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Spectral and Dynamical Invariants in a Complete Clustered Network

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Abstract: The main result of this work is a new criterion for the formation of good clusters in a graph. This criterion uses a new dynamical invariant, the performance of a clustering, that characterizes the quality of the formation of clusters. We prove that the growth of the dynamical invariant, the network topological entropy, has the effect of worsening the quality of a clustering, in a process of cluster formation by the successive removal of edges. Several examples of clustering on the same network are presented to compare the behavior of other parameters such as network topological entropy, conductance, coefficient of clustering and performance of a clustering with the number of edges in a process of clustering by successive removal.

Keywords: Networks, graphs, performance of a clustering, network topological entropy, conductance

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction and motivation

Complex networks are present in several areas such as physics, biology, economics, ecology and computer science. For example, at the molecular level the structure of DNA sequences and proteins spacial structure may be described in terms of graphs and networks. Social and economic agents organize themselves as complex network structures. Traffic flow and communication systems, such as internet and telephone webs are also described by complex networks. In the natural world we have the neural networks and the ecological food webs as examples of large complex networks.

A network may be defined in graph theory as a set of nodes or vertices and the edges or lines between them. There are many different methods to identify the edges and vertices, depending on the type of network you want to build. Unfortunately, the use of an experimental method for the revaluation of the whole network is very expensive in terms of time and resources; thus the development of theoretical methods is of great importance. The development of useful computational methods to assess the quality of connectivity, efficiency in complex networks is a goal of great interest. Graph topology can be described by a wide variety of parameters, some of which are discussed here. A complex network is a dynamical system and entropy is, perhaps, the most important numerical quantity that measures the complexity of a dynamical system. In 1948 at [12], Shannon established one entropy definition based on probabilities. In 1988 at [17], Milnor and Thurston developed a concept of topological entropy with the kneading theory and lately J. Sousa Ramos and his collaborators worked this notion, based on symbolic dynamics, see for example, [4], [5], [6], [18] and [19]. There are several concepts of network entropy, see [12]. We will use the one based on symbolic dynamics, as defined in Section 4. Symbolic dynamics is a fundamental tool available to describe complicated time evolution of a chaotic dynamical system.

The study of networks may be addressed in two main approaches. One is the behavior in each node (local dynamics), see [5] and [7], and the other is the topology of the network (global dynamics). In previous works, see

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[6] and [8], the authors studied some parameters that characterize the topology of the network, the clustering coefficient, the conductance, the synchronization interval for coupled systems and the performance of a clustering. In this paper we relate these invariants with the complexity of the systems, using the topological entropy defined for shifts of finite type associated with graphs. We define complete clustered network, denoted by CCN, that is, networks in which each cluster is a complete graph and we emphasize the effect of clusters formation for this special type of network in the network topological entropy.

The layout of this paper is as follows. In Section 2 we present some preliminaries notions, such as the adjacency matrix of a graph, a clustering of a graph, a complete clustered network and the respective construction process by successive removal. In Section 3 are presented some spectral and dynamical invariants: the clustering coefficient, the conductance, the network topological entropy and the performance of a clustering, which will be used to characterize the dynamical behavior of networks. In Section 4 we obtain explicitly formulas to compute the conductance of a complete graph and to compute the performance of a clustering. The monotonicity of the network topological entropy in relation to the construction process by successive removal is proved. We conclude also that, the value of the network topological entropy does not depend on the number of edges but on which edges are removed, i.e., depends on the network topology. Using these results, we prove a criterion for the construction of a good clustering. In Section 5 we present numerical simulations that illustrate the obtained results. We construct complete clustered networks by successive removal, starting with a complete graph with 15 nodes and simulate the formation of clusters in three different cases: two, three, and five clusters. We observe the behavior of the parameters conductance, coefficient of a clustering, the performance of a clustering and network topological entropy, in this process of clustering.

2 Preliminary concepts

In this section, we introduce some notions and basic results on graphs and discrete dynamical systems. Mathematically, networks are described by graphs (directed or undirected) and the theory of dynamical networks is a combination of graph theory and nonlinear dynamics. From the point of view of dynamical systems, we have a global dynamical system emerging from the interactions between the local dynamics of the individual elements. The tool of graph theory allows us to analyze the coupling structure between them, see [10].

A graph G is an ordered pair $G = (V,E)$, where $V = V(G)$ is a nonempty set of N vertices or nodes v_i and $E = E(G) \subset V(G) \times V(G)$ is a set of m_G pairs of vertices, $e_{ij} = (i,j)$, that are called edges or links, that connect two vertices v_i and v_j. We will only consider the case of undirected graphs, that means that the edge (i,j) is the same as the edge (j,i). N is called the order of the graph. For a graph with N vertices, the maximum cardinality of $E(G)$ is $\binom{N}{2} = \frac{N(N-1)}{2}$. If the graph G is not weighted, the adjacency matrix $A = A(G) = (a_{ij})$ is defined as follows,

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ and } v_j \text{ are connected} \\ 0, \text{ if } v_i \text{ and } v_j \text{ are not connected} \end{cases}.$$

