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# Homomorphism to $\mathbb{R}$ of Semidirect Products: A Dynamical Construction

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**Abstract:** We define a modified version of Erschler and Karlsson construction of homomorphisms from random walks [A. Erschler and A. Karlsson, Ann. Inst. Fourier (Grenoble) **60** (2010), no. 6, $2095-2113$]. We show that our definition is well adapted to semidirect products. The same construction is conjectured to work also for other classes of maps like quasi-morphisms.

**Keywords:** Random walks on groups, homomorphisms, semidirect product.

This paper is dedicated to the memory of Professor José Sousa Ramos.

## 1 Introduction

Since the seminal article of Avez [1], the fascinating world of random walks on finitely generated groups have emerged as a rich branch of mathematics. In that work the author introduced the entropy $h$ of a random walk on a group, arguably the most important numerical invariant associated to a group endowed with a probability measure, and connected entropy with the growth rate $v$. In his subsequent work [2], Avez put in evidence the strong connection between entropy and the existence of bounded harmonic functions on the group. The drift $\ell$ is another number that has emerged in the study of random walks. These numbers are linked via the fundamental inequality popularized by Vershik [9], $h \leq v\ell$. For a comprehensive reference in a more general setting see Kaimanovich and Vershik [8].

In this note we focus on a dynamical way of obtaining homomorphisms from a finitely generated group $G$ to $\mathbb{R}$. There are several ways to construct homomorphisms, depending on the setting. For example, if $G$ is amenable any homogeneous quasi-morphism to $\mathbb{R}$ is in fact a homomorphism, see [5]. Karlsson and Ledrappier in [7] proved that if a finite first moment random walk has zero entropy and positive drift, then there exists a non-trivial homomorphism to $\mathbb{R}$. Erschler and Karlsson [6] used a dynamical construction to establish the existence of such homomorphism in the case where the drift can also be zero. In this work we follow Erschler and Karlsson construction to define a homomorphism to $\mathbb{R}$ for a certain class of semidirect products. The original construction starts by making a judicious choice of a sequence of functions $T_n$, and of a numerical sequence $\alpha(n)$ such that the product $\alpha(n)T_n$ will have a sublimit $T$ which is precisely the desired homomorphism. We give now a brief description of this note. In section 2 we give some background, define the kernel $K_g^n(h)$ and establish its main properties. In section 3 we construct a twisted version of $T_n$, well adapted for semidirect products. We will show that our twisted sequence $\widetilde{T}_n$ verifies the same properties of Erschler and Karlsson construction and is invariant for a finite family of non-inner automorphisms $\{\phi_i\}$. As a consequence, we obtain the main result of [6] for the twisted maps $\widetilde{T}_n$, see proposition 3. In section 5, we state and prove our result (Theorem 1) about the existence of a homomorphism from a certain class of semidirect products to $\mathbb{R}$. Let $N$ be a finitely generated group and $\pi : Aut(N) \to Aut(N)/Inn(N) \simeq Out(N)$ denote the canonical homomorphism. Let $H$ be any group. The semidirect products $N \rtimes_\phi H$ we will work with are such that the group $Im(\pi \circ \phi) \subset Out(N)$ is finite and contains a set of representatives $\{\phi_i : i = 1, \ldots, k\}$ which is a subgroup of $Aut(N)$.

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A final word is in order. Both the authors of this work were students of José Sousa Ramos. His lectures were absolutely inspirational, using dynamical systems to connect almost every branch of mathematics. The lectures were very popular, bringing together many students with different backgrounds, from mathematics to physics and engineering. But the unforgettable facet of Sousa Ramos is certainly the kindness as he treated every person, always considering a student as a full human being. Very often his students became his friends and they all miss him a lot. One sentence encapsulates the main human qualities of professor Sousa Ramos, found by the second author almost two decades ago, while reading the British mathematician and philosopher Bertrand Russel:

*For my part, I think the important virtues are kindness and intelligence*.

Since then, in our imagination, this sentence became connected to José Sousa Ramos. This work is dedicated to the memory of our professor and friend.

