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Analytic Plane Sets are Locally $2n$-stars: A dynamically based Proof

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Abstract: A new, elementary proof of a well-known result, stating that the set of zeros of a two-variable analytic real function is locally a star with an even number of branches, is given. In contrast to other proofs available in the literature, ours is mainly based in some standard Poincaré-Bendixson theory.

Keywords: Poincaré-Bendixson theory, real analytic set, two-variable analytic real function, star, parity

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

In 1971, Dennis Sullivan discovered an important obstruction for a topological space to be the set of zeros of a holomorphic function [15, Corollary 2]: it must be locally homeomorphic to the cone over a polyhedron with even Euler characteristic. As a corollary, Sullivan proved that any plane analytic set (that is, the set of zeros of a two-variable real analytic function) is locally, up to some trivial cases, a topological star with evenly many branches.

As it turns out, this is a very useful tool when dealing with topological dynamics of bidimensional analytic flows. Two relatively recent examples are [13] and [14]. For instance, in [13] the star structure of analytic sets is needed to prove that ω-limit sets of analytic flows on the plane, the sphere or the projective plane are, roughly speaking, finite unions of topological circles. It is worth emphasizing that parity is critical as well, as it guarantees that, in these specific surfaces and for this type of flows, orbits cannot visit arcs of singular points in their ω-limit sets from both sides (this is not longer true in the torus or in proper subsets of the plane, for instance). This important point was missed in [13] and, as a consequence, some of the results stated there are not correct (this gap has been amended in [12]).

Another interesting application of the "star structure" theorem can be found in [11, Chapter X]: just with the help of some standard Poincaré-Bendixson techniques, one can prove what is arguably the key property of analytic dynamics in dimension two, namely, the local phase portrait at an isolated singular point (in the absence of periodic orbits in its vicinity and assuming that it is neither an attractor nor a repeller) consists of a finite union of hyperbolic, parabolic, and elliptic sectors. Remarkably enough, in the recent monographs [6] and [9] this decomposition theorem is proved using highly non-trivial and sophisticated desingularization methods.

To describe the topological structure of planar analytic sets, one can proceed in two steps: first, the local star structure is proved (this is the so-called Lojasiewicz theorem [10, Theorem 6.3.3, p. 168]); then one shows (Sullivan's theorem) that the number of branches is even. Lojasiewicz's theorem is a corollary of two classical results on real analyticity: the Weierstrass preparation theorem and the Hensel lemma. Their proofs, if somewhat cumbersome (especially in the case of Hensel's lemma), are elementary, see [10]. In contrast, all proofs of Sullivan's theorem we are aware of, including the original one, require advanced tools of algebraic topology [4,5,7,8], and hence are hard to follow for the non-specifically trained reader. The aim of this paper is to present a simple, dynamically based proof, of both Lojasiewicz's and Sullivan's theorems.

The paper is organized as follows. In Subsection 2.1 we introduce some notation and basic facts on analytic

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functions. Subsection 2.2 is devoted to collect the elementary results on qualitative theory of planar differential equations our proof is based on. The main theorem is precisely stated and proved in the last section.

2 Notation and elementary results

In this section we fix the notation and collect some definitions and properties which will be of use later.

2.1 Power series and analytic functions

We denote by $\mathbb{N}$ and $\mathbb{Z}^+$ the sets of non-negative and positive integers, respectively. Given any $n \in \mathbb{Z}^+$, any $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ and any $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we write $z^\alpha = z_1^{\alpha_1} \dots z_n^{\alpha_n}$ and $|z|^\alpha = |z_1|^{\alpha_1} \dots |z_n|^{\alpha_n}$.