As the graph is not directed, then $a_{ij} = a_{ji}$ and the matrix $A(G)$ is symmetric. The degree of a node v_i is the number of edges incident on it and is denoted by k_i, that is,

$$k_i = \sum_{j=1}^{N} a_{ij}.$$

Definition 1. *A clustering $C = \{C_1, C_2, ..., C_k\}$ of the graph $G = (V,E)$ is a pairwise disjoint collection of $k \leq N$ subsets of the vertex set V, where $\cup_{i=1}^{k} C_i = V$. Each C_i is called a cluster.*

For example, in the case of genetic algorithms, [23] describes an alternative method of construction of the clustering

A cluster C_i is identified with the induced subgraph of G, that is, the graph $G_{C_i} = (C_i, E(C_i))$. For each clustering $C = \{C_1, C_2, ..., C_k\}$, the set of intracluster edges is the union of sets of edges $E(C_i)$ and is defined by $Intra(C) = \cup_{i=1}^{k} E(C_i)$. The set of intercluster edges, $Inter(C)$, is the complement, in $E(G)$, of $Intra(C)$, that is, $Inter(C) = E(G) \backslash Intra(C)$, is the set of edges that connect different clusters. The cardinality of these sets is denoted by the correspondent lowercase $intra(C)$ and $inter(C)$, respectively.

Definition 2. *A complete clustered network, denoted by CCN, is a network with a clustering $C = \{C_1, C_2, ..., C_k\}$, where each induced subgraph $G_{C_i} = (C_i, E(C_i))$ is a complete graph.*

Some authors, see for example [1] and [13], call it a clique, meaning a set of pairwise adjacent vertices, i.e., the induced graph is complete with all possible edges. A clique is considered the perfect cluster model and methods to identify large cliques were motivated by sociological applications, see [1]. In social networks it is associated with the notion of a cohesive group that is identified with the following properties: familiarity and reachability among members, and robustness of the subgroup. Clearly, a clique is ideal with respect to all of these requirements since it induces a subgraph in which: each vertex has maximum degree, any pair of members has distance one between them and the vertex connectivity is maximal.

The focus of this paper is on the behavior of some graph invariants in the formation of CCN. In this work we will pay special attention to a particular type of clustering formation.

Definition 3. *A construction of a CCN by successive removal is a process in which, starting with a complete graph $G = (V,E)$, removing edges successively, a clustering $C = \{C_1, C_2, ..., C_k\}$ is achieved, where the clusters C_i, with $1 \leq i \leq k$, are complete graphs and $inter(C) = 0$.*

3 Spectral and dynamical invariants

This section is devoted to the presentation of the dynamical and spectral invariants: the clustering coefficient, the conductance, the network topological entropy and the performance of a clustering, which will be used to characterize the dynamics of networks.

3.1 Clustering coefficient

One aspect that characterizes a network is the global clustering coefficient. This parameter characterizes how densely clustered the edges in a network are. Clustering roughly means that, if the vertex i is connected to the vertex j and the vertex j with the vertex r then with high probability i is also connected to r. For a graph, the clustering coefficient c_i of a vertex v_i is given by the proportion of edges between the vertices within its neighborhood divided by the number of edges that could possibly exist between them. Being k_i the degree of a vertex v_i and denoting by $N_i = \{v_j \in V : e_{ij} \in E\}$ the neighborhood of vertex v_i, then one can define, see [22], the clustering coefficient c_i of a vertex v_i, with degree larger than one, by

$$c_i = \frac{\left|\{e_{jk}\}\right|}{\binom{k_i}{2}} = \frac{2\left|\{e_{jk}\}\right|}{k_i(k_i-1)} \tag{1}$$

where $\left|\{e_{jk}\}\right|$ is the cardinality of the set of edges

$$\{e_{jk} \in E;\ v_j, v_k \in N_i\}.$$

It is possible to calculate the clustering coefficient in terms of the adjacency matrix, see [22]. If $A = A(G) = (a_{ij})$ is the adjacency matrix, the clustering coefficient can be computed by

$$c_i = \frac{\sum_{j=1}^{N}\sum_{m=1}^{N} a_{ij}a_{jm}a_{mi}}{k_i(k_i-1)}. \tag{2}$$

The clustering coefficient of the network, see [3], characterizes the global clustering coefficient and is just the average of c_i over all vertices with degree larger than one

$$c = \frac{\sum_{i,k_i>1} c_i}{\sum_{i,k_i>1} 1}.$$

If all vertices have degree larger than one, then $c = \frac{1}{N}\sum_{i=1}^{N} c_i$.

3.2 Conductance

Bollobás in [2] has introduced the graph-theoretic concept of conductance, using the concept of conductance of a random walk from Jerrum and Sinclair, see [11]. This quantity should measure the ability to go from a small subset of vertices to its complementary. The conductance should detect the existence of funnels in the graph, i.e., subsets from which is hard to leave. This is particularly important in the identification of a good clustering because they should have the property of flowing well within and poorly flowing out. There are several definitions of conductance of a graph, see [9], depending on the type of graphs (directed, weighed) and on the used weigh.