## 2 Background

Let $G$ be a finitely generated group, endowed with a probability measure $\mu$. Denote by $e$ the identity of $G$. Let $X = \{s_1,\cdots,s_p\} \subset G$ be a finite set of generators. The word length of $g$ is defined to be

$$l(g) = \min\{n \in \mathbb{N} : g = s_1 \cdots s_n,\, s_i \in X\}.$$

The set of generators is called symmetric if $X^{-1} = X$. In that case the word length is generated in the usual way by the distance associated with the Cayley graph of $G$ and has the following properties:

$$l(gh) \leqslant l(g) + l(h) \tag{1}$$

$$l(g^{-1}) = l(g). \tag{2}$$

From (1) we have

$$l(gh) - l(h) \leq l(g)$$

and also

$$l(h) = l(g^{-1}gh) \leq l(g^{-1}) + l(gh).$$

Using (2) the last inequality can be written as

$$l(h) - l(gh) \leq l(g)$$

and we obtain

$$|l(gh) - l(h)| \leq l(g), \tag{3}$$

a formula that will be useful later.

**Definition 1.** *We say that $\mu$ has finite first moment if*

$$\sum_{g\in G} l(g)\mu(g) < +\infty.$$

Given probability measures $\mu$ and $\nu$ on $G$ we may define the convolution measure as follows

$$\mu * \nu(g) = \sum_{h\in G} \mu(h) h\nu(g)$$
$$= \sum_{h\in G} \mu(h)\nu(h^{-1}g)$$

where $h\nu(g) = \nu(h^{-1}g)$. Denote by $\mu^{*n}$ the $n$-fold convolution.

Using the word length define a sequence $L_n$ as

$$L_n = \sum_{h\in G} l(h)\mu^{*n}(h). \tag{4}$$

The sequence $L_n$ is subadditive

$$L_{n+m} \leq L_n + L_m.$$

Hence, the classical Fekete lemma ensures that the limit

$$\ell_\mu = \lim_{n\to+\infty} \frac{L_n}{n}$$

exists in $\mathbb{R}\bigcup\{-\infty,+\infty\}$. Since $l \geq 0$ and $L_n$ is subadditive we have $L_n \leq nL_1$ and since $\mu$ has finite first moment it follows that $\ell_\mu < +\infty$.

It will be usefull to define the following map

$$K_g^n(h) = g\mu^{*n}(h) - \mu^{*n}(h) \tag{5}$$

which plays the role of a kernel and verifies some useful properties, such as

1. $K_e^n(h) = 0$
2. $\sum_{h\in G} K_g^n(h) = 0$
3. $\sum_{h\in G} |K_g^n(h)| \leqslant 2$

Another important property is

$$\sum_{g\in G} K_g^n(h)\mu(g) = \mu^{*(n+1)}(h) - \mu^{*n}(h). \tag{6}$$

## 3 The modified map $\widetilde{T}_n$

For the rest of this section $G$ denotes a finitely generated group. Erschler and Karlsson [6] defined a sequence of maps $T_n : G \to \mathbb{R}$ by

$$T_n(g) = \sum_{h\in G} [l(gh) - l(h)]\mu^{*n}(h). \tag{7}$$

We will define a modified version of $T_n$ which is well adapted to semidirect products. As we will see, our modified maps, denoted $\widetilde{T}_n$, verify step by step Erschler and Karlsson construction. First, we need to set some more definitions.

The first step in our construction is to adapt the finite set of generators. Let $\{\phi_i : i = 1,\ldots,k\}$ be a finite subgroup of $Aut(G)$ with order $k$ whose non-identity elements are all non-inner automorphisms. There are plenty of examples of such groups, see section 5. In this note we refer to the finite group $\{\phi_i : i = 1,...,k\} \subset Aut(G)$ simply as a finite group of outer automorphisms, meaning that all the non-identity elements are non-inner. Note that this differs from the usual, since the elements of the group $Out(G) = Aut(G)/Inn(G)$ are not real automorphisms of $G$ but classes.

Let $X$ be a finite symmetric generating set of $G$. Then,

$$S = X \cup \bigcup_{i=1}^{k} \phi_i(X) \tag{8}$$

is also a finite symmetric generating set for $G$.

