Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090515

Syllable Permutations and Hyperbolic Lorenz Knots

Paulo Gomes[1,*], *Nuno Franco*[2] *and Luís Silva*[3]

[1] Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa, Rua Conselheiro Emídio Navarro, 1, 1959-007 Lisboa, Portugal
[2] Departamento de Matemática, Universidade de Évora, Largo dos Colegiais 2, 7004-516 Évora, Portugal
[3] Área Departamental de Matemática, Instituto Superior de Engenharia de Lisboa, Rua Conselheiro Emídio Navarro, 1, 1959-007 Lisboa, Portugal

Received: 16 Jan. 2015, Revised: 17 Apr. 2015, Accepted: 18 Apr. 2015
Published online: 1 Sep. 2015

Abstract: Lorenz knots are the knots corresponding to periodic orbits in the flow associated to the Lorenz system. This flow induces an iterated one-dimensional first-return map whose orbits can be represented, using symbolic dynamics, by finite words. As a result of Thurston's geometrization theorem, all knots can be classified as either torus, satellite or hyperbolic knots. Birman and Williams proved that all torus knots are Lorenz knots which can be represented by a class of words with a precise form. We consider about 20000 words corresponding to all non-trivial permutations of a sample of words associated to torus knots and, using the Topology and Geometry software *SnapPy*, we compute their hyperbolic volume, concluding that it is significantly different from zero, meaning that all these knots are hyperbolic. This leads us to conjecture that all knots in this family are hyperbolic.

Keywords: Lorenz knots, symbolic dynamics, hyperbolic knots

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

Lorenz knots

In 1963 Edward Lorenz introduced a simplified nonlinear model for convection in the atmosphere [6] which was at the origin of Chaos Theory and provided an example of what came to be known as a strange attractor. The *Lorenz system* is a three-dimensional flow determined by the Lorenz equations which, for the original parameters used by Lorenz, are

$$\begin{cases} x' = -10x + 10y \\ y' = 28x - y - xz \\ z' = -\frac{8}{3}z + xy \end{cases} \tag{1}$$

While initially interest was focused on the nonperiodic solutions of this system and their properties, it soon became apparent that closed orbits, corresponding to periodic solutions, also have interesting topological and geometric properties. The closed (periodic) orbits of the Lorenz system are called *Lorenz knots*, while *Lorenz links* are sets of (possibly linked) orbits.

Williams [7],[8] introduced the *Lorenz template* or knot-holder, which is a branched 2-manifold with boundary with one joining chart, one splitting chart and an expanding semi-flow defined on them. Fig. 1 represents the joining and splitting charts, while Fig. 2) depicts the Lorenz template built from gluing the bottom of the joining chart to the top of the splitting chart and the bottom of the splitting chart to the branches of the joining chart. Every Lorenz knot and link can be embedded in the Lorenz template, a conjecture formulated by Birman and Williams [3] and later proved through the work of Tucker [9]. Birman and Williams used the Lorenz template to study Lorenz knots and links [3]. This study was later continued by Birman and Kofman [4]. For a review on Lorenz knots and links, see [5].