In this paper we shall use the following definition of conductance, since we use undirected and non weighed graphs,

$$\Phi(G) = \min_{U\subset V} \frac{E(U, V\backslash U)}{\min\{|E_1(U)|, |E_1(V\backslash U)|\}}. \tag{3}$$

This definition, applied to a finite graph, should detect the cut of V, for which the number of edges going through is minimal relative to the size of U or its complementary.

3.3 Network topological entropy

Using results of symbolic dynamics theory, it is defined the network topological entropy, which incorporate the important dynamical properties of the system. The variation of network topological entropy with the parameters gives us a finer distinction between different states of complexity. We start from a representation of the network in terms of its adjacency matrix, $A = (a_{ij})$, where the matrix elements are all non-negative to denote the interaction strength between nodes i and j in the network. The introduction of this concept requires a strict and long construction, using tools of symbolic dynamics and algebraic graph theory, which is presented in the final appendix.

The topological entropy $h_{top}(X)$ of a shift dynamical system (X, σ) over some finite alphabet $\mathscr{A}$ is defined by

$$h_{top}(X) = \lim_{n\to\infty} \frac{\log tr\,(A^n(X))}{n}$$

and $h_{top}(X) = 0$ if $X = \emptyset$, where $A(X)$ is the transition matrix of X, see [16].

The Perron-Frobenius Theorem states that, if A is a finite irreducible adjacency matrix such that $A \neq 0$, then A has a positive eigenvector, called the Perron eigenvector, with corresponding positive eigenvalue, called Perron value, which is both geometrically and algebraically simple, see [15]. In our context, if the adjacency matrix $A \neq 0$ is irreducible and λ_A is the Perron value of A, then

$$h_{top}(X) = \log \lambda_A,$$

see [15]. We calculate the topological entropy of the associated dynamical system, which is equal to the logarithm of growth rate of the number of admissible words. If we have a network associated to a graph G which determine the shift space X, we established the next definition.

Definition 4. *The network topological entropy of the graph G is the quantity $h_{top}(X)$, that is,*

$$h_{top}(G) = h_{top}(X).$$

3.4 Performance of a clustering

The performance of a clustering should measure the quality of each cluster as well as the cost of the clustering. This bicriteria is based in a two-parameter definition of a (α,ε)-clustering, where α should measure the quality of the clusters and ε the cost of such partition, that is, the ratio of the intercluster edges to the total of edges in the graph. This concept of $(\alpha,\varepsilon)-$*clustering* was introduced by Kannan in [14].

Definition 5. *A partition $C = \{C_1, C_2, ..., C_k\}$ of V is called an $(\alpha,\varepsilon)-$clustering of a graph G if and only if:*

1. *the conductance of each cluster is at least α,*

$$\Phi(G(C_i)) \geq \alpha, \text{ for all } i = 1, ..., k$$

where $G(C_i) = (C_i, E_i)$ is the induced subgraph of G by the subset $C_i \subset V$;

2. *the fraction of intercluster edges to the total of edges is at most ε,*

$$\frac{inter\,(C)}{m_G} \leq \varepsilon.$$

By this definition the clustering is good if it maximizes α and minimizes ε. To accomplish both optimization problems is introduced in this paper a quantity by which it is possible to optimize both: minimize ε and simultaneously maximize α. It will be called the performance of a clustering.

Definition 6. *For an $(\alpha,\varepsilon)-$clustering C, of a graph G, define the performance of C by the ratio*

$$R_C = \frac{\varepsilon}{\alpha}.$$

Note that, the quality of the clustering C is as better as R_C is smaller.

4 Main results

In this section, we present the most important result of this work: a criterion for the formation of good clusterings. The proof of this result relies on an explicit formula for calculating the performance of a clustering, and the monotonicity of the network topological entropy in relation to the construction process by successive removal, according with Def.3.

The following result allow us an easy computation of the conductance of a complete graph and will be used in the proof of Proposition 1.

Lemma 1. *Let $G = (V,E)$ be a complete graph with N vertices. Then*

$$\Phi(G) = \begin{cases} \frac{N}{2(N-1)}, \text{ if } N \text{is even} \\ \\ \frac{N+1}{2(N-1)}, \text{ if } N \text{is odd} \end{cases}. \tag{4}$$

Proof. According to Eq.(3) and due to the fact that each vertex of G has degree $N-1$, we have

$$\Phi(G) = \min \Phi(U), \text{ for } U \subset V \text{ and } |U| \leq N/2 \tag{5}$$

where

$$\Phi(U) = \frac{|E(U, V-U)|}{|E_1(U)|}$$

for $U \subset V$, with $|U| = k \leq N/2$. Because each of the k vertices of U have degree $N-1$, and is joined with $N-k$ vertices of $V-U$, then

$$\Phi(U) = \frac{|E(U, V-U)|}{k(N-1)} = \frac{N-k}{N-1}.$$

The minimum over all $U \subset V$, with $|U| = k \leq N/2$ of $\frac{N-k}{N-1}$ gives us the conductance of G. By Eq.(5), we have

$$\Phi(G) = \min_{0<k\leq N/2} \frac{N-k}{N-1} = \begin{cases} \frac{N}{2(N-1)}, \text{ if } N \text{ is even} \\ \\ \frac{N+1}{2(N-1)}, \text{ if } N \text{ is odd} \end{cases},$$

as desired.