**Proposition 1.** *For every $g \in G$,*

$$l_S(g) = l_S(\phi_i(g)),\ \forall i = 1,...,k.$$

*Proof.* Since $\phi_i$ is an automorphism, $\phi_i(X)$ is also a generating set and

$$l_{\phi_i(X)}(\phi_i(g)) = l_X(g).$$

Let $g$ be an element of $G$ and $g = s_1...s_m$ a minimal decomposition of $g$ in $S = X \cup \bigcup_{i=1}^{k} \phi_i(X)$, with $m = l_S(g)$.

Observe that if $s_j \in X$ then $\phi_i(s_j) \in \phi_i(X)$ and so it is an element of $S$. On the other hand, since $\{\phi_i\}$ is a group, if $s_j = \phi_r(s'_j) \in \phi_r(X)$ then $\phi_i(\phi_r(s'_j)) = \phi_t(s'_j) \in \phi_t(X)$ is another element of $S$.

Now, suppose $l_S(\phi_i(g)) \neq l_S(g)$, for some $i$. From the above observation, for some $1 < j < m$, $\phi_i(s_j) = \phi_i^{-1}(s_{j+1})$ and we would have

$$\phi_i(s_j s_{j+1}) = e \Leftrightarrow s_j s_{j+1} = e$$

contradicting the fact that $l_S(g) = m$.

From now on we will use the finite generating set $S = X \cup \bigcup_{i=1}^{k} \phi_i(X)$ of $G$.

Now we use the finite subgroup $\{\phi_i\}$ of outer automorphisms and the generating set $S$ of $G$ to introduce the modified maps $\widetilde{T}_n$. Define a sequence of maps $\widetilde{T}_n = \widetilde{T}_{n,\{\phi_i\}} : G \to \mathbb{R}$:

$$\widetilde{T}_n(g) = \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G}[l(\phi_i(g)h) - l(h)]\mu^{*n}(h). \tag{9}$$

For simplicity, we write $\widetilde{T}_n$ instead of $\widetilde{T}_{n,\{\phi_i\}}$ and $l(g)$ instead of $l_S(g)$.

Note that this definition of $\widetilde{T}_n$ is a twisted version of Erschler-Karlsson definition, see [6, §3]. In fact, if $k = 1$ then the unique $\phi_i$ is the identity and we recover $T_n$ from (9).

The map $\widetilde{T}_n$ may also be defined using the kernel

$$\begin{aligned}
\widetilde{T}_n(g) &= \frac{1}{k}\sum_{i=1}^{k}\Big(\sum_{h\in G} l(\phi_i(g)h)\mu^{*n}(h) - \sum_{h\in G} l(h)\mu^{*n}(h)\Big)\\
&= \frac{1}{k}\sum_{i=1}^{k}\Big(\sum_{h\in G} l(h)\phi_i(g)\mu^{*n}(h) - \sum_{h\in G} l(h)\mu^{*n}(h)\Big)\\
&= \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G} l(h)(\phi_i(g)\mu^{*n}(h) - \mu^{*n}(h))\\
&= \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G} l(h)K^n_{\phi_i(g)}(h).
\end{aligned}$$

We will concentrate on a special class of probability measures $\mu$ on $G$ which are compatible with the finite subgroup of outer automorphisms $\{\phi_i\}$.

**Definition 2.** *A measure $\mu$ on $G$ is called $\{\phi_i\}$-invariant if*

$$\mu(\phi_i(g)) = \mu(g).$$

Note that, given a probability measure $\mu$ on $G$ and a finite subgroup $\{\phi_i : i = 1,...,k\} \subset Aut(G)$ of outer automorphisms, we may always obtain a $\{\phi_i\}$-invariant probability measure $\widetilde{\mu}$ out of $\mu$:

$$\widetilde{\mu}(g) = \frac{1}{k}\sum_{i=1}^{k}\mu(\phi_i(g)).$$

Moreover, if $\mu$ is non-degenerate and has finite first moment then $\widetilde{\mu}$ is also non-degenerate and has finite first moment.

Here are some properties of $\widetilde{T}_n$.