* Corresponding author e-mail: pgomes@adm.isel.pt

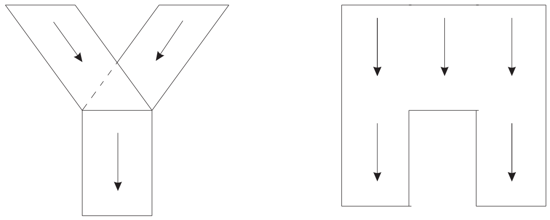

Fig. 1: Joining (left) and splliting (right) charts

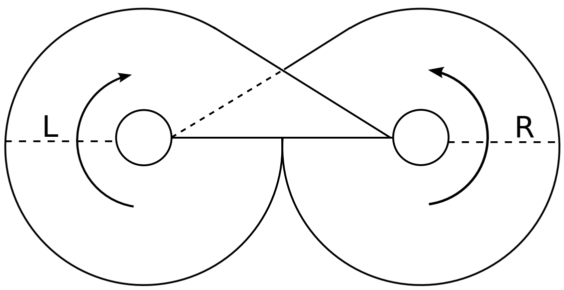

Fig. 2: The Lorenz template

Lorenz braids

An open n-braid can be informally thought of as a set of n strings that connect n points in a line (represented at the top in our diagrams) to n points in a (bottom) line parallel to the first, such that (see Fig. 4) each point at the top is connected to a single point at the bottom, distinct top points are connected to distinct bottom points and each string intersects any horizontal plane exactly once.

Each open n-braid determines a permutation π where $\pi(i) = j$ if point i at the top is connected by a string to point j at the bottom, for $i = 1,\ldots,n$. There is in general an infinite number of n-braids corresponing to the same permutation which differ in the pattern and number of string crossings.

Every knot and link is the closure of a (non-unique) open braid. In the case of Lorenz knots, if the Lorenz template is cut open along the dotted lines in Fig. 2, each knot and link on the template can be regarded as the closure of an open braid on the cut-open template. A braid is called *simple* if any two strings cross at most once and *positive* if in any crossing, the left string always crosses over the right string (Fig. 3).This definition of positive crossing follows Birman [3] instead of the usual convention in knot theory. In this figure we adopt the convention of drawing the overcrossing (L) strings as thicker lines than the undercrossing (R) strings. This convention will be used in all braid diagrams in this paper. The open braid associated in this way to the knot or link is called a *Lorenz braid* ([3]). Lorenz braids are simple positive braids composed of $n = p+q$ strings, where each of the p left (or L) strings crosses over at least one of the q right (or R) strings, with no crossings between strings in each subset L or R. These sets can be subdivided into subsets LL, LR, RL and RR according to the position of the startpoints and endpoints of each string. An example of a Lorenz braid with 7 strings, of which $p = 4$ are L strings and $q = 3$ R strings, is shown in Fig. 4.

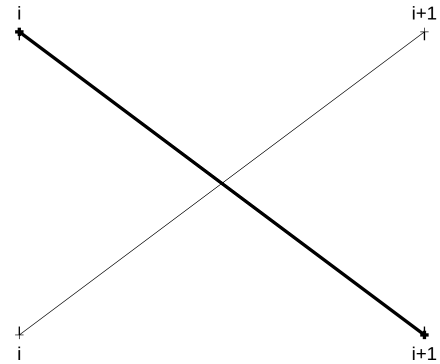

Fig. 3: Positive crossing

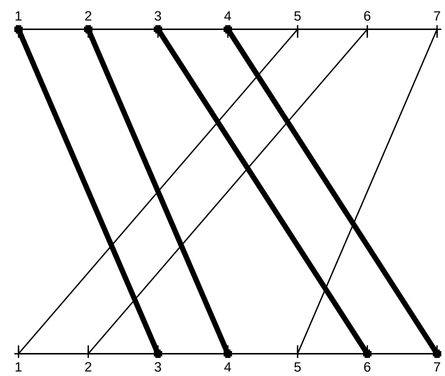

Fig. 4: A Lorenz braid

The set of braids on n strings can be given a group structure. Let σ_i, $i = 1,\ldots,n-1$ be the simple positive braid where the i-th string crosses over the $i+1$-string. The inverse of σ_i is the simple braid σ_i^{-1} where the i-th string crosses under the $i+1$-string. The *braid group on n strings B_n* is generated by σ_i, $i = 1,\ldots,n-1$ and it is defined by the presentation

$$B_n = \langle \sigma_1,\ldots,\sigma_{n-1} | \sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ i = 1,\ldots,n-2\rangle$$

In particular, all Lorenz braids can be expressed as products of these generators.