Consider a CCN with clusters C_i, $1 \leq i \leq k$. Recalling that $G_{C_i} = (C_i, E(C_i))$ is the induced subgraph of G, we denote by N_i the number of vertices in each C_i. As all clusters are complete graphs, the conductance of each cluster C_i is given by Eq.(4), according to Lemma 1.

Proposition 1. *Let $G = (V,E)$ be a CCN such that $C = \{C_1, C_2, ..., C_k\}$ is a clustering of G where each cluster C_i has N_i vertices, respectively, with $1 \leq i \leq k$. Let m_G be the total number of edges in G. Then C is an $(\alpha,\varepsilon)-$clustering of G with performance*

$$R_C = \frac{1 - \frac{1}{2m_G}\sum_{1\leq i\leq k} N_i(N_i-1)}{\min_{1\leq i\leq k} \Phi(G_{C_i})}, \tag{6}$$

where $\Phi(G_{C_i})$ is given by Eq.(4).

Proof. According that the number m_i of edges of each cluster C_i is given by $m_i = \frac{N_i(N_i-1)}{2}$, then

$$\begin{aligned}\varepsilon &= \frac{inter(C)}{m_G} = \frac{m_G - intra(C)}{m_G} = \frac{m_G - \sum_{1\le i\le k} m_i}{m_G} \\ &= 1 - \frac{1}{2m_G}\sum_{1\le i\le k} N_i(N_i-1).\end{aligned} \tag{7}$$

Considering this number ε and the number given by

$$\alpha = \min_{1\le i\le k} \Phi(G_{C_i}),$$

where $\Phi(G_{C_i})$ is given by Eq.(4), then according to Def.5 and Eq.(3), C is an $(\alpha,\varepsilon) - clustering$ of G with performance $R_C = \frac{\varepsilon}{\alpha}$, given by Eq.(6), as desired.

As proved in the next result, the construction of a CCN by successive removal, according to Def.3, will decrease the value of the network topological entropy.

Proposition 2. *If the graph G_2 is obtained from the graph G_1 by a process of successive removal of edges, then*

$$h_{top}(G_2) < h_{top}(G_1).$$

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$ be the adjacency matrices of the graphs G_1 and G_2, respectively. If the graph G_2 is obtained from the graph G_1, removing one edge, then the adjacency matrix B is obtained from the adjacency matrix A by replacing some entry $a_{ij} = 1$ by $b_{ij} = 0$. If, in the process of successive removal, the graph G_2 has a certain number of edges less than the graph G_1, then the matrix A has the same number of entries $a_{ij} = 1$ replaced by that same number of $b_{ij} = 0$. As the matrix A is symmetric, then B is also symmetric and $b_{ji} = 0$. So, the matrix A is equal to the matrix B plus some matrix with non negative entries. For any power n, we have $A^n = B^n + C$, for some matrix C which entries are all non negative. As $Tr(C) \ge 0$ and $Tr(A^n) = Tr(B^n) + Tr(C)$, then $Tr(A^n) > Tr(B^n)$. As a consequence, we obtain for all integer n,

$$\frac{\log tr(A^n)}{n} > \frac{\log tr(B^n)}{n}.$$

From Def.4 follows that $h_{top}(G_1) > h_{top}(G_2)$.

Note that, this result says that the monotonicity of the decreasing of the network topological entropy depends on the process of successive removal of edges. In fact, the value of the network topological entropy does not depend on the number of edges, see Fig.7, this value depends on which edges are removed, as we will show in the next section.

Theorem 1. *Let $G = (V,E)$ be a CCN, such that $C = \{C_1, C_2, ..., C_k\}$ is a clustering of G, constructed by successive removal. Then, the clustering C is as better, as smaller is the value of the network topological entropy.*

Proof. According to Def.6, the clustering C is better when the performance R_C is smaller. From Prop.1, we have

$$R_C = \frac{\varepsilon}{\alpha} = \frac{1 - \frac{1}{2m_G}\sum_{1\le i\le k} N_i(N_i-1)}{\min_{1\le i\le k} \Phi(G_{N_i})}$$

and in this quotient, the denominator is constant because the clustering C is fixed in all the construction process of the CCN. The numerator depends on the number of inter edges, for which we know, see Prop.2, that an amount of inter edges implies an amount on the topological entropy. Thus, if we decrease the number of inter edges we decrease R_C and also the topological entropy, as desired.

5 Numerical results and discussion

We defined above the clustering coefficient, the conductance, the network topological entropy and the performance of a clustering. We intend to understand the behavior of these parameters in the formation of a CCN, constructed by successive removal, as established in Def.3. Our approach is the simulation of such a construction and the observation of the evolution of these dynamical and spectral parameters. We start from a complete graph with 15 nodes and 105 edges and simulate the formation of clusters in three different cases:

i) CCN with 5 clusters, each one of them with 3 nodes;
ii) CCN with 3 clusters, one with 4, one with 5 and the other with 6 nodes;
iii) CCN with 2 clusters, one with 5 and the other with 10 nodes.