**Lemma 1.** *Let $\widetilde{T}_n(g)$ be defined as above, with $\mu$ $\{\phi_i\}$-invariant. We have*

1. $\widetilde{T}_n(e) = 0$
2. $|\widetilde{T}_n(g)| \leq l(g)$
3. $\widetilde{T}_n(\phi_i(g)) = \widetilde{T}_n(g), \forall i = 1,...,k$
4. $\sum_{g\in G}\widetilde{T}_n(g)\mu(g) = L_{n+1} - L_n$.

*Proof.* The first assertion is obvious. For the second property we use proposition 1 and (3)

$$|\widetilde{T}_n(g)| \leqslant \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G}|l(\phi_i(g)h) - l(h)|\mu^{*n}(h)$$

$$\leqslant \frac{1}{k}\sum_{i=1}^{k}\Big(\sum_{h\in G} l(\phi_i(g))\mu^{*n}(h) = l(g).$$

Property 3 is obvious from the definition and from the fact that $\{\phi_i\}$ is a group. And for the last one we use the fact that $\mu$ is $\{\phi_i\}$-invariant:

$$\sum_{g\in G}\widetilde{T}_n(g)\mu(g) = \frac{1}{k}\sum_{i=1}^{k}\sum_{g\in G}\sum_{h\in G} l(h)K^n_{\phi_i(g)}(h)\mu(g)$$

$$= \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G} l(h)\sum_{g\in G}[\phi_i(g)\mu^{*n}(h) - \mu^{*n}(h)]\mu(g)$$

$$= \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G} l(h)\sum_{u\in G}[u\mu^{*n}(h) - \mu^{*n}(h)]\mu(\phi_i^{-1}(u))$$

$$= \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G} l(h)\sum_{u\in G} K_u^n(h)\mu(u)$$

$$= \sum_{h\in G} l(h)(\mu^{*(n+1)}(h) - \mu^{*n}(h))$$

$$= L_{n+1} - L_n.$$

Define the defect

$$\mathrm{Def}_n(g_1,g_2) = \widetilde{T}_n(g_1g_2) - \widetilde{T}_n(g_1) - \widetilde{T}_n(g_2).$$

Although $\widetilde{T}_n$ is not a homomorphism, we can bound the defect.

**Lemma 2.** *We have*

$$|Def_n^T(g_1,g_2)| \leqslant l(g_1)\sum_{h\in G}|K_{g_2}^n(h)|. \quad (10)$$

*Proof.* From the definition of $\widetilde{T}_n$

$$|\mathrm{Def}_n(g_1,g_2)| = |\frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G}(l(\phi_i(g_1g_2)h) - l(\phi_i(g_1)h) - l(\phi_i(g_2)h) + l(h))\mu^{*n}(h)|$$

$$= \frac{1}{k}|\sum_{i=1}^{k}\sum_{h\in G}[l(\phi_i(g_1)\phi_i(g_2)h) - l(\phi_i(g_2)h)]\mu^{*n}(h) - \sum_{h\in G}[l(\phi_i(g_1)h) - l(h)]\mu^{*n}(h)|$$

$$= \frac{1}{k}|\sum_{i=1}^{k}\sum_{h\in G}[l(\phi_i(g_1)h) - l(h)]\phi_i(g_2)\mu^{*n}(h) - \sum_{h\in G}[l(\phi_i(g_1)h) - l(h)]\mu^{*n}(h)|$$

$$= \frac{1}{k}|\sum_{i=1}^{k}\sum_{h\in G}[l(\phi_i(g_1)h) - l(h)]K_{\phi_i(g_2)}^n(h)$$

$$\leqslant \frac{1}{k}\sum_{i=1}^{k}\sum_{h\in G}|l(\phi_i(g_1)h) - l(h)||K_{\phi_i(g_2)}^n(h)|$$

$$\leqslant \frac{1}{k}\sum_{i=1}^{k} l(\phi_i(g_1))\sum_{h\in G}|K_{\phi_i(g_2)}^n(h)|$$

$$\leqslant l(g_1)\sum_{h\in G}|K_{\phi_i(g_2)}^n(h)|.$$

In [6], the authors explain the existence of a sequence $\beta(n)$ such that $\sum_{h\in G}|K_g^n(h)| \le C(g)\beta(n)$. See [6, §3] for further details.