A *real power series in n real variables centered at* $z_0 \in \mathbb{R}^n$ is a formal expression of the type

$$\sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha \tag{1}$$

with $a_\alpha \in \mathbb{R}$. We say that the series (1) is *absolutely convergent* at $z \in \mathbb{R}^n$ if for a bijection $\phi : \mathbb{N} \to \mathbb{N}^n$ the series $\sum_{n=0}^\infty |a_{\phi(n)}| \, |z - z_0|^{\phi(n)}$ is convergent (as a numerical series). Recall that if a numerical series converges absolutely then it also converges unconditionally; hence, if the series (1) is absolutely convergent, then the numerical series $\sum_{n=0}^\infty a_{\phi'(n)} (z - z_0)^{\phi'(n)}$ converges for any bijection $\phi' : \mathbb{N} \to \mathbb{N}^n$ and its sum does not depend on the rearrangement. Thus, if a power series like (1) converges absolutely at a point z, then we can speak about its *sum* at that point.

Analytic functions are those which can be expressed as a power series around any point in their domain. More formally:

Definition 2.1. Let $f : U \to \mathbb{R}$ be a function defined on an open subset $U \subset \mathbb{R}^n$. The function f is said to be *analytic* at a point $z_0 \in U$ if there exist an open neighbourhood of z_0, $V \subset U$, and a sequence of real numbers $(a_\alpha)_{\alpha \in \mathbb{N}^n}$, such that the power series $\sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ is absolutely convergent at any $z \in V$ and its sum coincides with $f(z)$. We say that f is *analytic (on U)* if it is analytic at any point of U.

Typically, functions $f : U \subset \mathbb{R}^n \to \mathbb{R}^m$ are described as vectors $f = (f_1, f_2, \dots, f_m)$, with each $f_i : U \to \mathbb{R}$ being a real function. We say that such a vector function is *analytic* if all its components f_i are analytic according to the above definition.

Analytic functions behave well under algebraic operations: the sum, the product, the division and the composition of analytic functions are analytic (where they are well defined). The same can be said about their calculus: any analytic function is continuously differentiable and any partial derivative of first order at any point of its domain can be computed differentiating formally each term of its representation as an absolutely convergent power series, and we obtain again an absolutely convergent power series. In particular, any analytic function is of class C^∞. We refer the reader to [10] for the details.

Two important (and also elementary) properties of real analytic functions are presented below; we will used them repeatedly in the sequel. The first of them works for analytic functions defined on any open subset of $\mathbb{R}^n$, the second one is true only in the one-dimensional case.

Proposition 2.2. If $f : U \to \mathbb{R}$ is an analytic function, $U \subset \mathbb{R}^n$ is connected and f vanishes at an open subset V of U, then it vanishes at the whole U.

Proof. It follows, after using a standard connectedness argument, from the relation between the coefficients of a power series representing an analytic function and its partial derivatives, see [10, Remark 2.2.4] □.

Proposition 2.3. Let $f : I \to \mathbb{R}$ be an analytic function defined on an open interval $I \subset \mathbb{R}$. If f vanishes at a sequence of points accumulating in I, then f vanishes at the whole interval I.

Proof. This is an easy application of Rolle's theorem, see [10, pp. 11–14] □.

The result closing this subsection, while elementary as well and already mentioned in the seminal Bendixson paper [3], is of an altogether different calibre. It relies on the fact that the ring of local convergent power series is a unique factorization domain; a detailed proof can be found, for instance, in [13].

Theorem 2.4. Let $f = (f_1, f_2) : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ be an analytic function and $w \in U$ be a zero of f. Then there are an open neighbourhood of w, $W \subset U$, and analytic functions $k, h_1, h_2 : W \to \mathbb{R}$ such that:

(i) $f_1 = kh_1$ and $f_2 = kh_2$ in W;
(ii) $h = (h_1, h_2)$ has no zeros in $W \setminus \{w\}$.

2.2 Analytic differential equations

Next we list some basic, well-known results on analytic plane vector fields (that is, analytic functions mapping open subsets of $\mathbb{R}^2$ into $\mathbb{R}^2$) and their associated flows. Much of what we are going to say is also true for C^1 or even locally Lipschitz vector fields, but we are only interested in the analytic case. Good references are [1,6], among others. In what follows, the analytic vector field $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ will remain fixed.