At each step of this construction we calculate all the parameters under study. In our process of construction of a clustering we just remove intercluster edges, keeping in all clusters complete graphs, see Figs.1, 2 and 3.

The results are drawn separately so that each image has one of the parameters: performance (R), conductance (ϕ), network topological entropy ($top.ent$) and the clustering coefficient (c), evolving with the number of edges. For case i), a CCN with 5 clusters, we observe in Fig.4, that the evolution of parameters network topological entropy, conductance and performance, with the growth of the edges is increasing. The clustering coefficient has a different behavior, it decreases from 1 to a certain turning point, see [6], and then begins to globally grow up to 1. This is due to the fact that, if the graph G is complete then the clustering coefficient $c = 1$, see Eq.(4) of Lemma 1. When G is disconnected the clustering coefficient is again $c = 1$, since the obtained clusters are always complete. Concerning the conductance, it is maximal and is given by Eq.(3), when the graph G is complete. When G is disconnected the conductance is zero. For the cases ii) and iii) the observations are similar, see Figs.5 and 6.

The introduction of the concept of performance of a clustering, made in this paper, allowed to quantify and

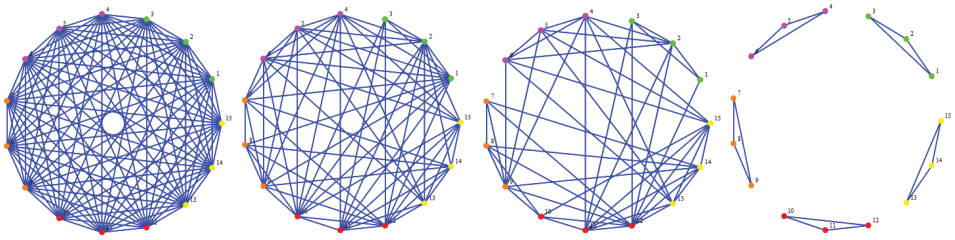

Fig. 1: Construction by successive removal of a clustering with 5 clusters.

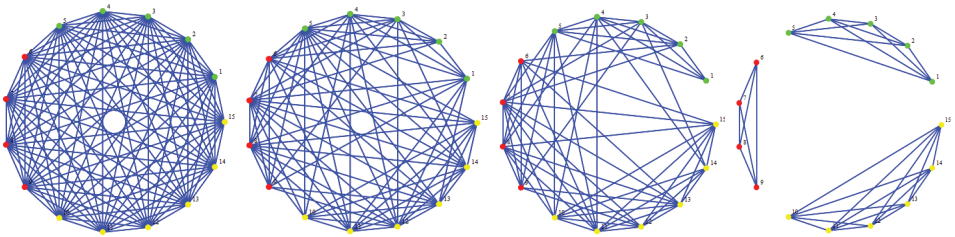

Fig. 2: Construction by successive removal of a clustering with 3 clusters.

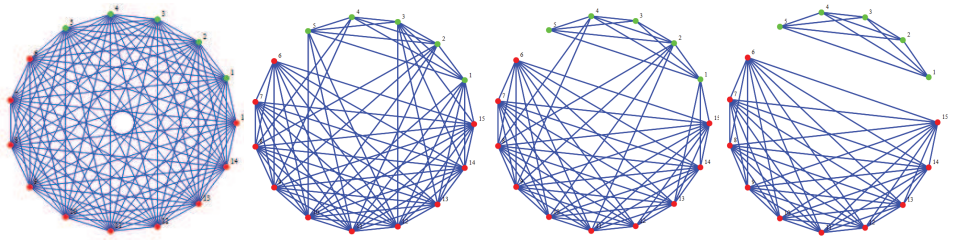

Fig. 3: Construction by successive removal of a clustering with 2 clusters.

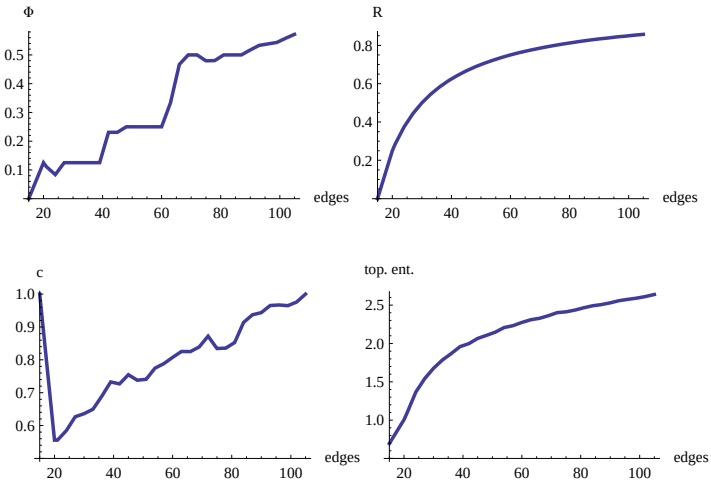

Fig. 4: Conductance (ϕ), performance (R), clustering coefficient (c) and network topological entropy ($top.ent$) evolving with the number of edges for CCN with 5 clusters.