**Lemma 3.** *If $\sum_{h\in G}|K_g^n(h)| \leqslant C(g)\beta(n)$ then there exists a constant $C_o$ such that $\sum_{h\in G}|K_g^n(h)| \leqslant l(g)C_o\beta(n)$.*

*Proof.* First note that for any pair $g_1,g_2 \in G$ we have

$$\sum_{h\in G}(g_1g_2\mu^{*n}(h) - g_1\mu^{*n}(h)) =$$

$$= \sum_{h\in G}(\mu^{*n}((g_1g_2)^{-1}h) - \mu^{*n}(g_1^{-1}h))$$

$$= \sum_{h\in G}(\mu^{*n}(g_2^{-1}g_1^{-1}h) - \mu^{*n}(g_1^{-1}h))$$

$$= \sum_{h\in G}(g_2\mu^{*n}(g_1^{-1}h) - \mu^{*n}(g_1^{-1}h))$$

$$= \sum_{u\in G}(g_2\mu^{*n}(u) - \mu^{*n}(u))$$

where in the last line we made the change of variables $u = g_1^{-1}h$. Suppose now $g$ has $l(g) = m$. We have $g = s_1\cdots s_m$, for some $s_i \in S$ and

$$\sum_{h\in G}|K_g^n(h)| = \sum_{h\in G}|K_{s_1\cdots s_m}^n(h)|$$

$$= \sum_{h\in G}|s_1\cdots s_m\mu^{*n}(h) - \mu^{*n}(h)|$$

$$= \sum_{h\in G}|s_1\cdots s_m\mu^{*n}(h) - s_1\cdots s_{m-1}\mu^{*n}(h) + s_1\cdots s_{m-1}\mu^{*n}(h) + \cdots + s_1\mu^{*n}(h) - \mu^{*n}(h)|$$

$$\leqslant \sum_{h\in G}|s_1\cdots s_m\mu^{*n}(h) - s_1\cdots s_{m-1}\mu^{*n}(h)| + \sum_{h\in G}|s_1\cdots s_{m-1}\mu^{*n}(h) - s_1\cdots s_{m-2}\mu^{*n}(h)| + \cdots + \sum_{h\in G}|s_1\mu^{*n}(h) - \mu^{*n}(h)|$$

$$\leqslant \sum_{h\in G}|s_m\mu^{*n}(h) - \mu^{*n}(h)| + \sum_{h\in G}|s_{m-1}\mu^{*n}(h) - \mu^{*n}(h)| + \cdots + \sum_{h\in G}|s_1\mu^{*n}(h) - \mu^{*n}(h)| = \sum_{i=1}^{m}\sum_{h\in G}|K_{s_i}^n(h)|.$$

By hypothesis $\sum_{h\in G}|K_g^n(h)| \leqslant C(g)\beta(n)$, so

$$\sum_{i=1}^{m}\sum_{h\in G}|K_{s_i}^n(h)| \leqslant \sum_{i=1}^{m}C(s_i)\beta(n)$$

$$\leqslant \max_{s\in S}C(s)\sum_{i=1}^{m}\beta(n)$$

$$\leqslant \max_{s\in S}C(s)\beta(n)m$$

Writing $C_o = \max_{s\in S}C(s)$ and recalling that $m = l(g)$ the result follows.

It is convenient to introduce the following notation

$$\gamma(n) = \max_{s\in S}|\widetilde{T}_n(s)|. \quad (11)$$

**Proposition 2.** *Let $\mu$ be a probability measure on $G$ $\{\phi_i\}$-invariant. Suppose that $g$ and $\phi_i(g)$ are in the support of $\mu$. Then,*

$$|\widetilde{T}_n(g)| \leqslant C_o\beta(n)l^2(g) + l(g)\gamma(n). \tag{12}$$

*Proof.* Let $g, \phi_i(g) \in \operatorname{supp}\mu \subset G$, $i = 1,\dots,k$. Write $g = s_1\cdots s_m$, $m = l(g)$, with $s_1,\dots,s_m \in S$. By lemmas 2 and 3, we have