Consider the autonomous system

$$\dot{z} = f(z). \tag{2}$$

The zeros of f are also called the *singular points* of (2); the rest of the points are called *regular*. Given any $z \in U$, the system (2) admits a unique maximal solution

$\varphi_z : I_z \to U$ satisfying $\varphi_z(0) = z$. The interval I_z is open and the function φ_z is analytic; moreover, the *flow* of (2) defined by (whenever it makes sense) $\varphi(t,z) = \varphi_z(t)$ is analytic as well. The image of φ_z, $\gamma_z = \varphi_z(I_z)$, is called the *orbit of* (2) *through* z. If J is a subinterval of I_z, then we say that $\varphi_z(J)$ is a *semi-orbit* of (2). If z is a singular point of (2), then $I_z = \mathbb{R}$ and $\gamma_z = \{z\}$. If z is regular and φ_z is a periodic function, then we say that γ_z is a *periodic orbit*. The orbits of the system (2) foliate the phase space U, that is, two distinct orbits do not intersect each other.

For any $z \in U$, $I_z = (a_z, b_z)$, the *ω-limit set* of the point z (or of the orbit γ_z) is the set

$$\omega_f(z) = \omega_f(\gamma_z) = \{w \in U : \exists\, t_n \to b_z \text{ such that } \gamma_z(t_n) \to w\}.$$

The previous definition is a good one: if $\gamma_{z'} = \gamma_z$, then $\omega_\varphi(z) = \omega_\varphi(z')$. The *$\alpha$-limit set* $\alpha_f(z) = \alpha_f(\gamma_z)$ is analogously defined replacing $t_n \to b_z$ by $t_n \to a_z$. Points in $\alpha_\varphi(z)$ and in $\omega_\varphi(z)$ are called *limit points* of the orbit γ_z. In general, ω-limit sets (respectively α-limit sets) are closed subsets of U and are the union sets of some orbits of (2). If the ω-limit set of z is nonempty, then $b_z = \infty$, and the analogous statement holds for its α-limit set.

An analytic embedding $\lambda : J \to U$ of an open interval $J \subset \mathbb{R}$ is called an analytic *transverse local section* of f if the vectors $\dot{\lambda}(s)$ and $f(\lambda(s))$ are linearly independent for any $s \in J$. Of course, if $w \in U$ is a regular point of (2) and $f(w)$ is linearly independent to $v \in \mathbb{R}^2$, then $\lambda : (-\varepsilon, \varepsilon) \to U$ defined by $\lambda(s) = w + sv$ is an analytic transverse local section provided that $\varepsilon > 0$ is small enough.

Let $f_1 : U_1 \to \mathbb{R}^2$, $f_2 : U_2 \to \mathbb{R}^2$ be two analytic vector fields and let $\varphi_1 : \Delta_1 \to \mathbb{R}^2$ and $\varphi_2 : \Delta_2 \to \mathbb{R}^2$ their associated flows. We say that f_1 is *analytically conjugate* to f_2 if there is an analytic diffeomorphism $h : U_1 \to U_2$ such that the domains $I_{1,z}$ for $\varphi_{1,z}$ and $I_{2,h(z)}$ for $\varphi_{2,h(z)}$ are equal and $h(\varphi_1(t,z)) = \varphi_2(t, h(z))$ for every $(t,z) \in \Delta_1$.

Theorem 2.5 (the flow box theorem). Let $\lambda : J \to U$ be an analytic transverse local section of (2), assume that $[c,d] \subset J$ and $c < 0 < d$, and write $\lambda(0) = w$. Then there exist $\varepsilon > 0$, an open neighbourhood W of w in U and an analytic diffeomorphism $h : W \to (-\varepsilon, \varepsilon) \times (c,d)$ such that:

(i) $\lambda(J) \cap W = \lambda((c,d))$ and $h(\lambda(s)) = (0,s)$ for any $s \in (c,d)$;
(ii) h is an analytic conjugacy between $f|_W$ and the constant vector field $g : (-\varepsilon, \varepsilon) \times (c,d) \to \mathbb{R}^2$ given by $g(x,y) = (1,0)$.

Proof. See for example [6, Theorem 1.12] □.