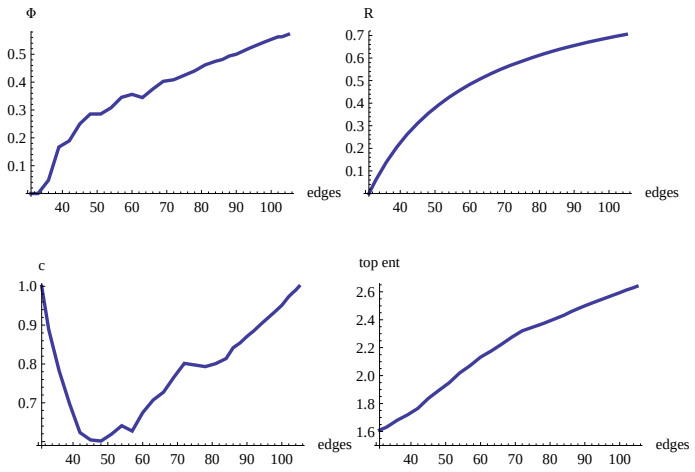

Fig. 5: Conductance (ϕ), performance (R), clustering coefficient (c) and network topological entropy ($top.ent$) evolving with the number of edges for CCN with 3 clusters.

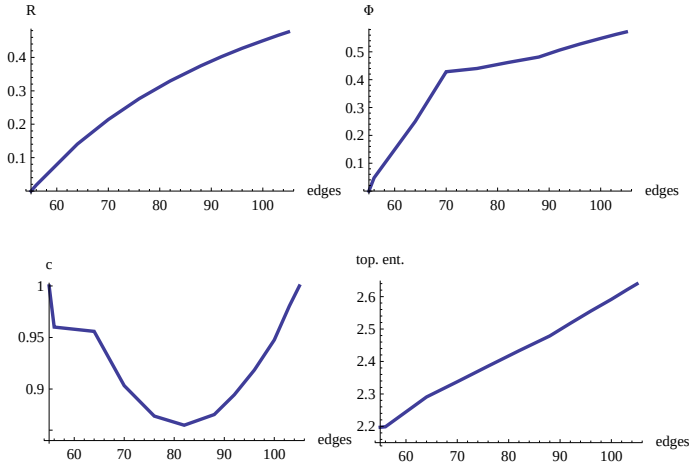

Fig. 6: (Performance (R), conductance ϕ), clustering coefficient (c) and network topological entropy ($top.ent$) evolving with the number of edges for CCN with 2 clusters.

compare different clusterings, establishing a certain hierarchy. The main result of this paper is the effect of the dynamical invariant, network topological entropy, in the performance of a clustering. The fact that the increase of the network topological entropy decrease the performance of a clustering, in the process of successive removal, is not altogether surprising, but as far as the authors are aware, this result has never been formulated so far. The increasing of the complexity of a system, characteristic measured by the network topological entropy, announces a certain disorder in the set of nodes. A good clustering should be precisely the opposite: tidying up of the nodes in well behaved and stable compartments.

Despite the monotonous decrease of the network topological entropy with the successive removal of edges, we can be observed that the network topological entropy depends also on several features of the network topology. In Fig.7 we can see examples of graphs with the same number of edges (63) and the same number of vertices (15) but with different configurations and different networks topological entropies. In this case (from left to the right in the Fig.7) $h(G_1) = 2.17514$, $h(G_2) = 2.27353$ and $h(G_3) = 2.31013$. So, it remains an open problem to study how the topology of the network determines the network topological entropy.

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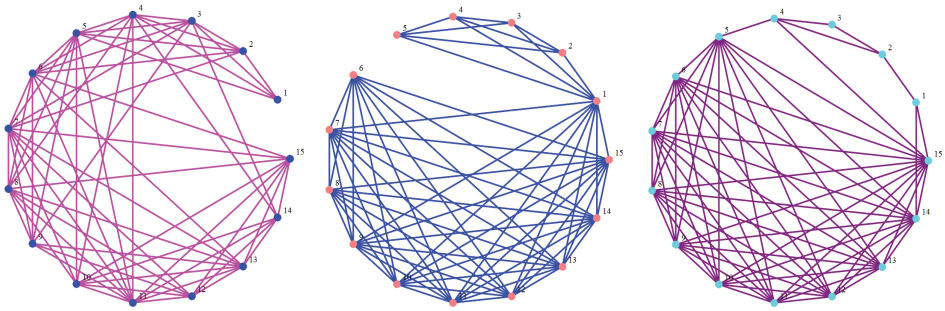

Fig. 7: Three different networks with the same number (63) of edges, but different network topological entropies.