$$\begin{aligned}
&|\mathrm{Def}_n(s_1\cdots s_{m-1}, s_m)| \leq \\
&\leqslant \frac{1}{k}\sum_{i=1}^{k} l(\phi_i(s_1\cdots s_{m-1}))\sum_{h\in G}|K^n_{\phi_i(s_m)}(h)| \\
&= \frac{1}{k}\sum_{i=1}^{k} l(s_1\cdots s_{m-1})\sum_{h\in G}|K^n_{\phi_i(s_m)}(h)| \\
&\leqslant \frac{1}{k} l(s_1\cdots s_{m-1})\sum_{i=1}^{k} l(\phi_i(s_m))C_0\beta(n) \\
&= (m-1)C_o\beta(n).
\end{aligned}$$

So

$$\begin{aligned}
&|\widetilde{T}_n(g)| = |\widetilde{T}_n(s_1\cdots s_m)| = \\
&= |\widetilde{T}_n(s_1\cdots s_m) - \widetilde{T}_n(s_1\cdots s_{m-1}) - \widetilde{T}_n(s_m) + \\
&+\widetilde{T}_n(s_1\cdots s_{m-1}) - \widetilde{T}_n(s_1\cdots s_{m-2}) - \widetilde{T}_n(s_{m-1}) + \cdots \\
&\cdots + \widetilde{T}_n(s_1 s_2) - \widetilde{T}_n(s_1) - \widetilde{T}_n(s_2) + \sum_{i=1}^{m}\widetilde{T}_n(s_i)| \\
&\leqslant C_o\beta(n)((m-1)+(m-2)+\cdots+1) + \sum_{i=1}^{m}|\widetilde{T}_n(s_i)| \\
&\leqslant C_o\beta(n)((m-1)+(m-2)+\cdots+1) + m\gamma(n) \\
&\leqslant C_o\beta(n)m^2 + m\gamma(n).
\end{aligned}$$

**Definition 3.** *We say that $\mu$ has finite second moment if*

$$\sum_{g\in G} l(g)^2\mu(g) < +\infty.$$

Now we can state the main result of Erschler and Karlsson for the modified maps $\widetilde{T}_n$ defined in (9). Let $\Delta_{n_k}L = L_{n_k+1} - L_{n_k}$, with $L_{n_k}$ as in (4).

**Proposition 3.** *Suppose that $\mu$ is non-degenerate $\{\phi_i\}$-invariant, has finite second moment and for some sequence $n_k$ it holds that $\Delta_{n_k}L > 0$ and*

$$\lim_{k\to+\infty}\frac{\beta(n_k)}{\Delta_{n_k}L} = 0.$$

*Then the group $G$ admits a non-trivial homomorphism to $\mathbb{R}$.*

The proof goes exactly along the same lines of [6, §4]. For the sake of completeness we reproduce here the details.

By proposition 2 we know that

$$|\widetilde{T}_n(g)| \leqslant \gamma(n)l(g) + C_o l^2(g)\beta(n).$$

By lemma 1 and since $\mu$ has finite second moment, we have

$$\begin{aligned}
0 < \Delta_{n_k}L &= \sum_{g\in G}\widetilde{T}_n(g)\mu(g) \leq \sum_{g\in G}|\widetilde{T}_n(g)|\mu(g) \\
&\leq \gamma(n)\sum_{g\in G} l(g)\mu(g) + C_o\beta(n)\sum_{g\in G} l^2(g)\mu(g) \\
&\leq C_1\gamma(n) + C_2\beta(n).
\end{aligned}$$

Under the above conditions, for every $\varepsilon > 0$ there exists $n_k$ such that $\beta(n_k) \leqslant \varepsilon\,\Delta_{n_k}L$ and we have

$$0 < \Delta_{n_k}L \leq C_1\gamma(n_k) + C_2\varepsilon\,\Delta_{n_k}L.$$

Take $\varepsilon$ with $C_1\varepsilon < \frac{1}{2}$. For $n_k$ big enough, we have

$$0 < \frac{1}{2}\Delta_{n_k}L \leq (1 - C_1\varepsilon)\Delta_{n_k}L \leq C_2\gamma(n_k). \tag{13}$$