Remark 2.6 ([2, Theorem 1.1, p. 45]). Let $z \in U$ be a regular point whose orbit is not periodic, let $a < b$ be two points in I_z and consider the semi-orbit $\varphi_z([a,b])$. Then, using compactness and the previous theorem, it is possible to find a neighbourhood W of $\varphi_z([a,b])$ in U and a small $\varepsilon > 0$ such that f is (analytically) conjugate in W to the constant vector field $g : (a-\varepsilon, b+\varepsilon) \times (-1,1) \to \mathbb{R}^2$ given by $g(x,y) = (1,0)$.

One of the landmarks of bidimensional qualitative theory of differential equations is the famous Poincaré-Bendixson theorem. It deals with the asymptotical behaviour of the orbits of (2).

Theorem 2.7 (the Poincaré-Bendixson theorem). Let $z \in U$ and suppose that the set $\gamma_z^+ = \{\varphi_z(t) : t \geq 0\}$ (respectively $\gamma_z^- = \{\varphi_z(t) : t \leq 0\}$) is contained in a compact subset of U. Then either the set $\omega_\varphi(z)$ (respectively $\alpha_\varphi(z)$) contains some singular point or it is a periodic orbit itself.

Proof. See for instance [6, Theorem 1.25] or [1, Theorem 13, p. 92] □.

The following classical corollary of the Poincaré-Bendixson theorem is of special interest:

Theorem 2.8. If γ is a periodic orbit of (2) enclosing (as a Jordan curve) a region $B \subset U$, then B contains a singular point of (2).

Proof. See [6, Theorem 1.31] or [1, Theorem 16, p. 97] □.

In standard proofs of the Poincaré-Bendixson Theorem the result below is stated as a preliminary lemma. Its proof can be found for example in [6, Lemma 1.29] or [1, Theorem 11, p. 90].

Theorem 2.9. Let $z \in U$ and assume that γ_z^+ (respectively, γ_z^-) is contained in a compact subset of U. If the ω-limit or the α-limit set of an orbit $\gamma \subset \omega_f(z)$ (respectively, $\gamma \subset \alpha_\varphi(z)$) contains some regular point, then γ is periodic and $\omega_\varphi(z) = \gamma$ (respectively $\alpha_\varphi(z) = \gamma$).

The proof of our last lemma is very simple, so we include it:

Lemma 2.10. Let $k : U \to \mathbb{R}$ be an analytic function and consider the system $\dot{z} = g(z)$, with $g = kf$. Let z be a regular point of $\dot{z} = g(z)$ and $\phi_z : J_z \to U$ be the maximal solution of $\dot{z} = g(z)$ passing through z. Then there exist an open interval $0 \in L \subset \mathbb{R}$ and an analytic function $\tau : L \to \mathbb{R}$ such that $\tau(0) = 0$ and $\varphi_z(t) = \phi_z(\tau(t))$ for all $t \in L$.

Proof. If z is a regular point of $\dot{z} = g(z)$, then it cannot be a zero of k. Therefore, there is an open interval $0 \in J \subset J_z$ where the function $F = 1/(k \circ \phi_z)$ is well defined and analytic.

Let us consider now the maximal solution of Cauchy's problem

$$\begin{cases} \dot{\tau} = F(\tau) \\ \tau(0) = 0. \end{cases}$$

This maximal solution is a function τ defined in an open interval L containing 0 and whose evaluations belongs to J. It is easy to check that the composition $\phi_z \circ \tau$ is a solution of

$$\begin{cases} \dot{z}(t) = f(z) \\ z(0) = z, \end{cases}$$

hence we have $L \subset I_z$ and $\varphi_z(t) = \phi_z(\tau(t))$ for all $t \in L$ □.

3 The local structure of analytic sets

We devote this final section to state and prove the main result of the paper.