The *Lorenz map* is the first-return map induced on the *branch line* (the horizontal line in Fig. 2, where the two branches of the joining chart meet) by the semi-flow on the Lorenz template. The branch line can be mapped onto the interval $[-1,1]$; the resulting Lorenz map f is a one-dimensional map from $[-1,1]\setminus\{0\}$ onto $[-1,1]$,

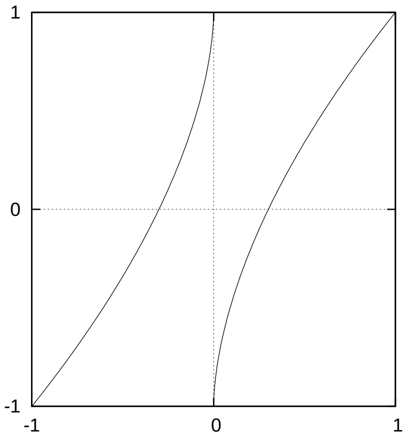

Fig. 5: Lorenz map

with one discontinuity at 0 and stricly increasing in each of the subintervals $[-1,0[$ and $]0,1]$ (Fig. 5).

Symbolic dynamics for the Lorenz map

Let $f^j = f \circ f^{j-1}$ be the j-th iterate of the Lorenz map f anf f^0 be the identity map. We define the *itinerary* of a point x under f as the symbolic sequence $(i_f(x))_j$, $j = 0,1,\ldots$ where

$$(i_f(x))_j = \begin{cases} L \text{ if } f^j(x) < 0 \\ 0 \text{ if } f^j(x) = 0 \\ R \text{ if } f^j(x) > 0 \end{cases}.$$

The itinerary of a point in $[-1,1]\setminus\{0\}$ under the Lorenz map can either be an infinite sequence in the symbols L,R or a finite sequence in L,R terminated by a single symbol 0 (because f is undefined at $x = 0$). The *length* $|X|$ of a finite sequence $X = X_0 \ldots X_{n-1}0$ is n, so it can be written as $X = X_0 \ldots X_{|X|-1}0$. A sequence X is periodic if $X = (X_0 \ldots X_{p-1})^\infty$ for some $p > 1$. If p is the least integer for which this holds, then p is the (least) period of X.

The space Σ of all finite and infinite sequences can be ordered in the lexicographic order induced by $L < 0 < R$: given $X,Y \in \Sigma$, let k be the first index such that $X_k \neq Y_k$. Then $X < Y$ if $X_k < Y_k$ and $Y < X$ otherwise.

The *shift map* $s : \Sigma \setminus \{0\} \to \Sigma$ is defined as usual by $s(X_0X_1\ldots) = X_1\ldots$ (deleting the first symbol). From the definition above, an infinite sequence X is periodic *iff* there is $p > 1$ such that $s^p(X) = X$. We will also define the shift operator on finite aperiodic words. Given a p-periodic sequence $X = (X_0X_1\ldots X_{p-1})^\infty$ where $w = X_0X_1\ldots X_{p-1}$ is a finite aperiodic word of length p, we define $s(X_0X_1\ldots X_{p-1}) = X_1\ldots X_{p-1}X_0$. Then $s(X) = (X_1\ldots X_{p-1}X_0)^\infty = (s(X_0X_1\ldots X_{p-1}))^\infty$. The sequence $w, s(w), \ldots, s^{p-1}(w)$ will also be called the *orbit* of w and a word in the orbit of w will be generally called a shift of w.

Given a finite aperiodic word w of length n, its associated Lorenz braid can be constructed as follows: order the successive shifts $s(w), s^2(w), \ldots, s^n(w) = w$ lexicographically and let the ordered words be $w_1,\ldots,w_n$, such that $w_1 < \ldots < w_n$ lexicographically. Associate these words to the n startpoints and endpoints of a Lorenz $n-$braid. Each string in the braid connects the startpoint corresponding to $s^k(w)$ to the endpoint corresponding to $s^{k+1}(w)$. Fig. 6 exemplifies this for $w = LRRLR$. The permutation associated to the braid is defined by $\pi(i) = j$ if $s(w_i) = w_j$, $i = 1,\ldots,n$. The corresponding Lorenz knot is the closure of the braid, as shown in Fig. 7.