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Appendix

Consider a finite set of symbols, denoted by $\mathscr{A}$, which is called the alphabet. Often, the choice falls on the first natural numbers:

$$\mathscr{A} = \{1,2,...,r\}.$$

A finite sequence of elements of $\mathscr{A}$ is called a block, or a word and the set of bi-infinite sequences, $x = (x_i)_{i\in\mathbb{Z}}$, of elements of $\mathscr{A}$, denoted by $\mathscr{A}^{\mathbb{Z}}$, is called the full $\mathscr{A}$-shift:

$$\mathscr{A}^{\mathbb{Z}} = \left\{x = (x_i)_{i\in\mathbb{Z}} : x_i \in \mathscr{A} \text{ for all } i \in \mathbb{Z}\right\}.$$

The shift map σ on the full shift $\mathscr{A}^{\mathbb{Z}}$ maps a point x to the point $y = \sigma(x)$ whose ith coordinate is $y_i = x_{i+1}$. If F is a collection of blocks over $\mathscr{A}$, we will think of as being forbidden blocks. Define X_F to be the subset of sequences in $\mathscr{A}^{\mathbb{Z}}$ which do not contain any block in F. Any subset of the full shift $\mathscr{A}^{\mathbb{Z}}$ such that $X = X_F$, for some family F of forbidden blocks over $\mathscr{A}$, is called a shift space X, or simply shift.

Definition 7. *A shift of finite type is a shift space that can be described by a finite set of forbidden blocks.*

This definition allows us to construct a shift of finite type X, through a directed graph whose bi-infinite sequences of vertices, connected by some edge, will correspond to the elements of X.

Definition 8. *Let A be the adjacency matrix of a graph G. The vertex shift space $X_A = X_G$ is the shift space with alphabet $\mathscr{A} = \{1,2,...,r\}$, defined by*

$$X_A = X_G = \left\{x = (x_i)_{i\in\mathbb{Z}} \in \mathscr{A}^{\mathbb{Z}} : a_{kj} = 1,\ \forall i \in \mathbb{Z}\right\},$$

with $k = x_i$ and $j = x_{i+1}$.

Every directed graph G, with its adjacency matrix A, describes a certain symbolic dynamical system, a shift of finite type $X_A = X_G$.

If, instead of considering the forbidden blocks, consider the blocks allowed, you can define the concept of language. It is called the language of X the collection $B(X)$

$$B(X) = \cup_{n=0}^{\infty} B_n(X),$$

where $B_n(X)$ is the set of all blocks, of length n, that occur for some element of X, see [16].

We introduce a very useful result on the number of walks in graphs, see [20]. Let G a graph on N vertices with associated adjacency matrix A. Then, the $(i,j)^{\text{th}}$ entry of A^k is the number of *k-walk* from i to j. In particular, the entries along the main diagonal of A^k are the numbers of closed *k-walks* in G.

As a consequence of this result to a shift of finite type space $X_A = X_G$, with A the adjacency matrix of directed graph G, with the set of vertices $V = \{1,2,...,r\}$, the following relationship is valid,

$$|B_n(X)| = \sum_{i=1}^{r}\sum_{j=1}^{r} (a_{ij})^n. \tag{8}$$

Remark. In the case where X is a full shift in an alphabet $\mathscr{A}$ of r symbols, follows that $|B_n(X)| = r^n$.

Since the entropy evaluates the complexity of a system, it is natural to introduce the concept of entropy, of a shift space X, as a rate of variation of the number of blocks of length n, $B_n(X)$, present in the language $B(X)$.

Definition 9. *Let X be a shift space. The entropy $h(X)$ over the finite alphabet $\mathscr{A}$ is defined by*

$$h(X) = \lim_{n\to\infty} \frac{\log|B_n(X)|}{n}.$$

Since the trace of a square matrix is the sum of the diagonal elements, this result shows that the number of closed walks with length k, in G, is the trace of the power

matrix A^k, $tr(A^k)$, entropy can be defined through the following expression,

$$h(X) = \lim_{n\to\infty} \frac{\log tr\,(A^n(X))}{n}.$$

Using the above observation is valid the following results.

Lemma 2. *Let X be a full shift in an alphabet of r symbols. Then the entropy of X is given by $h(X) = \log r$.*

Lemma 3. *Let $A \neq 0$ be a nonnegative matrix having a positive eigenvector v. Then the corresponding eigenvalue λ is positive, and there are positive constants c_0 and d_0 such that,*

$$c_0\ \lambda^n \leq \sum_{i=1}^{r}\sum_{j=1}^{r} (a_{ij})^n \leq d_0\ \lambda^n.$$

Hence if G is a graph whose adjacency matrix is A, then $h(X_G) = \log\lambda$.

Proof. Considering Eq.8, the inequalities are obtained. To show the final statement, just keep in mind the definition of entropy.

The Perron-Frobenius Theorem states that, if A is a finite irreducible adjacency matrix such that $A \neq 0$, then A has a positive eigenvector, called the Perron eigenvector, with corresponding positive eigenvalue, called Perron value, which is both geometrically and algebraically simple, see [15].

Theorem 2. *Let $X_A = X_G$ be a shift of finite type, where A is a finite irreducible adjacency matrix of G. Then $h(X_G) = \log\lambda_A$.*

Proof. Considering the previous lemmas and the Perron-Frobenius Theorem the result is proved.

