\delta \longrightarrow 0$ in (11), we get

$$\mathscr{P}_{\mu,\nu}(C(x) - f(x), t) \geq_{L^*} \mathscr{P}'_{\mu,\nu}(\phi(x,0), \varphi(2)(\varphi(8)-\alpha))$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2^n x$ and $2^n y$ in (3), respectively, we get

$$\begin{aligned}
&\mathscr{P}_{\mu,\nu}\Big(\frac{f(2^n(2x+y))}{8^n} + \frac{f(2^n(2x-y))}{8^n} - \frac{2f(2^n(x+y))}{8^n} - \frac{2f(2^n(x-y))}{8^n} - \frac{12f(2^n(x))}{8^n}, t\Big) \\
\geq_{L^*}\ & \mathscr{P}'_{\mu,\nu}(\phi(2^n x, 2^n y), \varphi(8^n)t)
\end{aligned} \tag{12}$$

for all $x, y \in X$ and $t > 0$. Since $\lim_{n\longrightarrow\infty} \mathscr{P}'_{\mu,\nu}(\phi(2^n x, 2^n y), \varphi(8^n)t) = 1_{L^*}$, it follows that C fulfills (4).

To prove the uniqueness of the cubic function C, assume that there exists a cubic function $D : X \longrightarrow Y$ which satisfies (4). Fix $x \in X$. Clearly, $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$ for all $n \in \mathbb{N}$. It follows from (4) that

$$\begin{aligned}
&\mathscr{P}_{\mu,\nu}\Big(C(x) - D(x), t\Big) \\
=\ & \mathscr{P}_{\mu,\nu}\Big(\frac{C(2^n x)}{8^n} - \frac{D(2^n x)}{8^n}, t\Big) \\
\geq_{L^*}\ & \mathbf{M}\Big\{\mathscr{P}_{\mu,\nu}\Big(\frac{C(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\Big), \mathscr{P}_{\mu,\nu}\Big(\frac{D(2^n x)}{8^n} - \frac{f(2^n x)}{8^n}, \frac{t}{2}\Big)\Big\} \\
\geq_{L^*}\ & \mathscr{P}'_{\mu,\nu}(\phi(2^n x, 0), \varphi(2) \times \varphi(8^n)(\varphi(8)-\alpha)t) \\
\geq_{L^*}\ & \mathscr{P}'_{\mu,\nu}\Big(\phi(x,0), \frac{\varphi(2) \times \varphi(8^n)(\varphi(8)-\alpha)t}{\alpha^n}\Big)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Since $\lim_{n\to\infty} \frac{\varphi(2)\times\varphi(8^n)(\varphi(8)-\alpha)t}{\alpha^n} = \infty$, we get

$$\lim_{n\to\infty} \mathscr{P}'_{\mu,\nu}\Big(\phi(x,0), \frac{\varphi(2) \times \varphi(8^n)(\varphi(8)-\alpha)t}{\alpha^n}\Big) = 1_{L^*}$$

for all $x \in X$ and $t > 0$. Therefore, it follows that $\mathscr{P}_{\mu,\nu}(C(x) - D(x), t) = 1_{L^*}$ for all $t > 0$ and so $C(x) = D(x)$. This completes the proof.

Corollary 1.*Let X be a linear space, $(Z, \mathscr{P}'_{\mu,\nu}, \mathbf{M})$ be a φ-IRN-space, $(Y, \mathscr{P}_{\mu,\nu}, \mathbf{M})$ be a complete φ-IRN-space in which $\varphi(t) = t$ and p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}
&\mathscr{P}_{\mu,\nu}(f(2x+y) + f(2x-y) \\
&-2f(x+y) - 2f(x-y) - 12f(x), t) \\
\geq_{L^*}\ & \mathscr{P}'_{\mu,\nu}((\|x\|^p + \|y\|^q)z_0, t)
\end{aligned}$$

for all $x, y \in X$ and $t > 0$, $f(0) = 0$ and $p, q < 3$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathscr{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathscr{P}'_{\mu,\nu}(\|x\|^p z_0, 2(8-2^p)t))$$

for all $x \in X$ and $t > 0$.

*Proof.*Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x,y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 1 by $\alpha = 2^p$.

Corollary 2.*Let X be a linear space, $(Z, \mathscr{P}'_{\mu,\nu}, \mathbf{M})$ be a φ-IRN-space, $(Y, \mathscr{P}_{\mu,\nu}, \mathbf{M})$ be a complete φ-IRN-space in which $\varphi(t) = t$ and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\begin{aligned}
&\mathscr{P}_{\mu,\nu}(f(2x+y) + f(2x-y) \\
&-2f(x+y) - 2f(x-y) - 12f(x), t) \\
\geq_{L^*}\ & \mathscr{P}'_{\mu,\nu}(\varepsilon z_0, t)
\end{aligned}$$

for all $x, y \in X$ and $t > 0$ and $f(0) = 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$\mathscr{P}_{\mu,\nu}(f(x) - C(x), t) \geq_{L^*} \mathscr{P}'_{\mu,\nu}(\varepsilon z_0, 14t)$$

for all $x \in X$ and $t > 0$.