Given any positive integer $r \in \mathbb{Z}^+$, we say that a topological space is an *r-star* if it is homeomorphic to $S_r = \{z \in \mathbb{C} : z^r \in [0,1]\}$. If X is an r-star and $h : S_r \to X$ is an homeomorphism, then the image of the origin under h is called a *vertex of the star*. Note that the vertex of a star is uniquely defined except in the cases $r = 1, 2$, when X is just a closed arc and the vertexes are, respectively, its endpoints (for $r = 1$) or its interior points (for $r = 2$).

Theorem 3.1. Let $U \subset \mathbb{R}^2$ be open and connected and $f : U \to \mathbb{R}$ be an analytic function. Let $C = \{z \in U : f(z) = 0\}$ be the set of zeros of f. Then either $C = U$ or given any non-isolated point of C, $z \in C$, there exists a neighbourhood V of z and an $n \in \mathbb{Z}^+$ such that $V \cap C$ is a $2n$-star with vertex z.

Remark 3.2 For the sake of simplicity the above result has been stated for analytic functions on the real plane but, obviously, the same local obstruction works as well for analytic functions defined on analytic surfaces.

Proof. If C has non-empty interior (as a subset of U), then f is identically zero (Proposition 2.2). In what follows we will assume that $\mathrm{int}(C) = \emptyset$.

Let $z_0 \in U$ be a non-isolated point of C. We will construct an analytic system

$$\dot{z} = h(z) \tag{3}$$

in a specific open neighbourhood W of z_0 such that either all its points are regular points for h (and then we will see that, in a neighbourhood of z_0, C reduces to an arc with z_0 in its interior), or z_0 is the only singular point of (3) in W and there is a compact neighbourhood of z_0, $V \subset W$, that can be written as a finite union of evenly many sectors, with the additional property that $C \cap V$ is the union of $\{z_0\}$ and the orbits separating the adjoining sectors.

To define this system we proceed in two steps. Firstly, we consider the system $\dot{z} = g(z)$ given by $g(z) = (-\frac{\partial f}{\partial y}(z), \frac{\partial f}{\partial x}(z))$. Secondly, by virtue of Theorem 2.4, there exist a neighbourhood of z_0, $W \subset U$, and analytic functions $k, h_1, h_2 : W \to \mathbb{R}$ such that $g_1 = kh_1$, $g_2 = kh_2$ and $h = (h_1, h_2)$ has no zeros in $W \setminus \{z_0\}$. The function h is the vector field we are looking for.

Notice that, replacing f by f^2 if needed, there is no loss of generality in assuming that $\frac{\partial f}{\partial x}(z) = \frac{\partial f}{\partial y}(z) = 0$ for all $z \in C$. Moreover, f is a first integral for the system (3), that is, if $z : J \subset \mathbb{R} \to W$ is any of its solutions, then $f \circ z$ is constant. Indeed, if $t \in J$ is such that $z(t)$ is a singular point of $\dot{z} = g(z)$, then both partial derivatives of first order of f vanish at $z(t)$; otherwise we apply Lemma 2.10 to guarantee that $\dot{z}(t)$ and $g(z(t))$ are proportional vectors. Therefore, we get in any case

$$\frac{d(f \circ z)}{dt}(t) = \left\langle \left(\frac{\partial f}{\partial x}(z(t)), \frac{\partial f}{\partial y}(z(t)) \right), \dot{z}(t) \right\rangle = 0,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of $\mathbb{R}^2$.

Obviously there are two options for z_0: either it is an isolated singular point of (3) or it is a regular one. We will distinguish these two cases in the following reasoning.

We first handle the case when all points of W are regular for (3). According to the flow box theorem, W can be chosen in such a way that all the orbits of (3) accumulate at the boundary of W and intersect (the image of) a transverse local section $\lambda : J \to W$ with $0 \in J$ and $\lambda(0) = z_0$ at exactly one point. Let $s \in J$ and consider the maximal solution of (3) passing through $z_s = \lambda(s)$, $\varphi_{z_s}(t)$. The composition $f \circ \varphi_{z_s}$ is constant; when $s = 0$ that constant is necessarily zero. Taking into account Theorem 2.3 and the fact that C has empty interior, we get that $f \circ \varphi_{z_s}$ do not vanish if $s \in J \setminus \{0\}$ is close enough to 0. Thus, choosing if necessary a smaller neighbourhood W of z_0, we get that the zeros of f in W are exactly the points of the orbit γ_{z_0}.