It follows that, for $n_k$ sufficiently big

$$\gamma(n_k) > 0.$$

Define the coefficient

$$\alpha(n_k) = \frac{1}{\gamma(n_k)}.$$

It follows from (13) that

$$0 < \alpha(n_k)\,\Delta_{n_k}L \leq 2C_2.$$

Hence

$$0 \leq \alpha(n_k)\beta(n_k) = \alpha(n_k)\,\Delta_{n_k}L\,\frac{\beta(n_k)}{\Delta_{n_k}L} \leqslant 2C_2\frac{\beta(n_k)}{\Delta_{n_k}L}.$$

We conclude that

$$\alpha(n_k)\beta(n_k) \to 0. \tag{14}$$

Define a new sequence

$$\widetilde{T}_n^{\alpha}(g) = \alpha(n)\widetilde{T}_n(g). \tag{15}$$

By lemma 2, we have

$$|\mathrm{Def}_{n_k}(g_1, g_2)| \leqslant l(g_1)C(g_2)\beta(n_k)\alpha(n_k).$$

Taking the limit as $k \to +\infty$ we conclude that

$$\widetilde{T}_{\alpha}(g_1 g_2) = \widetilde{T}_{\alpha}(g_1)\widetilde{T}_{\alpha}(g_2).$$

As explained in [6], the homomorphism $\widetilde{T}_{\alpha}$ is well defined and nontrivial.

## 4 Semidirect products

In this section we show that the modified maps construction is well adapted to semidirect products. Our setting will be the following.

Let $N$ be a finitely generated group and $H$ any other group. Let $\mu$ be a probability measure on $N$. Given a homomorphism $\phi : H \to Aut(N)$, we may form the semidirect product $N \rtimes_\phi H$. Let $X$ be a symmetric generating set of $N$. Let $Inn(N)$ denote the inner automorphisms of $N$ and $Out(N)$ the outer automorphisms. Then, we have

$$H \xrightarrow{\phi} Aut(N) \xrightarrow{\pi} Aut(N)/Inn(N) \simeq Out(N)$$

and $Im(\pi \circ \phi)$ is a subgroup of $Out(N)$.

We require the following conditions to be satisfied:

C1. $Im(\pi \circ \phi) \subset Out(N)$ is a finite subgroup and has a set of representative $\{\phi_i : i = 1, \ldots, k\} \subset Aut(N)$ with a subgroup structure.
C2. $S = X \cup \bigcup_{i=1}^{k} \phi_i(X)$, with $X$ a symmetric generating set of $N$.
C3. $\mu$ is non-degenerate and has finite second moment.
C4. $\mu(\phi_i(g)) = \mu(g)$ for every $\phi_i \in \{\phi_i\}$.

There is a plenty supply of examples of a finitely generated group $N$ with a finite subgroup $\{\phi_i\} \subset Aut(N)$ whose non-identity elements are non-inner automorphisms. We give two examples.

*Example 1.* Let $\mathbb{F}_2 = \langle a, b\rangle$ denote the free group on two generators. Since $Inn(\mathbb{F}_2) = \mathbb{F}_2/Z(\mathbb{F}_2) = \mathbb{F}_2$ is torsion free, every finite order automorphism in not inner. For instance, the automorphisms

$$\alpha : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases} \quad \beta : \begin{cases} a \mapsto b \\ b \mapsto a^{-1}b^{-1} \end{cases} \quad \gamma : \begin{cases} a \mapsto b^{-1} \\ b \mapsto a \end{cases}$$

have order 2, 3 and 4, respectively. Therefore, the cyclic subgroups

$$\langle\alpha\rangle, \ \langle\beta\rangle, \ \langle\gamma\rangle$$

are finite subgroups of $Aut(\mathbb{F}_2)$ whose non-identity elements are non-inner automorphisms.

*Example 2.* Let $\bigoplus_{i=1}^{m} C_2$ denote the direct sum of $m$ copies of the cyclic group with 2 elements. Let $G$ be any finitely generated group. Since $\bigoplus_{i=1}^{m} C_2$ is abelian, every automorphism is outer. Then, a semidirect product $N = \bigoplus_{i=1}^{m} C_2 \rtimes_\psi G$ has a subgroup of $Aut(N)$ whose non-identity elements are non-inner automorphisms.