*Proof.*Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x,y) = \varepsilon z_0$. Then the corollary is followed from Theorem 1 by $\alpha = 1$.

Theorem 2.*Let X be a linear space, $(Z, \mathscr{P}'_{\mu,\nu}, \mathbf{M})$ be a φ-IRN-space and $\phi : X \times X \longrightarrow Z$ be a function such that, for some $0 < \alpha < \varphi(16)$,*

$$\mathscr{P}'_{\mu,\nu}(\phi(2x,0),t) \geq_{L^*} \mathscr{P}'_{\mu,\nu}(\alpha\phi(x,y),t) \tag{13}$$

for all $x,y \in X$ and $t > 0$, $f(0) = 0$ and

$$\lim_{n\to\infty} \mathscr{P}'_{\mu,\nu}(\varphi(2^n x, 2^n y), \varphi(16^n)t) = 1_{L^*}$$

for all $x,y \in X$ and $t > 0$. Let $(Y, \mathscr{P}_{\mu,\nu}, \mathbf{M})$ be a complete φ-IRN-space. If $f : X \to Y$ is a mapping such that

$$\mathscr{P}_{\mu,\nu}(f(2x+y) + f(2x-y) \tag{14}$$
$$-4f(x+y) - 4f(x-y) - 24f(x) + 6f(y),t)$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}(\phi(x,y),t), \tag{15}$$

for all $x,y \in X$ and $t > 0$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mathscr{P}_{\mu,\nu}(f(x) - Q(x),t) \tag{16}$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}(\phi(x,0), \varphi(2)(\varphi(16) - \alpha)t)) \tag{17}$$

for all $x,y \in X$ and $t > 0$

*Proof.*The proof is similar with the proof of Theorem 1.

Corollary 3.*Let X be a linear space, $(Z, \mathscr{P}'_{\mu,\nu}, \mathbf{M})$ be a φ-IRN-space, $(Y, \mathscr{P}_{\mu,\nu}, \mathbf{M})$ be a complete φ-IRN-space in which $varphi(t) = t$ and p,q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathscr{P}_{\mu,\nu}(f(2x+y) + f(2x-y)$$
$$-4f(x+y) - 4f(x-y) - 24f(x) + 6f(y),t)$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}((\|x\|^p + \|y\|^q)z_0,t)$$

$x,y \in X$ and $t > 0$, $f(0) = 0$ and $p,q < 4$, then there exists a unique quartic mapping $Q : X \to Y$ such that

$$\mathscr{P}_{\mu,\nu}(f(x) - Q(x),t)$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}(\|x\|^p z_0, 2(16 - 2^p)t))$$

for all $x \in X$ and $t > 0$.

*Proof.*Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x,y) = (\|x\|^p + \|y\|^q)z_0$. Then the corollary is followed from Theorem 2 by $\alpha = 2^p$.

Corollary 4.*Let X be a linear space, $(Z, \mathscr{P}'_{\mu,\nu}, \mathbf{M})$ be a φ-IRN-space, $(Y, \mathscr{P}_{\mu,\nu}, \mathbf{M})$ be a complete φ-IRN-space in which $varphi(t) = t$ and let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that*

$$\mathscr{P}_{\mu,\nu}(f(2x+y) + f(2x-y)$$
$$-4f(x+y) - 4f(x-y) - 24f(x) + 6f(y),t)$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}(\varepsilon z_0,t)$$

for all $x,y \in X$ and $t > 0$ and $f(0) = 0$, then there exists a unique quartic mapping $C : X \to Y$ such that

$$\mathscr{P}_{\mu,\nu}(f(x) - Q(x),t)$$
$$\geq_{L^*} \mathscr{P}'_{\mu,\nu}(\varepsilon z_0, 30t)$$

for all $x \in X$ and $t > 0$.

*Proof.*Let $\varphi : X \times X \longrightarrow Z$ be defined by $\varphi(x,y) = \varepsilon z_0$. Then the corollary is followed from Theorem 2 by $\alpha = 1$.

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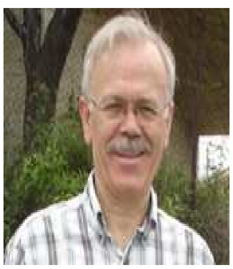

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