A dynamical system (M, f) consists of a compact metric space M together with a continuous map $f : M \to M$. When $M = X$, a shift space and $f = \sigma$, the shift map, then (X, σ) is called a shift dynamical system.

Let (M, f) be a dynamical system and $S \subset M$, $n \in N$ and $\varepsilon > 0$. S is a $(n;\varepsilon)$-spanning set if for every $x \in X$ there exists $y \in S$ such that $d(f^j(x); f^j(y)) < \varepsilon$ for all $0 \leq j \leq n$. Denote by $r_n(f;\varepsilon)$ be the size of the $(n;\varepsilon)$-spanning set, for f with fewest number of elements.

Definition 10. *The topological entropy of f, $h_{top}(f)$, is given by,*

$$h_{top}(f) = \lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup\frac{\log r_n(f;\varepsilon)}{n}.$$

Lemma 4. *If (X, σ) is a shift dynamical space, then $r_n(\sigma; 2^{-k}) = |B_{n+2k}(X)|$.*

By the previous equality, in the case of a shift dynamical space, the topological entropy coincide with the value of the entropy of the shift space , see [21], i.e.,

$$h_{top}(f) = h(X).$$

If we have a network associated to a graph G which determine the shift space X, we will call network topological entropy of G to the quantity $h_{top}(X)$, i.e.,

$$h_{top}(G) = h_{top}(X).$$

References

[1] B. Balasundaram, S. Butenko, I.V. Hicks, Operations Research **59 (1)**, 133-142 (2011).
[2] B. Bollobás, Volume Estimates and Rapid Mixing, Flavors of Geometry, MSRI Publications, **31**, 1997.
[3] B. Bollobás, O.M. Riordan, Handbook of Graphs and Networks: From the Genome to the Internet, Wiley-VCH, 2003.
[4] A. Caneco, C. Grácio, C. Ramos, S. Fernandes, J.L. Rocha, in Proc. of Int. Workshop on Nonlinear Maps and Applications, NOMA'07, 42-45, 2007.
[5] A. Caneco, C. Grácio, J.L. Rocha, J. of Nonlinear Mathematical Physics **15 (3)**, 102-111 (2008).
[6] A. Caneco, C. Grácio, S. Fernandes, J.L. Rocha, C.Ramos, ISAST Transactions on Computers and Intelligent Systems **2 (1)**, 47-52 (2009).
[7] A. Caneco, C. Grácio, J.L. Rocha, Int. J. Bifurcation and Chaos **19 (11)**, 1-14 (2009).
[8] A. Caneco, S. Fernandes, C. Grácio, J.L. Rocha, Dynamics, Games and Science, eds. M.M. Peixoto, A. Pinto, D.A.J. Rand, Springer, 2010.
[9] S. Fernandes, C. Grácio, C. Ramos, Nonlinear Dynamics **61**, 435-442 (2010).
[10] L. Hogben, Electronic Journal of Linear Algebra **14**, 12-31 (2005).
[11] M.R. Jerrum, A.J. Sinclair, SIAM J. Comput. **18**, 1149 (1989).
[12] L. Ji, B-H. Wang, W-X. Wang, T. Zhou, Chin. Phys. Lett. **25 (11)**, 4177 (2008).
[13] C.R. Johnson, C. Negron, Electronic Journal of Linear Algebra **18**, 146-161 (2009).
[14] R. Kannan, S. Vempala, A. Vetta, in Proc. of the Symposium on Foundations of Computer Science, 2000.
[15] B. Kitchens, Symbolic Dynamics, Springer-Verlag, Berlin, Heidelberg, New York, 1998.
[16] D. Lind, B. Marcus, An Introduction to Symbolic Dynamics and Codings, Cambridge University Press, Cambridge, 1995.
[17] J. Milnor, W. Thurston, Dynamical Systems (College Park, MD, 1986/87), Lecture Notes in Math., 1342, Springer, Berlin, 465-563, 1988.
[18] J.L. Rocha, J. Sousa Ramos, J. Difference Equ. Appl. **9**, 319-335 (2003).
[19] J.L. Rocha, J. Sousa Ramos, Int. J. Math. Math. Sci. **38**, 2019-2038 (2004).
[20] G. Royle, C. Godsil, Algebraic Graph Theory, Springer, 2001.

[21] C. Spandl, Electronic Notes in Theoretical Computer Science **167**, 131-155 (2007).
[22] D.J. Watts, S.H. Strogatz, Nature **393**, 440-442 (1998).
[23] L. Zhenguo, W. Xiao-Ming, C. Shih-Fu, Segmentation using superpixels: A bipartite graph partitioning approach, Computer Vision and Pattern Recognition (CVPR), 2012 IEEE Conference on, 789-796 (2012).

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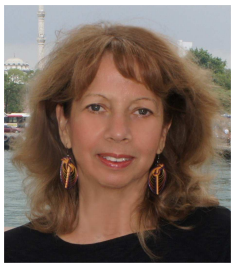

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