A (finite or infinite) sequence X is called *L-maximal* if $X_0 = L$ and for $k > 0$, $X_k = L \Rightarrow s^k(X) \leq X$, and *R-minimal* if $X_0 = R$ and for $k > 0$, $X_k = R \Rightarrow X \leq s^k(X)$. An infinite periodic sequence $(X_0\ldots X_{n-1})^\infty$ with least period n is L-maximal (resp. R-minimal) if and only if the finite sequence $X_0\ldots X_{n-1}0$ is L-maximal (resp. R-minimal).

The *braid index* of a (Lorenz) link is the minimum number of strings necessary to represent it as the closure of an open braid. It is a knot invariant. The *trip number* is the number of syllables (subwords of type L^aR^b with maximal length). The trip number of a Lorenz link is the sum of the trip numbers of its components. After an initial conjecture by Birman and Williams [3], Franks and Williams [10] proved that the braid index of a Lorenz link is equal to its trip number t. In [3], for each Lorenz knot, a braid with minimal number (t) of strings was presented, as a product of generators:

$$\Delta^2 \prod_{i=1}^{t-1} (\sigma_1 \ldots \sigma_i)^{n_i} \prod_{i=t-1}^{1} (\sigma_{t-1} \ldots \sigma_i)^{m_{t-i}}$$

where Δ^2 is the full-twist in t strings, and, denoting by π the permutation associated with the Lorenz braid, the exponents are

$$n_i = \#\{j : \pi(j) - j = i+1 \wedge \pi(j) < \pi^2(j)\}$$
$$m_i = \#\{j : j - \pi(j) = i+1 \wedge \pi(j) > \pi^2(j)\}$$

This minimal t-braid will be used in computations below.

Thus, each finite aperiodic L,R word determines a Lorenz braid and a minimal braid which represent the same Lorenz knot (their closures are equivalent).

Given a pair of relatively prime integers p,q, the $T(p,q)$ *torus knot* is (isotopic to) a closed curve on the surface of an unknotted torus T^2 that intersects a circle of constant latitude p times and a circle of constant longitude q times. Birman and Williams [3] proved that every torus knot is a Lorenz knot.

A *satellite knot* is defined by taking a nontrivial knot C (companion) and another nontrivial knot P (pattern) contained in a solid unknotted torus T and not contained in a $3-$*ball* in T; the satellite knot is the image of P under an homeomorfism that takes the core of T onto C.