Now we consider the case when z_0 is an isolated singularity of h (in fact, because of the way we have defined W, it is the only singularity of h). Since f is a first integral, if the α-limit set or the ω-limit set of a solution $z(t)$ of (3) contains z_0, then the function $f \circ z$ must be identically zero. As a consequence, $\{z_0\}$ cannot be, at the same time, the α-limit and the ω-limit set of any solution $z(t)$ of (3), that is, the system (3) admits no homoclinic orbits (except $\{z_0\}$ itself). Indeed, if we suppose the contrary, then the image of $z(t)$ (together with z_0) defines a Jordan curve. Therefore, the Poincaré-Bendixson theorem and Theorem 2.8 imply that all orbits in the region enclosed by this Jordan curve are homoclinic as well and C has non-empty interior, a contradiction.

We claim that the system (3) admits no sequences of periodic orbits $(J_n)_{n\in\mathbb{N}}$ satisfying $J_n \subset B(z_0, 1/n) \cap W$ for all n. We argue to a contradiction by assuming that such a sequence does exist. Recall that any periodic orbit in W encloses z_0 (by Theorem 2.8), so given any two of them one encloses the other; in particular we can assume that J_n encloses J_{n+1} for every n. Besides, since z_0 is not an isolated zero of f, given an arbitrary n one finds a $z \neq z_0$ in the region enclosed by J_n such that $f(z) = 0$. Say that z belongs to the annulus between consecutive curves J_m and J_{m+1}, $m \geq n$. By the Poincaré-Bendixson theorem, two possibilities arise for the orbit of z: either it is a periodic orbit consisting of zeros of f or it spirals towards two periodic orbits (both consisting of zeros of f) included in the fixed annulus. Therefore, one can also consider a new sequence of periodic orbits $(J'_n)_{n\in\mathbb{N}}$ such that each J'_n consists of zeros of f and verifies that J_n encloses J'_n and J'_n encloses J'_{n+1}. Consequently, the analytic function $\tau \mapsto f(z_0 + \tau(1,0))$ vanishes at a sequence of points $(\tau_n)_n$ converging to 0 so, by Proposition 2.3, it vanishes in a full open interval containing 0, say $(-\delta, \delta)$. Now realize that any orbit of any point near enough to z_0 must spiral around z_0, hence it must intersect the segment $\{z_0 + \tau(1,0) : \tau \in (-\delta, \delta)\}$. We conclude that f vanishes in a neighbourhood of z_0, contradicting that C has empty interior.

As a consequence of the above claim, there is a small neighbourhood W' of z_0 such that the system (3), when restricted to W', has no periodic orbits. (Note that this does not exclude the possibility that the initial system has a sequence of periodic orbits accumulating at z_0; however, since their diameters cannot be too small, they become non-periodic after being "cut off" by W'.) To ease the notation we still call the new neighbourhood and the corresponding system W and (3), respectively.

Next, let us consider a small enough $r > 0$ such that $V = \mathrm{Cl}(B(z_0,r)) \subset W$. In the absence of homoclinic and periodic orbits, taking into account that z_0 is the only singular point of (3), and using the Poincaré-Bendixson theorem and Theorem 2.9, the semi-orbits of the system in V can be easily described. Namely, if $z \in \mathrm{int}(V) \setminus \{z_0\}$ and γ_z is the orbit of (3) through z, then the connected component of $\mathrm{Cl}(V) \cap \gamma_z$ containing z is either a closed arc with endpoint in ∂V or (together with z_0) a closed arc whose endpoints are z_0 and a point of ∂V. More precisely, let $z \in \mathrm{int}(V) \setminus \{z_0\}$, let $\varphi_z : (a_z, b_z) \to W$ be the maximal solution of (3) with $\varphi_z(0) = z$, and write $\varphi_z((a_z,b_z)) = \gamma_z$. Then the component of $\gamma_z \cap V$ containing z is either $\varphi_z([a'_z,b'_z])$, $\varphi_z((a_z,b'_z])$ (with $a_z = -\infty$), or $\varphi_z([a'_z,b_z))$ (with $b_z = \infty$) for some $a_z < a'_z < 0 < b'_z < b_z$; points $\varphi_z(a'_z), \varphi_z(b'_z)$ belong to ∂V; and we have $\lim_{t\to-\infty} \varphi_z(t) = z_0$ and $\lim_{t\to\infty} \varphi_z(t) = z_0$, respectively, in the last two cases.