We may now state our main result regarding semidirect products.

**Theorem 1.** *Let $N$ and $H$ be two groups as above and suppose conditions $C1 - C4$ are satisfied. Suppose for some sequence $n_k$ it holds that $\Delta_{n_k} L > 0$ and*

$$\lim_{k\to+\infty} \frac{\beta(n_k)}{\Delta_{n_k} L} = 0.$$

*Let $R : H \to \mathbb{R}$ be any homomorphism. Then, there is a unique nontrivial homomorphism*

$$T \rtimes R : N \rtimes_\phi H \to \mathbb{R}$$

*such that the following diagram commutes*

$$\begin{array}{ccccc} N & \xrightarrow{id_N \times e_H} & N \rtimes_\phi H & \xleftarrow{e_N \times id_H} & H \\ & {\scriptstyle T}\searrow & \downarrow {\scriptstyle T \rtimes R} & \swarrow{\scriptstyle R} & \\ & & \mathbb{R} & & \end{array}$$

*Proof.* By proposition 3, there is a nontrivial homomorphism

$$T : N \to \mathbb{R}.$$

According to [4, Proposition 27, p.240], we need to prove that

$$T(\phi_y(x)) = R(y)T(x)R(y^{-1}) \tag{16}$$

for all $x \in N$ and all $y \in H$.

Now, since $(\mathbb{R}, +)$ is abelian, condition (16) above becomes

$$T(\phi_y(x)) = T(x)$$

We need to consider two different cases.

**Case 1:** $\phi_y$ is an inner automorphism of $N$.

In that case, there exists $z \in N$ such that $\phi_y(x) = zxz^{-1}$ and we have

$$T(\phi_y(x)) = T(zxz^{-1}) = T(z) + T(x) - T(z) = T(x)$$

so, (16) is trivially verified.

**Case 2:** $\phi_y$ is an outer automorphism of $N$.

In that case, by hypothesis, $Im(\pi \circ \phi)$ is finite and by construction, each map $\widetilde{T}_n$ is invariant under composition with $\phi_y$, see lemma 1, property 3. Therefore, the limit of $T_n^\alpha$ in (15) is also invariant under composition with $\phi_y$ and the result follows.

## 5 Conclusions

In our construction we have used a finite group of outer automorphisms of $N$. It would be interesting to relax this condition. This would imply, of course, a different sequence of maps $\widetilde{T}_n$.

There is a vast literature about quasi-morphisms defined on groups. In his PhD thesis [3], the first author has shown that quasi-morphisms $Q : G \to \mathbb{R}$ can be used instead of the length function in the definition of $T_n$:

$$T_n(g) = \sum_{h \in G} (Q(gh) - Q(h))\mu^{*n}(h).$$

It would be also interesting to use the setting of quasi-morphisms and prove a similar result to theorem 1 for semidirect products.

Another interesting direction to explore beyond this note is the following. Let $G$ be a finitely generated group and $f : G \to \mathbb{R}$ a generalized word length, *i.e*, a function satisfying properties (1) and (2), with finite first moment which is not a quasi-morphism. It should be possible to define a twisted sequence $\widetilde{T}_n$ for this setting and obtain an equivalent result of theorem 1. Moreover, if $(N, \preceq)$ is left orderable, what conditions should one impose in order to obtain a limit map $\widetilde{T} : N \to \mathbb{R}$ which is not only a group homomorphism but also satisfies the following property

$$g \preceq h \Rightarrow \widetilde{T}(g) \leq \widetilde{T}(h),$$

that is, $\widetilde{T}$ is a poset homomorphism?

These questions are examples of the ongoing work being carried out by the authors.

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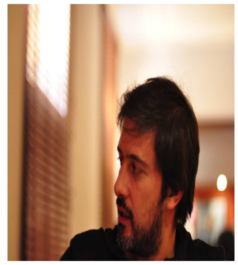

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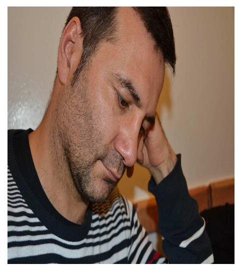

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