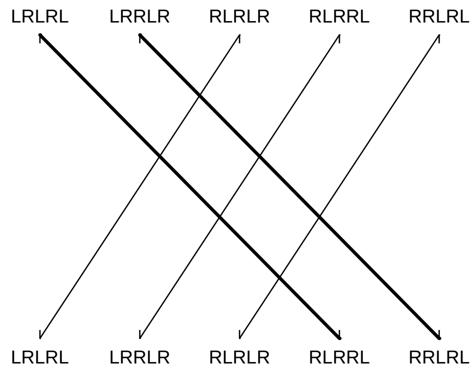

Fig. 6: Lorenz braid corresponding to $w = LRRLR$

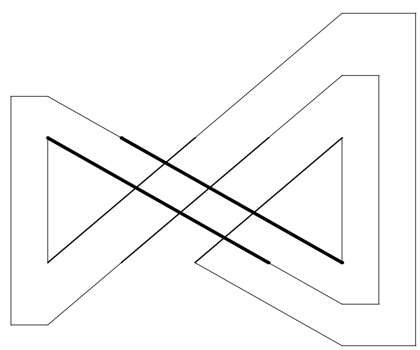

Fig. 7: Lorenz knot corresponding to $w = LRRLR$

Hyperbolic knots

A *hyperbolic 3-manifold* is a 3-manifold with a complete Riemannian metric of constant curvature -1. Such a manifold may be decomposed into *ideal hyperbolic tetrahedra*. The *hyperbolic volume* is the sum of the volumes of these tetrahedra; it is a topological invariant of the manifold [1]. A knot is called *hyperbolic* if its complement in S^3 is a hyperbolic 3-manifold. A knot is therefore hyperbolic if the volume of its complement is nonzero. Thurston proved that all knots that are neither satellite knots nor torus knots are hyperbolic [13]. Birman and Kofman [4] listed hyperbolic Lorenz knots taken from a list of the simplest hyperbolic knots. In a previous article the authors generated and tested families of Lorenz knots that are a generalization of some of those that appear in Birman and Kofman's list and conjectured that the families tested are hyperbolic [12].

2 Torus knots and syllable permutations

As mentioned above, all torus knots are Lorenz knots. Since $T(q,p) = T(p,q)$ we will consider torus knots $T(p,q)$ with $p < q$ only. Lorenz knots corresponding to orbits in the Lorenz template which are represented by evenly distributed words in the alphabet $\{L,R\}$ are torus knots [3]. Also, given a torus knot $T(p,q)$ there is an evenly distributed word with p symbols L and q symbols R that represents it. There is thus a bijection between torus knots and cyclic permutation classes of evenly distributed words. We will call the L-maximal word that represents $T(p,q)$ the *standard word* $w(p,q)$ for $T(p,q)$. These are also the words generated in the symbolic Farey tree (see [11]). Let $k > 0$ be the quotient of the integer division of q by p and $0 < r < p$ the remainder ($q = kp + r$). Then $w(p,q)$ has p syllables: r of type LR^{k+1} and $p - r$ of type LR^k, evenly distributed. This aperiodic word is unique for each p,q satisfying the conditions above [3].

Any word resulting from permuting the syllables of the standard word of $T(p,q)$ is aperiodic and therefore corresponds to a Lorenz knot with the same trip number. Let $n_L(w)$ and $n_R(w)$ be the number of letters L and R in the word w, respectively. If the word w resulting from syllable permutation was not aperiodic there would be a subword v of w such that $w = v^j$ for some $j > 1$, then we would have $p = n_L(w) = jn_L(v)$ and $q = n_R(w) = jn_R(v)$ and p,q wouldn't be relatively prime. Since any cyclic permutation of a finite word represents the same knot we will consider the L-maximal form of all distinct permutations corresponding to each torus knot.

For example, for the standard word *LRRLRLRRLRLRRLRLR* corresponding to $T(7,10)$ the syllable permutations in L-maximal form are *LRRLRRLRRLRLRLRLR*, *LRRLRRLRLRRLRLRLR* and *LRRLRRLRLRLRRLRLR*.

3 Computational test

Description

For each aperiodic L-maximal word tested, p is the number of symbols L, q is the number of symbols R, k and r are respectively the quotient and remainder of the integer division of q by p.

We first generated, using the CAS *Maxima*, all the words in the alphabet L,R satisfying the following conditions:

– p and q are relatively prime
– $p < q$
– $5 \leq p \leq 19$ and $6 \leq q \leq 100$
– Each word has p syllables with the form LR^b, with $b \in \{k, k+1\}$.

Remark. For words with $1 \leq p \leq 4$, all the syllable permutations of each word are equivalent to the standard word (they have the same $L-$maximal form), therefore they were not included in the test.

The list of words generated contains 196552 different words. It includes all the evenly distributed (standard) words that are known to correspond to torus knots $T(p,q)$ with $5 \leq p \leq 19$ and $6 \leq q \leq 100$.

The generated words were converted, also using a *Maxima* script, into minimal Birman-Williams braids in a format suitable as input for the Geometry and Topology software *SnapPy*. This program was then used to compute the hyperbolic volume of the complement of the knot corresponding to each braid.