Among the semi-orbits of (3) in V, those having $\{z_0\}$ as their α-limit or ω-limit sets consist of zeros of f. We claim that there are only finitely many of them. If the opposite is true, then any circle centered in z_0 with radius s less than r would contain infinitely many zeros of f. Applying Proposition 2.3 to the analytic function $\tau \mapsto f(z_0 + (s\cos\tau, s\sin\tau))$, we get that f vanishes in the whole circle and, since s is arbitrary, in the whole V. A similar argument allows us to assume (using if necessary a smaller r) that the zeros of f contained in V (apart from z_0) are exactly those in the semi-orbits having z_0 as a limit point. Since z_0 is not an isolated point of C, the family of these special semi-orbits cannot be empty: we denote them by $\Gamma_1, \Gamma_2, \dots \Gamma_m$ and assume that they are counterclockwise ordered. It only rests to show that m is even.

Let $z_i \in \Gamma_i \cap \partial V$, $i = 1, 2 \dots, m$. Taking advantage of analyticity once more, there is no loss of generality in assuming that ∂V is locally transverse to (3) at these points and $\Gamma_i \setminus \{z_i\} \subset \mathrm{int}(V)$ for any i. We call Γ_i *outward* or *inward* according to, respectively, $\lim_{t\to-\infty} \varphi_{z_i}(t) = z_0$ or $\lim_{t\to\infty} \varphi_{z_i}(t) = z_0$. We prove that m is even by showing that the semi-orbits Γ_i are consecutively outward and inward.

Assume, for instance, that Γ_i is inward. Let A be the counterclockwise arc in ∂V with endpoints z_i and z_{i+1} (here, we identify $m+1$ and 1; if $m = 1$, then $A = \partial V$). Let $(p_n)_n$ be a sequence of points in A monotonically converging to z_i. Since ∂V is locally transverse to (3) at z_i, we can assume that there are semi-orbits Υ_n entering V at p_n and escaping from V at corresponding points $q_n \in A$. Observe that the points q_n are reversely ordered as those in the sequence $(p_n)_n$, hence they converge to a point $q \in A$. Let t_n be the escaping time of Υ_n, that is $\varphi_{p_n}(t_n) = q_n$. Since $\varphi_{z_i}(t)$ is well defined (and inside V) for any $t \geq 0$, the continuity of the flow at z_i implies $t_n \to \infty$ as $n \to \infty$. Finally, the continuity of the flow at q guarantees that $\varphi_q(t)$ is well defined, and inside V, for any $t \leq 0$, that is, $q = z_{i+1}$ and Γ_{i+1} is outward (see Remark 2.6). This finishes the proof. □.

Remark 3.3. Note that the two last paragraphs of the above argument can be disposed of: since W contains neither homoclinic nor periodic orbits, and only finitely many heteroclinic orbits, z_0 must admit a neighbourhood consisting of a finite number n of hyperbolic sectors. The point z_0 is, alternatively, the α-limit set and the ω-limit set of the orbits limiting these sectors, hence n is even (instead, we can use the Poincaré index formula [6, Proposition 6.32] to deduce that the topological index of z_0 is the integer $1 - n/2$, so n is even).

Yet, as indicated at the beginning of the paper, the only elementary proof of the "sectors" theorem we are aware of is based on the "star structure" theorem. Thus, in order to avoid a circular argument, we are bound to (implicitly) use desingularization and, in a sense, the simple profile of our proof is lost.

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