Hardware and software

– CPU: Intel Core i7 CPU M620 @2.67GHz
– Memory (RAM): 8GB
– OS: Debian GNU/Linux 7.5 (http://www.debian.org/)
– Maxima 5.27.0 (http://maxima.sourceforge.net/)
– SnapPy 2.1 (snappy.computop.org)
– Python 2.7 (http://www.python.org/)

Results

1. The volumes computed by *SnapPy* for the evenly distributed words are all less or equal than the volume of an ideal regular tetrahedron ($v_3 = 1.101494\ldots$) and thus less than the minimum volume of a knot complement ($2v_3 = 2.02988\ldots$) [2]. This is expected, since the corresponding knots are torus knots (non-hyperbolic) and therefore the computed volume should be zero. The small non-zero values can be attributed to numerical errors.
2. The volumes corresponding to all the other words in the list are greater than $3.41791\ldots$ and therefore greater than the minimum volume of a knot complement, which can be interpreted as the corresponding knots being hyperbolic.

4 Conclusion and conjecture

The results from the computational test provide support for the following conjecture on the classification of torus knots:

Conjecture 1. All Lorenz knots corresponding to non-trivial syllable permutations of standard torus knot words are hyperbolic.

References

[1] C. Adams, M. Hildebrand, J. Weeks, Hyperbolic Invariants of Knots and Links. *Trans. Amer. Math. Soc.* **326**, 1-56 (1991)
[2] C. Adams, Hyperbolic Knots, In: W. Menasco, M. Thistlethwaite (Eds.), *Handbook of Knot Theory*. Elsevier, 2005
[3] J. Birman, R. Williams, Knotted Periodic Orbits in Dynamical Systems–1: Lorenz Equations. *Topol.* **22**, 47–82 (1983)
[4] J. Birman, I. Kofman, A New Twist on Lorenz Links. *J. Topol.* **2**, 227–248 (2009)
[5] J. Birman, Lorenz knots. arXiv:1201.0214 [math.GT] http://arxiv.org/abs/1201.0214
[6] E. Lorenz, Deterministic Nonperiodic Flow. *Journal of the Atmospheric Sciences* **20**, 130-141 (1963)
[7] R. Williams, The structure of Lorenz attractors, In: P. Bernard, T. Ratiu (Eds.), *Turbulence Seminar, Berkeley 1976/77*, Lecture Notes in Mathematics, Volume 615. Springer, 1977
[8] R. Williams, The Structure of Lorenz Attractors. *Publications Mathématiques de l'I.H.E.S.* **50**, 73-99 (1979)
[9] W. Tucker, A Rigorous ODE Solver ans Smale's 14th Problem. *Found. Comput. Math.* **2**, 53-117 (2002)
[10] J. Franks, R. Williams, Braids and the Jones Polynomial. *Trans. Amer. Math. Soc.* **303**, 97-108 (1987)
[11] N. Franco, L. Silva, Effective computation of the multivariable Alexander polinomial of Lorenz links. *Physica D* **237**, 3322–3328 (2008)
[12] P. Gomes, N. Franco, L. Silva, Families of hyperbolic Lorenz knots, In: C. Grácio et al. (Eds.), *Nonlinear Maps and their Applications*, Springer Proceedings in Mathematics & Statistics, vol. 57. Springer, 2014
[13] E. Thurston, Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry. *Bull. Amer. Math. Soc. (N. S.)* **6**, 357-381 (1982)

Paulo Gomes is a PhD student in Mathematics at the University of Évora, Portugal. His research interests are in the areas of dynamical systems, symbolic dynamics and knot theory.

Nuno Franco received the PhD degree in Mathematics at Universit de Bourgogne, Dijon, France. His research interests are in the areas of Artin groups, braid and knot theory and dynamical sistems. A computational and algorithmic approach is used in these fields. He has published research articles in reputed international journals of mathematics and physics.

Luís Silva received the PhD degree in Mathematics at Instituto Superior Técnico of Lisbon, under the supervision of J. Sousa Ramos. His research interests are in the areas of low dimensional autonomous and nonautonomous dynamical systems, symbolic dynamics, topological invariants, renormalization and applications to knot theory. He has published research articles in reputed international journals.